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STRONG CONVERGENCE THEOREMS FOR FAMILIES OF NONEXPANSIVE MAPPINGS BY BROWDER'S TYPE ITERATIONS

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ABSTRACT. In this paper, we study Browder's type iterations for nonexpansive semigroups in Banach spaces. Then, we study strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces. We also give strong convergence theorems for the nonexpansive semigroups in Banach spaces by the viscosity approximation method.

1. INTRODUCTION

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Then, a mapping $T : C \to C$ is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Let x be an element of C and for each t with $0 < t < 1$, let x_t be a unique element of C satisfying $x_t = tx + (1-t)Tx_t$. In 1967, Browder [5] proved the following strong convergence theorem.

Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty bounded closed convex subset of H and let T be a nonexpansive mapping of C into itself. Let x be an element of C and for each t with $0 < t < 1$, let x_t be a unique element of C satisfying*

$$x_t = tx + (1-t)Tx_t.$$

Then, $\{x_t\}$ converges strongly to the element of $F(T)$ nearest to x as $t \downarrow 0$.

Reich [18] and Takahashi and Ueda [31] extended Browder's result to those of a Banach space. Using the idea of Shimizu and Takahashi [19, 20] and the notion of sequence of means, Shioji and Takahashi [21] proved the strong convergence of Browder's type sequences for nonexpansive semigroups (see also [22, 23, 24]). On the other hand, Domingues Benavides, Acedo and Xu [10] proved Browder's type strong convergence theorems for uniformly asymptotically regular one-parameter nonexpansive semigroups. Acedo and Suzuki [14] generalized Domingues Benavides, Acedo and Xu's results concerning the condition of the sequences in real numbers. Author [1] studied Browder's type iterations for nonexpansive semigroups and proved strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Hilbert spaces by using the idea of [5, 10, 14, 29, 30]. The author [1] also proved strong convergence theorems for the nonexpansive semigroups by the viscosity approximation

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method. Furthermore, author [2] proved strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in a smooth Banach space which satisfies Opial's condition.

In this paper, we study Browder's type iterations for nonexpansive semigroups in Banach spaces. Then, we study strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces by using the idea of [1, 5, 10, 14, 29, 30]. Furthermore, we also give strong convergence theorems for the nonexpansive semigroups in Banach spaces by the viscosity approximation method.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{Z}^+$ and $\mathbb{R}^+$ the set of all nonnegative integers and the set of all nonnegative real numbers, respectively.

Let E be a real Banach space with norm $\|\cdot\|$. We denote by B_r the set $\{x \in E : \|x\| \leq r\}$. Let E^* be the dual space of a Banach space E. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$. Let C be a closed subset of a Banach space and let T be a mapping of C into itself. We denote by $F(T)$ the set $\{x \in C : x = Tx\}$.

We denote by I the identity operator on E. The duality mapping J from E into 2^{E^*} is defined by

$$J(x) = \{y^* \in E^* : \langle x, y^* \rangle = \|x\|^2 = \|y^*\|^2\}, x \in E.$$

From the Hahn-Banach theorem, we see that $J(x) \neq \emptyset$ for all $x \in E$.

Let E be a smooth Banach space. Then, J is said to be weakly sequentially continuous at zero if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

We say that a Banach space E satisfies *Opial's condition* [16] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{1}$$

for each $y \in E$ with $y \neq x$. If E is reflexive Banach space with weakly sequentially continuous duality mapping, then E satisfies Opial's condition. Each Hilbert space and the sequence spaces ℓ^p with $1 < p < \infty$ satisfy Opial's condition (see [16]). Though an L^p-space with $p \neq 2$ does not usually satisfy Opial's condition, each separable Banach space can be equivalently renormed so that it satisfies Opial's condition (see [11, 16]). In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [13]. It is well known that this condition is equivalent to the analogous condition of $\varlimsup$ (see [4]).

Proposition 2.1. *Let H be a Hilbert space. Let $\{x_n\}$ be a sequence in H converging weakly to $x \in H$. Then,*

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{2}$$

for each $y \in E$ with $y \neq x$.

