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Integrals with respect to non additive measures

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1 Introduction

Integrals with respect to non additive measures integral have been studied as non linear integrals in various field [4, 5, 25, 37, 38, 39, 31]. The Choquet integral and Sugeno integral are the representatives of those ingagral and the generalization of the Choquet integral and Sugeno integral have been studied [1, 24]. On the other hand, the integrals which are not the generalization of the Choquet or Sugeno integral have been studied In this paper, we overview the definitions and basic properties of the following integrals:

- Sugeno integral [28]
- Choquet integral [2]
- Shilklet integral [26]
- Pan integral [33]
- Lower integral and Upper integral (Lehler integral) [10, 11, 12, 34, 35, 36].

We present the generalization of those integrals and show the direction of the investigation in the future.

2 Sugeno integral and Choquet integral

In this section, we will show the definition and basic properties of Choquet integral and Sugeno integral with respect to a non additive measure as the preliminaries.

Let S be a universal set, $(S, \mathcal{S})$ be a measurable space and $\mathcal{S}$ be σ-algebra of S.

Definition 1. Let $\mu : \mathcal{S} \longrightarrow R_+$.

We say that μ is a non additive measure if

1. $\mu(\emptyset) = 0$,

2. $A \subset B,\ A, B \in \mathcal{S} \Rightarrow \mu(A) \leq \mu(B)$.

We say that a function $f : S \to R_+$ is measurable if $\{x | f(x) \geq a\} \in \mathcal{S}$ for all $a \geq 0$.
$\mathcal{F}^+(S)$ denotes a set of non negative measurable function that is ,
$\mathcal{F}^+(S) = \{f | f : S \to R_+, f : \text{measurable}\}$.
We will show thae definition of Sugeno integral [28].

Definition 2. Let μ be a non additive measure on $(S, \mathcal{S})$ with $\mu(S) = 1$.
Let $f : S \to [0, 1]$ be a measurable function.
A Sugeno integral of f with respect to μ is defined by

$$(S) \int f d\mu := \vee_{r \in [0,1]} (r \wedge \mu_f(r))$$

where $\mu_f(r) := \mu(\{x | f(x) > r\})$.

The domein of Sugeno integral is restricted to a class of function $f : S \to [0, 1]$. Extended Sugeno integral is proposed as follows:

Definition 3. μ be a non additive measure on $(S, \mathcal{S})$ with $\mu(S) = \infty$.
Let $f : S \to [0, \infty)$ be a measurable function.
The extended Sugeno integral of f with respect to μ is defined by

$$(S) \int f d\mu := \quad \vee_{r \in [0,\infty)} (r \wedge \mu_f(r)),$$

where $\mu_f(r) := \mu(\{x | f(x) > r\})$.

Sugeno integral uses only the order of each value of function, not requires addtion or multiplication. Sugeno integral is useful for problem with ordinal scale, but it is weak for transformation of scale.

Even if μ is $\sigma-$additive, Sugeno integral is not coincide with Lebesgue integral. That is, Sugeno integral is not the extension of classical integral.

Next we will show the definiton of Choquet integral.

Definition 4. Let μ be a non additive measure on $(S, \mathcal{S})$.
Let $f : S \to [0, \infty)$ be a measurable function.
Choquet integral of f with respect to μ is defined as

$$(C) \int f d\mu := \int_0^\infty \mu_f(r) dr$$

where $\mu_f(r) := \mu(\{x | f(x) > r\})$.

The similar idea to Choquet integral is presented by Vitali 1925 [32]. König [9] is also presented the same integral, called horizontal Integral.

If $|S| < \infty$,

$$(C)\int f(\theta)\,d\mu = \sum_{i=1}^{n}[f(\theta_i) - f(\theta_{i-1})]\mu(F_i).$$

3 Upper and lower integral, Pan integral and Shilklet integral

First we will show the definition of Upper and Lower integral proposed by Wang et al [34, 36].

Definition 5. Let μ be a non additive measure on $(S, \mathcal{S})$.

Let $f : S \to [0, \infty)$ be a measurable function.

