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A C^{*}-algebraic approach to supersymmetry

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This note is a supplement of my talk given at "Seminar on Applications of the Renormalization Group (RG) Methods in Mathematical Sciences" at RIMS of Kyoto University, September 11–13, 2013.

I propose two classes of supersymmetric C^{*}-dynamical systems. The former referred to as Case (I) is intended for exact supersymmetry hidden in fermion lattice systems. The latter referred to as Case (II) is intended for the usual meaning of supersymmetry between fermions and bosons. In my talk I explained Case (II), while no mention was made on Case (I). In this note I will focus on Case (I) giving summary of its framework.

Take a quadruple $(\mathcal{F}, \gamma, \alpha_t, \delta)$, where $\mathcal{F}$ is a unital graded C^{*}-algebra with a grading automorphism γ, $\{\alpha_t\,;\, t\in\mathbb{R}\}$ is a one-parameter group of *-automorphisms of $\mathcal{F}$, and δ is a superderivation of $\mathcal{F}$. Assume that

$$\alpha_t\cdot\gamma=\gamma\cdot\alpha_t.$$

Let $\mathcal{A}_\circ$ be a unital γ-invariant *-subalgebra of $\mathcal{F}$. Let $\delta:\mathcal{A}_\circ\mapsto\mathcal{F}$ be a linear map such that it is odd with respect to the grading:

$$\delta\cdot\gamma=-\gamma\cdot\delta \quad \text{on } \mathcal{A}_\circ,$$

and the graded Leibniz rule holds:

$$\delta(AB)=\delta(A)B+\gamma(A)\delta(B) \quad \text{for every } A,B\in\mathcal{A}_\circ.$$

We call this δ a superderivation of $\mathcal{F}$. Define the conjugation of δ as

$$\delta^{*}(A):=-\big(\delta\big(\gamma(A^{*})\big)\big)^{*} \quad \text{for } A\in\mathcal{A}_\circ.$$

If a superderivation δ_{s} satisfies

$$\delta_{\mathrm{s}}=\delta_{\mathrm{s}}^{*} \text{ on } \mathcal{A}_\circ,$$

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then it is said to be hermite. For a (non hermite) superderivation δ defined on $\mathcal{A}_\circ$, let

$$\delta_{s,1} := \delta + \delta^*, \quad \delta_{s,2} := i(\delta - \delta^*) \ \text{ on } \mathcal{A}_\circ.$$

The above $\delta_{s,1}$ and $\delta_{s,2}$ are hermite superderivations defined on $\mathcal{A}_\circ$. Conversely, any superderivation and its conjugate can be written as

$$\delta = \frac{1}{2}(\delta_{s,1} - i\delta_{s,2}), \quad \delta^* = \frac{1}{2}(\delta_{s,1} + i\delta_{s,2}) \text{ on } \mathcal{A}_\circ.$$

If a state on $\mathcal{F}$ is invariant under δ, then it is said to be supersymmetric.

We shall state our assumptions on C^*-dynamics. We divide them into two, the basic ones and the additional ones.

The basic assumptions:

Assume that the time evolution $\{\alpha_t\,;\ t \in \mathbb{R}\}$ is strongly continuous.

$$\lim_{t\to 0} \|\alpha_t(F) - F\| \longrightarrow 0 \text{ for every } \ F \in \mathcal{F}.$$

Let d_0 denote the infinitesimal generator of $\{\alpha_t\,;\ t \in \mathbb{R}\}$ and let $\mathcal{D}_{d_0}$ be its domain,

$$\mathcal{D}_{d_0} := \left\{X \in \mathcal{F};\ \lim_{t\to 0} \frac{1}{t}\big(\alpha_t(X) - X\big) \text{ exists in the norm } \right\}, \quad d_0(X) := -i\frac{d}{dt}\alpha_t(X)\Big|_{t=0} \in \mathcal{F} \ \text{ for } X \in \mathcal{D}_{d_0}.$$

From the γ-invariance of α_t it follows that

$$\gamma(\mathcal{D}_{d_0}) \subset \mathcal{D}_{d_0}, \quad d_0 \cdot \gamma = \gamma \cdot d_0 \ \text{ on } \mathcal{D}_{d_0}.$$

Assume that

$$\mathcal{A}_\circ \text{ is norm dense in } \ \mathcal{F}.$$

Assume that

$$\mathcal{A}_\circ \subset \mathcal{D}_{d_0}.$$

When we deal with supersymmetry, this inclusion should be generically strict. Finally we put the following crucial condition. It will be called 'differentiability of the superderivation'.

$$\delta(\mathcal{A}_\circ) \subset \mathcal{A}_\circ.$$

We shall formulate supersymmetric C^*-dynamics based on the basic assumptions stated above.