Banach space E is said to be smooth if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each x and y in S_1, where $S_1 = \{u \in E : \|u\| = 1\}$. The norm of E is said to be uniformly Gâteaux differentiable if for each y in S_1, the limit is attained uniformly for x in S_1. We know that if E is smooth, then the duality mapping is single-valued and norm to weak star continuous and that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping is single-valued and norm to weak star, uniformly continuous on each bounded subset of E.

A closed convex subset C of a Banach space E is said to have normal structure if for each bounded closed convex subset K of C which contains at least two points, there exists an element of K which is not a diametral point of K. It is well-known that a closed convex subset of a uniformly convex Banach space has normal structure and a compact convex subset of a Banach space has normal structure (see [30]). We also know that uniformly smooth Banach space has normal structure (see [30]). Every weakly compact convex subset of a Banach space satisfying Opial's condition has normal structure (see [12]). We note that closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings if for every bounded closed convex subset K of C, every nonexpansive mapping on K, has a fixed point. We also know that every weakly compact convex subset with Opial property has fixed point property.

Let C be a nonempty closed convex subset of E and let K be a nonempty subset of C. A mapping P of C onto K is said to be sunny if $P(Px+t(x-Px)) = Px$ for each $x \in C$ and $t \geq 0$ with $Px+t(x-Px) \in C$. P is a retraction if $Px = x$ for each $x \in K$. We know from [7, Theorem 3] and [17, Lemma 2.7] the following lemma (see also [30]).

Lemma 2.2 ([7, 17]). *Let E be a smooth Banach space, let C be a convex subset of E and let K be a subset of C. Then, a retraction P of C onto K is sunny and nonexpansive if and only if*

$$\langle x-Px, J(y-Px)\rangle \leq 0 \quad \text{for all} \quad x\in C \quad \text{and} \quad y \in K.$$

Hence, there is at most one sunny nonexpansive retraction of C onto K.

If there is a sunny nonexpansive retraction of C onto K, K is said to be a sunny nonexpansive retract of C. The following theorem related to the existence of nonexpansive retractions was proved in [8, 9].

Theorem 2.3 ([8, 9]). *Let E be a reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. If T has a fixed point in every nonempty bounded closed convex subset of E such that T leaves invariant, then $F(T)$ is a nonexpansive retract of C.*

Let μ be a mean on positive integers $\mathbb{N}$, i.e., a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. We know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n : n\in\mathbb{N}\} \leq \mu(f) \leq \sup\{a_n : n \in \mathbb{N}\}$$

for each $f = (a_1, a_2, \ldots) \in l^\infty$. Occasionally, we use $\mu_n(a_n)$ instead of $\mu(f)$. So, a Banach limit μ is a mean on $\mathbb{N}$ satisfying

$$\mu_n(a_n) = \mu_n(a_{n+1}).$$

Let $f = (a_1, a_2, \ldots) \in l^\infty$ and let μ be a Banach limit on $\mathbb{N}$. Then,

$$\underline{\lim}_{n\to\infty}\, a_n \leq \mu(f) = \mu_n(a_n) \leq \overline{\lim}_{n\to\infty}\, a_n.$$

In particular, if $a_n \to a$, then $\mu(f) = \mu_n(a_n) = a$ (see [28, 30]). The following lemma was proved in [31] (see also [18, 28]).

Lemma 2.4 ([31]). *Let C be a nonempty closed convex subset of a Banach space with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence in E and let μ be a Banach limit. Let $z \in C$. Then,*

$$\mu_n\|x_n - z\|^2 = \min_{y\in C} \mu_n\|x_n - y\|^2$$

if and only if $\mu_n\langle y - z, J(x_n - z)\rangle \leq 0$ for each $y \in C$, where J is the duality mapping of E.

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

Let S be a semitopological semigroup. A semitopological semigroup S is called right (resp. left) reversible if any two closed left (resp. right) ideals of S have nonvoid intersection. If S is right reversible, $(S, \leq)$ is a directed system when the binary relation "$\leq$" on S is defined by $s \leq t$ if and only if $\{s\} \cup \overline{Ss} \supset \{t\} \cup \overline{St}$, $s, t \in S$, where $\overline{A}$ is the closure of A. A commutative semigroup S is a directed system when the binary relation is defined by $s \leq t$ if and only if $\{s\} \cup (S + s) \supset \{t\} \cup (S + t)$.