Upper integral of f with respect to μ is defined as

$$(U)\int f d\mu := \lim_{\epsilon \to +0} U_\epsilon$$

where

$U_\epsilon := \sup\{\sum_{j=1}^{\infty} \lambda_j \mu(A_j) | f - \epsilon \leq \sum_{j=1}^{\infty} \lambda_j 1_{A_j} \leq f, A_j \in \mathcal{S}\}.$

Lower integral of f with respect to μ is defined as

$$(L)\int f d\mu := \lim_{\epsilon \to +0} L_\epsilon$$

where

$L_\epsilon := \inf\{\sum_{j=1}^{\infty} \lambda_j \mu(A_j) | f \leq \sum_{j=1}^{\infty} \lambda_j 1_{A_j} \leq f + \epsilon, A_j \in \mathcal{S}\}.$

The similar integral to upper integral is proposed by by Lehler [10, 11].

Definition 6.

Let be $S := \{1, 2, \ldots, n\}$ and $\mathcal{S} = 2^S$, μ: a non additive measure on $(S, \mathcal{S})$ and

$$\Phi_\mu := \{\phi : R_+^n \to R_+ | \text{homogeneous,concave s. t.} \varphi(1_A) \geq \mu(A) \text{for} A \in \mathcal{S}\}.$$

Let $f : S \to [0, \infty)$ be measurable.

A Lehler integral of f with respect to μ is defined by

$$(Le)\int f d\mu := \inf_{\phi \in \Phi_\mu} \varphi(f)$$

We have a proposition below as the Basic properties of Lehler integral [10].

Proposition 7. *Let be $S := \{1, 2, \ldots, n\}$ and $\mathcal{S} = 2^S$, μ: a non additive measure on $(S, \mathcal{S})$ and $f : S \to [0, \infty)$ be measurable.*

1. $(Le)\int f d\mu = \max\{\sum_{A\subset S} \alpha_A \mu(A) | \sum_{A\subset S} \alpha_A 1_A = f, \alpha_A \geq 0\}$

2. $(Le)\int f d\mu = \min_{P: additive, P\geq\mu} \int f dP$

3. *If μ is super modular, then*
   $(Le)\int f d\mu = (C)\int f d\mu$
   for all $f \in \mathcal{F}^+(S)$.

It follows from the proposition above that

$$(Le)\int f d\mu = (U)\int f d\mu.$$

Next we will introduce Pan integral by Yang [33] and Shilklet integral by Shilklet [26].

Definition 8. *(Pan integral)*

Let μ be a non additive measure on $(S, \mathcal{S})$. Let $f : S \to [0, \infty)$ be measurable.

The pan integral of f with respect to μ is defined as

$$(P)\int f d\mu = \sup\{\sum_{i=1}^{n} c_i \mu(A_i) | \sum_{i=1}^{n} c_i 1_{A_i} \leq f, A_i \textit{is a partition of } S\}.$$

Definition 9. *(Shilklet integral)*

Let μ be a non additive measure on $(S, \mathcal{S})$. Let $f : S \to [0, \infty)$ be measurable.

The Shilklet integral of f with respect to μ is defined as

$$(Sh)\int f d\mu = \sup\{c\mu(A) | c1_A \leq f\}.$$

The example of a comparison among the integrals is shown as following:

Example 1. *Example a group of worker by Mesiar and Stupnanova [16]*

Let $S = \{x_1, x_2, x_3\}$ be a group of workers. The daily performances are expressed as the non additive measure below:

$\mu(\{x_1\}) = 2$, $\mu(\{x_2\}) = 3$, $\mu(\{x_3\}) = 4$,

$\mu(\{x_1, x_3\}) = 4$, $\mu(\{x_1, x_2\}) = 7$, $\mu(\{x_2, x_3\}) = 5$,

$\mu(S) = 8$

The availability of single workers in working days is a function from S expressed below:

$f(x_1) = 5$,

$f(x_2) = 4,$

$f(x_3) = 3.$

The problem is how to organize the working plan to attain the maximal global performance, being constraint by one of the following " working laws ".

