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微小振動子の入力場位相揺らぎに対するロバストフィルタリング及び LQG 制御

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The quantum Kalman filter and LQG controller provides enough specification for accurate estimation and control in linear quantum system, though this holds only in the case when a trustful mathematical model of the system is available. In this paper, we consider an optomechanical oscillator which is continuously monitored via an optical laser, although the laser phase is unknown. For this system, we propose a robust scheme of both filtering and control against the phase uncertainty of the optical laser. The filter estimates the uncertain parameter using an auxiliary estimator that adaptively changes the filtering and control algorithm so that it aquires the better estimation and control performance near to that of the optimal one. The robustness is demonstrated in a numerical simulation.

1 Introduction

To realize various nano-architectures including quantum information processors [9], we need accurate measurement and control of the system under consideration. *Quantum filtering theory* [2, 3, 4] and its application to quantum feedback control [13, 14] serve basic methodologies satisfying these requirements and have actually shown the potential usefulness [1, 6, 11]. For linear quantum systems, especially, the filter and

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the controller have completely the same form as the traditional Kalman filter and LQG controller, thus such a *quantum Kalman filter* and a *quantum LQG controller* are enough implementable in reality.

However, as in the classical case (we here call "classical" only because they are not quantum systems), the quantum Kalman filter and the quantum LQG controller work well only when a trustful mathematical model of the system under consideration is available. Fortunately, for this long-standing issue, the classical system theory has provided a number of solutions; they are largely classified to two approaches.

The first is based on the notion of "soft sensing", that is, the estimation algorithm is modified, without adding a real sensor, so that the filter and the controller acquire robustness against the model uncertainty [8, 10].

The second is the "hardware sensing" method that simply introduces actual sensors so as to obtain more information utilizable for the better estimation under uncertainties. This method is, classically more or less, never inferior to the above soft sensing technique, if one affords to buy an expensive high-quality sensor required. In the quantum case, however, we cannot always have such a hopeful statement. This is due to the unavoidable *quantum back-action* property; that is, introducing a hardware sensor must bring additional noise to the system, thus it sometimes happens that the estimation performance gets worse. The hardware sensing method for robust quantum filtering and control has, presumably because of this fact, not yet been examined.

In this paper, despite of the back-action issue mentioned above, we apply the idea of hardware sensing to the quantum case and propose a new configuration of the quantum Kalman filter and the quantum LQG controller that has robustness against a certain practical uncertainty.

Let us first describe the uncertainty we treat in this paper. Fig.1 depicts a general linear quantum system coupling with an optical laser field. The output light is measured using so-called *balanced homodyne detector*, and the signal generated is then processed in the filter to calculate the estimate of the system variable. Also we can control the system with the estimate so that the system has nice property. In reality, an uncertainty appears in this input laser field; that is, the phase of the input laser cannot be determined in any real experiment. As shown later on, the phase uncertainty is explicitly contained in all the system, the filter and the controller equations, and thus it may cause large degradation of the estimation and the

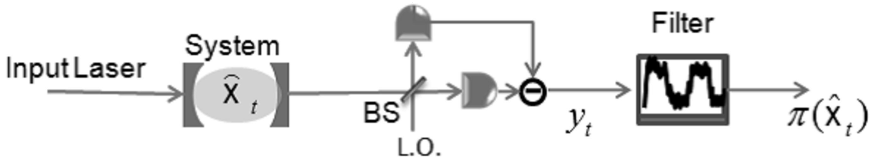

図 1 Schematic of the quantum Kalman filter. First, the system couples with a input laser. The output laser is mixed, by a beam splitter(BS), with a strong reference light called the local oscillator(LO), and two outputs are then respectively detected via a photon detector. This whole measurement configuration is called the homodyne detection. Finally, the quantum Kalman filter uses the output to calculate the best estimation $\pi(\hat{\mathbf{x}}_t)$.

control performance. It should be mentioned that this sort of *indirect measurement* scheme through a laser is commonly used for various quantum systems, for instance an atomic system in the cavity QED setup [5, 7] and an opto-mechanical system in an optical interferometer [12]. In this sense, the robust Kalman filtering and LQG control technique under the phase uncertainty of the input laser should be widely useful and significant.