Let C be a nonempty closed convex subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a nonexpansive semigroup on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;
(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$.

We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, i.e., $F(\mathcal{S}) = \bigcap_{t\in S} F(T(t))$.

3. Lemma

In this section, we give some lemmas which plays an important role in the proof of our main results (see also [1, 2, 27]).

Lemma 3.1 ([2]). *Let C be a nonempty closed convex subset of a Banach space E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let m be a positive integer and let $t \in S$. Let u be an element of C and for each α with $0 < \alpha < 1$, let $Q(u, \alpha)$ be the unique element of C satisfying*

$$Q(u, \alpha) = \alpha u + (1 - \alpha)(T(t))^m Q(u, \alpha).$$

Assume that E is smooth. Then, for every $v \in F(\mathcal{S})$,

$$\|Q(u,\alpha) - v\|^2 \leq \langle u - v, J(Q(u,\alpha) - v)\rangle \tag{3}$$

and

$$\langle u - Q(u,\alpha), J(v - Q(u,\alpha))\rangle \leq 0 \tag{4}$$

hold.

Lemma 3.2 ([2]). *Let C be a nonempty closed convex subset of a Banach space E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ and let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$. Let $u \in C$, let $t \in S$, and let $\{Q(u,n)\}$ be the sequence defined by*

$$Q(u,n) = \alpha_n u + (1-\alpha_n)(T(t))^{m_n} Q(u,n)$$

for each $n \in \mathbb{N}$. Assume that E is smooth. Then, the following hold:

(i) If for every $u \in C$, $\{Q(u,n)\}$ has a subsequence converging strongly to an element (say Pu) of $F(\mathcal{S})$, then P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

(ii) If for every $u \in C$, every subsequence of $\{Q(u,n)\}$ has a subsequence converging strongly to an element of $F(\mathcal{S})$, then $\{Q(u,n)\}$ converges strongly to an element (say Pu) of $F(\mathcal{S})$ and P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Lemma 3.3 ([2]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1-\alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that E is smooth, the normalized duality mapping J of E is weakly sequentially continuous at zero and C has the Opial property. Assume also that $\{x_n\}$ converges weakly to some $x \in F(\mathcal{S})$. Then, $\{x_n\}$ converges strongly.

Lemma 3.4 ([3]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1-\alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that C has the fixed point property for nonexpansive mappings. Let μ be a Banach limit and let $g(y) = \mu_n \|y - x_n\|$. Put

$$K = \{z \in C : g(z) = \min_{y \in C} g(y)\}$$

and assume $K \cap F(\mathcal{S}) \neq \emptyset$. Then, every subsequence of $\{x_n\}$ has a subsequence converging strongly to an element in $K \cap F(\mathcal{S})$.

4. STRONG CONVERGENCE THEOREMS

In this section, we study strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces.

Let C be a nonempty closed convex subset of a Banach space E, let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. We say that a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically regular if

$$\lim_{s \in S} \|T(h)T(s)x - T(s)x\| = 0$$

for all $h \in S$ and $x \in C$ (see also [29, 30]). The following lemma plays an important role in the proof of our main results (see [1, 14]).

Lemma 4.1 ([2]). *Let C be a nonempty closed convex subset of a Banach space E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Assume that $\mathcal{S} = \{T(t) : t \in S\}$ is asymptotically regular, that is,*

$$\lim_{t \in S} \|T(h)T(t)x - T(t)x\| = 0$$

for all $h \in S$ and $x \in C$. Then,

$$F(T(h)) = F(\mathcal{S})$$

for each $h \in S$.

We say that a nonexpansive semigroup $\mathcal{S} = \{T(t) : t \in S\}$ is uniformly asymptotically regular if for every $h \in S$ and for every bounded subset K of C,

$$\lim_{s \in S} \sup_{x \in K} \|T(h)T(s)x - T(s)x\| = 0.$$

holds.