1. *(Working law 1)*

   Only one group of workers can work for a fixed time period;

   Then we we have the Shilklet integral,

   $(Sh) \int f d\mu = \sup\{c\mu(A) | c1_A \le f\}$
   $= f(x_1)\mu(\{x_1, x_2\}) = 4 * 7 = 28$

2. *(Working law 2)*

   Several disjoint groups of workers can work, each for its fixed time period;

   Then we have the Pan integral,

   $(P) \int f d\mu = f(x_2)\mu(\{x_1, x_2\}) + f(x_3)\mu(\{x_3\})$
   $= 4 * 7 + 3 * 4 = 40$

3. *(Working law 3)*

   One group of workers starts to work, a worker after stopping his work cannot start again;
   Then we have the Choquet integral,

   $(C) \int f d\mu = (f(x_1) - f(x_2))\mu(\{x_1\}) + (f(x_2) - f(x_3))\mu(\{x_1, x_2\}) + f(x_3)\mu(\{x_1, x_2, x_3\})$
   $= 1 * 2 + 1 * 7 + 3 * 8 = 33$

4. *(Working law 4)*

   There is no constraint.

   The we have the Upper integral or Lehler integral.

   $(U) \int f d\mu = (f(x_1) - f(x_2))\mu(\{x_1\}) + f(x_2)\mu(\{x_1, x_2\}) + f(x_3)\mu(\{x_3\})$
   $= (5 - 4) * 2 + 4 * 7 + 3 * 4 = 42$

4 Generalization of integrals

The generalization of the integral with respect to non additive measure are major theme of non-additive measure theory. In this section we introduce two direction of the generalization: the generalization of binary operations and indeterminate integral or decomposition integral.

4.1 Generalized binary operation -pseudo addition, pseudo multiplication

We define a generalized fuzzy integral in terms of a pseudo-addition $\oplus$ and a pseudo-multiplication $\boxdot$. Formally, $\oplus$ and $\boxdot$ are binary operators that generalize addition and multiplication, and also max and min. We want to recall that generalized fuzzy integrals have been investigated by Benvenuti et al. 2002 in [1].

Similar approach : Murofushi, Sugeno 1991. [19]

Note that we will use $k \in (0, \infty)$ in the rest of this paper.

Definition 10. *A pseudo-addition $\oplus$ is a binary operation on $[0, k]$ or $[0, \infty)$ fulfilling the following conditions:*

(A1) $x \oplus 0 = 0 \oplus x = x$.

(A2) $x \oplus y \leq u \oplus v$ *whenever* $x \leq u$ *and* $y \leq v$.

(A3) $x \oplus y = y \oplus x$.

(A4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

(A5) $x_n \to x, y_n \to y$ *implies* $x_n \oplus y_n \to x \oplus y$.

A pseudo-addition $\oplus$ is said to be strict if and only if $x \oplus y < x \oplus z$ whenever $x > 0$ and $y < z$, for $x, y, z \in (0, k)$; and it is said to be Archimedean if and only if $x \oplus x > x$ for all $x \in (0, k)$.

Definition 11. *A pseudo-multiplication $\boxdot$ is a binary operation on $[0, k]$ or $[0, \infty)$ fulfilling the conditions:*

(M1) *There exists a unit element* $e \in (0, k]$ *such that* $x \boxdot e = e \boxdot x = x$.

(M2) $x \boxdot y \leq u \boxdot v$ *whenever* $x \leq u$ *and* $y \leq v$.

(M3) $x \boxdot y = y \boxdot x$.

(M4) $(x \boxdot y) \boxdot z = x \boxdot (y \boxdot z)$.

(M5) $x_n \to x, y_n \to y$ *implies* $x_n \boxdot y_n \to x \boxdot y$.

Example 2.

1. The maximum operator $x \vee y$ is a non Archimedean pseudo-addition on $[0, k]$.

2. The sum $x + y$ is an Archimedean pseudo-addition on $[0, \infty)$.

3. The Sugeno operator $x +_\lambda y := 1 \wedge (x + y + \lambda xy)$ $(-1 < \lambda < \infty)$ is an Archimedean pseudo-addition on $[0,1]$.