Definition. The following set of relations is referred to as infinitesimal supersymmetric dynamics:

$$\delta \cdot \delta = \mathbf{0} \quad \text{on} \quad \mathcal{A}_\circ,$$

$$d_0 = \delta^* \cdot \delta + \delta \cdot \delta^* \quad \text{on} \quad \mathcal{A}_\circ.$$

The additional assumptions:

We shall list additional assumptions upon the infinitesimal supersymmetric dynamics defined above.

Assume that

$$\overline{d_0|_{\mathcal{A}_\circ}} = d_0,$$

where the bar on $d_0|_{\mathcal{A}_\circ}$ denotes the norm closure. Next we require that

$$\delta : \mathcal{A}_\circ \mapsto \mathcal{F} \quad \text{is norm-closable.}$$

This means that for any sequence $\{A_n \in \mathcal{A}_\circ\}$,

$$\text{if} \quad \lim_{n\to\infty} A_n = 0 \quad \text{and also} \quad \lim_{n\to\infty} \delta(A_n) = B \quad \text{in norm,} \quad \text{then} \quad B = 0.$$

We denote the closure of δ by $\overline{\delta}$ and the extended domain of $\overline{\delta}$ by $\mathcal{D}_{\overline{\delta}}$. We may further assume that

$$\delta^* : \mathcal{A}_\circ \mapsto \mathcal{F} \quad \text{is norm-closable.}$$

Similarly we may assume that

$$\delta_{s,1} : \mathcal{A}_\circ \mapsto \mathcal{F} \quad \text{is norm-closable,} \quad \delta_{s,2} : \mathcal{A}_\circ \mapsto \mathcal{F} \quad \text{is norm-closable.}$$

Finally we assume that the time evolution $\{\alpha_t\,;\, t \in \mathbb{R}\}$ commutes with the action of δ:

$$\alpha_t(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta}} \quad \text{and} \quad \overline{\delta} \cdot \alpha_t = \alpha_t \cdot \delta \quad \text{on} \quad \mathcal{A}_\circ \quad \text{for every } t \in \mathbb{R}.$$

Similarly

$$\alpha_t(\mathcal{A}_\circ) \subset \mathcal{D}_{\overline{\delta^*}} \quad \text{and} \quad \overline{\delta^*} \cdot \alpha_t = \alpha_t \cdot \delta^* \quad \text{on} \quad \mathcal{A}_\circ \quad \text{for every } t \in \mathbb{R}.$$

We obtain the following results.

Theorem. *Assume the infinitesimal supersymmetric dynamics. Assume that all the additional assumptions are satisfied. Then the set of supersymmetric states with respect to the superderivation* δ *is a face in the state space of* $\mathcal{F}$. *Namely, if a supersymmetric state* φ *on* $\mathcal{F}$ *is written as a finite convex sum of states* $\varphi = \sum_i \lambda_i \varphi_i$, *where* $\lambda_i > 0$, $\sum_i \lambda_i = 1$, *and each* φ_i *is a state on* $\mathcal{F}$, *then each* φ_i *is supersymmetric.*

Theorem. *Assume the infinitesimal supersymmetric dynamics. Assume that all the additional assumptions are satisfied. Suppose that* φ *is an even supersymmetric state on* $\mathcal{F}$ *with respect to the hermite superderivation* δ_{s}. *Let* $(\pi_\varphi, \mathscr{H}_\varphi, \Omega_\varphi)$ *denote the GNS triplet of* φ. *Let* Q_{s} *denote the self-adjoint supercharge operator implementing* δ_{s} *on the GNS Hilbert space* $\mathscr{H}_\varphi$. *Let* $\mathfrak{M}_\varphi$ *denote the von Neumann algebra in* $\mathfrak{B}(\mathscr{H}_\varphi)$ *associated with* $(\pi_\varphi, \mathscr{H}_\varphi, \Omega_\varphi)$. *Then* Q_{s} *is affiliated to* $\mathfrak{M}_\varphi$.