Several authors proved strong convergence theorems for uniformly asymptotically regular one-parameter nonexpansive semigroups by Browder's type iterations (see also [6, 10, 14, 27]). We study strong convergence theorems for uniformly asymptotically regular nonexpansive semigroups in Banach spaces by using the idea of [1, 5, 10, 27, 29, 30].

Theorem 4.2 ([2]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that E is smooth, the normalized duality mapping J of E is weakly sequentially continuous at zero and C has the Opial property. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Theorem 4.3 ([3]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$, let $t \in S$, and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that C has the fixed point property for nonexpansive mappings. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

5. Deduced theorems

We can deduce some strong convergence theorems from our main results.

We know that $f : C \to C$ is said to be a contraction on C if there exists $r \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq r\|x - y\|$$

for each $x, y \in C$. Using [26] and Theorem 4.2, we obtain the following strong convergence theorem by the viscosity approximation methods (see also [1, 2, 3, 15]).

Theorem 5.1 ([2]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let f be a contraction on C. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $t \in S$ and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that E is smooth, the normalized duality mapping J of E is weakly sequentially continuous at zero and C has the Opial property. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Using [26] and Theorem 4.3, we also obtain the following strong convergence theorem by the viscosity approximation methods (see also [1, 2, 3, 15]).

Theorem 5.2 ([3]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a locally weakly compact convex subset of E, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a uniformly asymptotically regular nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let f be a contraction on C. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $t \in S$ and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)(T(t))^{m_n} x_n$$

for each $n \in \mathbb{N}$. Assume that C has the fixed point property for nonexpansive mappings. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Let C be a nonempty closed convex subset of E. A family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(i) For each $t \in \mathbb{R}^+$, $T(t)$ is nonexpansive;
(ii) $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{R}^+$;
(iii) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

In the case when $S = \mathbb{R}^+$, that is, $\mathcal{S}$ is a uniformly asymptotically regular one-parameter nonexpansive semigroup, we have the following strong convergence theorem for the semigroup by Theorem 4.2 (see also [10, 14]).

Theorem 5.3 ([2]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$ and let $t \in (0, \infty)$. Let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)T(t^{m_n})x_n$$

for each $n \in \mathbb{N}$. Assume that E is smooth, the normalized duality mapping J of E is weakly sequentially continuous at zero and C has the Opial property. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

We also have the following strong convergence theorems for a one-parameter nonexpansive semigroup by Theorem 4.3 (see also [10, 14]).

Theorem 5.4 ([3]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a locally weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $u \in C$ and let $t \in (0, \infty)$. Let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n u + (1 - \alpha_n)T(t^{m_n})x_n$$

for each $n \in \mathbb{N}$. Assume that C has the fixed point property for nonexpansive mappings. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Using [26] and Theorem 5.3, we obtain the following strong convergence theorem by the viscosity approximation methods (see also [1, 2, 10, 14, 15, 26]).

Theorem 5.5 ([2]). *Let E be a Banach space, let C be a locally weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let f be a contraction on C. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $t \in (0, \infty)$ and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t^{m_n})x_n$$

for each $n \in \mathbb{N}$. Assume that E is smooth, the normalized duality mapping J of E is weakly sequentially continuous at zero and C has the Opial property. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

Using [26] and Theorem 5.4, we obtain the following strong convergence theorem by the viscosity approximation methods (see also [1, 2, 10, 14, 15, 26]).

Theorem 5.6 ([3]). *Let E be a Banach space whose norm is uniformly Gâteaux differentiable, let C be a locally weakly compact convex subset of E. Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let f be a contraction on C. Let $\{m_n\}$ be a sequence in $\mathbb{Z}^+$ such that $m_n \to \infty$ or $m_n \to N$ for some $N \in \mathbb{Z}^+$. Let $\{\alpha_n\}$ be a sequence in $\mathbb{R}$ such that $0 < \alpha_n < 1$, and $\alpha_n \to 0$. Let $t \in (0, \infty)$ and let $\{x_n\}$ be the sequence defined by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)T(t^{m_n})x_n$$

for each $n \in \mathbb{N}$. Assume that C has the fixed point property for nonexpansive mappings. Then, $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $F(\mathcal{S})$.

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