Proposition 12. *(Ling, 1965, [13])*

If a pseudo-addition $\oplus$ is Archimedean, then there exists a continuous and strictly increasing function $g : [0,k] \to [0,\infty]$ such that $x \oplus y = g^{(-1)}(g(x)+g(y))$, where $g^{(-1)}$ is the pseudo-inverse of g defined by

$$g^{(-1)}(u) := \begin{cases} g^{(-1)}(u) & \text{if } \ u \le g(k) \\ k & \text{if } \ u > g(k). \end{cases}$$

The function g is called an additive generator of $\oplus$.

Definition 13. *Let μ be a fuzzy measure on a fuzzy measurable space $(S, \mathcal{S})$; then, we say that μ is a $\oplus$-measure or a $\oplus$-decomposable fuzzy measure if $\mu(A \cup B) = \mu(A) \oplus \mu(B)$ whenever $A \cap B = \emptyset$ for $A, B \in \mathcal{S}$.*

A $\oplus$-measure μ is called normal when either $\oplus = \vee$, or $\oplus$ is Archimedean and $g \circ \mu$ is an additive measure. Here, g corresponds to an additive generator of $\oplus$.

Definition 14. *Let $k \in (0,\infty)$, let $\oplus$ be a pseudo-addition on $[0,k]$ or $[0,\infty)$ and let $\boxdot$ be a pseudo-multiplication on $[0,k]$ or $[0,\infty)$; then, we say that $\boxdot$ is $\oplus$-fitting if*

(F1) *$a \boxdot x = 0$ implies $a = 0$ or $x = 0$,*

(F2) *$a \boxdot (x \oplus y) = (a \boxdot x) \oplus (a \boxdot y)$.*

Under these conditions, we say that $(\oplus, \boxdot)$ is a pseudo-fitting system.

Let $\oplus$ be a pseudo-addition; then, we define its pseudo-inverse $-_\oplus$ as

$$a -_\oplus b := \inf\{c | b \oplus c \ge a\}$$

for all $(a,b) \in [0,k]^2$.

Definition 15. *Benvenuti.Mesiar.Vivona.2002 [1]*

For any $r > 0$ and $A \in \mathcal{S}$, the basic simple function $b(r,A)$ is defined by $b(r,A)(x) = r$ if $x \in A$ and $b(r,A)(x) = 0$ if $x \notin A$.

Then, we say that a function f is a simple function if it can be expressed as

$$f := \oplus_{i=1}^{n} b(a_i, A_i) \text{ for } a_i > 0. \tag{1}$$

where $A_i \in \mathcal{S}$.

Expression 1 is called a comonotonic additive representation of f, if $\oplus = +$,$A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_n$.

Definition 16. *Let μ be a fuzzy measure on a measurable space $(S, \mathcal{S})$, and let $(\oplus, \boxdot)$ be a pseudo-fitting system. Then, the generalized fuzzy integral (GF-integral) of a measurable simple function $f := \oplus_{i=1}^{n} b(a_i, A_i)$, with $a_i > 0$ and $A_1 \supsetneq A_2 \supsetneq \dots A_n$, $A_i \in \mathcal{X}$, is defined as follows:*

$$(GF) \int f d\mu := \oplus_{i=1}^{n} a_i \boxdot \mu(A_i).$$

The GF-integral of a simple function is well defined [1].

Example 3.

1. When $\oplus = +$ and $\boxdot = \cdot$, the generalized fuzzy integral is a Choquet integral.

2. When $\oplus = \vee$ and $\boxdot = \wedge$, the generalized fuzzy integral is a Sugeno integral.

4.2 Decomposition Integral or indeterminate integral

Decomposition integral by Even, Y. Lehrer, E. 2011 [12] is one of the approach which unifies the Choquet and the Lehler Integrals. Similar approach is shown by Wang et al. 2003 [35] as indeterminate integral. We present the definitions of the decomposition integral and the indeterminate integral and show that both are essentially the same.

Definition 17. *Let μ be a fuzzy measure on $(S, \mathcal{S})$.*
We say that the subset $\mathcal{H}$ of $\mathcal{S}$ ($\mathcal{H} \subset \mathcal{S}$) is a decomposition system on $(S, \mathcal{S})$ if $\cup_{H \in \mathcal{H}} H = S$.
The function $I_{\mathcal{H},\mu} : \mathcal{F}^+(S) \to [0, \infty)$ given by

$$I_{\mathcal{H},\mu}(f) := \sup\{\sum_{i=1}^{n} c_i \mu(A_i) | \sum_{i=1}^{n} b(c_i, A_i) \leq f, A_i \in \mathcal{H}\}$$

is called a decomposition integral.

The next proposition shows that the decomposition integral is a one of the generalization of the integrals.

Proposition 18. *Let $\mathcal{H}$ be a decomposition system on $(S, \mathcal{S})$.*

1. *If $\mathcal{H} := \{\{A\} | A \in \mathcal{S} \setminus \{\emptyset\}\}$, then a decomposition integral $I_{\mathcal{H},\mu}(f)$ is the Shilklet integral.*

2. *If $\mathcal{H} := \{C | C is a finite chain in \mathcal{S}\}$, a decomposition integral $I_{\mathcal{H},\mu}(f)$ is the Choquet integral.*

3. *If $\mathcal{H} := \{\mathcal{P} | \mathcal{P} is a finite partition of (S, \mathcal{S})\}$, a decomposition integral $I_{\mathcal{H},\mu}(f)$ is the Pan integral.*

4. If $\mathcal{H} := \{\mathcal{D} | \mathcal{D}$ is a finite non-empty subset of $\mathcal{S}\}$, a decomposition integral $I_{\mathcal{H},\mu}(f)$ is the Lehler integral.

Next we will present the indeterminate integral. In the following, we suppose that S is a finite set.

Definition 19. *Let $f \in \mathcal{F}^+(S)$.*

A set function $\alpha_f : \mathcal{S} \to [0, \infty)$ with $\alpha_f(\emptyset) = 0$ is called the decomposition of f if

$$\sum_{x \in A} \alpha_f(A) = f(x)$$

for $x \in S$.

Then f is expressed as

$$f = \sum_{A \subset X} \alpha_f(A) 1_A$$

The indeterminate integral with respect to μ under the decomposition α_f is defined by

$$(I) \int f d\mu_{|\alpha_f} = \sum_{A \subset X} \alpha_f(A)\mu(A).$$

The family of all indeterminate integral denotes $(I) \int f d\mu_{|\alpha_f}$, that is,

$$(I) \int f d\mu = \{\sum_{A \subset X} \alpha_f(A)\mu(A) | \sum_{A \subset X} \alpha_f(A) 1_A = f\}.$$

The next proposition is obvious from the definitions.

Proposition 20. *Let μ be a fuzzy measure on $(S, \mathcal{S})$, $\mathcal{H}$ be a decomposition system on $(S, \mathcal{S})$ and $I_{\mathcal{H},\mu}$ be a decomposition integral on $\mathcal{F}^+(S)$.*

Then we have

$$(I) \int f d\mu \subset \{I_{\mathcal{H},\mu}(f) | \mathcal{H} \subset \mathcal{S}\}.$$

It is easy to extend the definition of the indeterminate integral to the infinite S. In this case, the indeterminate integral is $\{I_{\mathcal{H},\mu}(f) | \mathcal{H} \subset \mathcal{S}\}$.

5 Concluding remarks

We have shown the definition and the basic properties of the Choquet. the Sugeno, the Shilkret, the Pan, the Lower integrals and Upper integrals with respect to a non additive measure and their generalizations. Several problems to be solved near future are listed below.

1. The convergence theorems of integral, which depend on the continuity of μ have been proved separately [4, 31, 28]. Unified approach will be possible. The problem is to find a ultimate generalization.

2. There are very few study of integrals under topological setting [7, 22, 23, 30]. The problem is to find a suitable topological assumption for μ, which might be called regularity.

3. There are very few study of integrals extending to negative value [6, 20, 21]. Extension to negative value or extension of lattice structure will be a problem to be considered in the near future.

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