We next mention briefly about the hardware sensing scheme for robust Kalman filtering and LQG control. This is built up by first dividing the input laser before it goes into the system and then putting another homodyne detector along the second optical path to obtain further information; see Fig. 3 in Section 3. A (probabilistic) back-action is brought when one does this additional measurement. Nonetheless, the hardware sensing scheme proposed allows us to estimate the unknown parameter itself; the result can then be fed back to adaptively change the filtering and control algorithm and, eventually, the filter and the controller acquire robustness property despite of the back-action mentioned above. We call this scheme the *adaptive Kalman filter* and the *adaptive LQG controller.*

This paper is organized as follows. In Section 2, the standard quantum Kalman filter and LQG controller are described by taking an opto-mechanical oscillator as a system, then the uncertainty issue is explained. Section 3 is devoted to show the configuration and the algorithm of the adaptive Kalman filter and LQG controller. Its robustness property is further given. Finally, in Section 4, we numerically demonstrate the efficiency of our method.

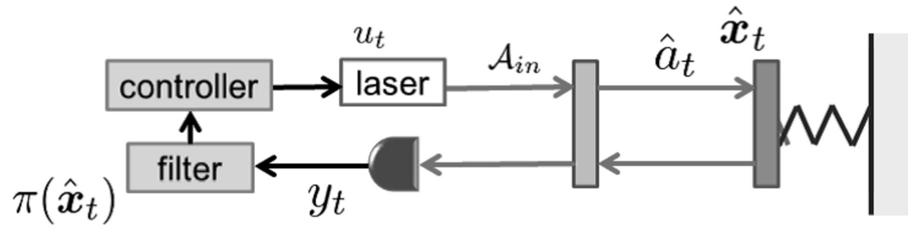

図 2 Schematic of the optomechanical oscillator. The right end mirror of the Fabry-Perot cavity serves as the oscillator, where $\hat{\mathbf{x}}_t$ is the system variable, $\hat{a}_t$ is an annihilation operator corresponding to the intra-cavity, and $\mathcal{A}_{in}$ is the mean amplitude of the input laser.

2 The quantum Kalman filter and LQG controller

The system we focus on is an optomechanical oscillator. Consider now a Fabry-Perot cavity composed of a moving mirror which interacts with an input field A_{in_t}. The mirror is modeled with position operator $\hat{q}_t$, momentum operator $\hat{p}_t$, mass m, and resonant frequency ω_m. The intra-cavity optical field $\hat{a}$ decays at rate γ due to coupling through the partially transmitting mirror to the output beam A_{out_t}. The intra-cavity optical field assumed to be resonant at the input beam's carrier frequency ω_0, at which point the roundtrip length is $2L$. The mirror position $\hat{q}_t$ is defined relative to this equibirium position. The operators obey canonical commutation relations: $[\hat{q}_t, \hat{p}_t] = i\hbar$, $[\hat{a}_t, \hat{a}_t^\dagger] = 1$.

Removing mean fields and defining amplitude and phase quadrature operators for the fluctuations that remain, $A_{in_t} = \mathcal{A}_{in} + (\hat{\xi}_{1_t} + i\hat{\xi}_{2_t})$, $A_{out_t} = \mathcal{A}_{in} + (\hat{\eta}_{1_t} + i\hat{\eta}_{2_t})$, and $\hat{a}_t = \alpha + \hat{a}_{1_t} + i\hat{a}_{2_t}$ where $\alpha = \mathcal{A}_{in}\sqrt{2/\gamma}$, We can obtain the system dynamics with the variable $\hat{\mathbf{x}}_t = (\hat{q}_t, \hat{p}_t, \hat{a}_{1_t}, \hat{a}_{2_t})^\top$ which is found in a form of so-called quantum stochastic differential equation:

$$d\hat{\mathbf{x}}_t = A\hat{\mathbf{x}}_t dt + B(\mathbf{u}_t dt + d\hat{\xi}_t), \tag{1}$$

$$A = \begin{bmatrix} 0 & \frac{1}{m} & 0 & 0 \\ -m\omega^2 & 0 & 2\hbar\kappa(\alpha_1) & 2\hbar\kappa(\alpha_2) \\ -\kappa(\alpha_2) & 0 & -\gamma & 0 \\ \kappa(\alpha_1) & 0 & 0 & -\gamma \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{2\gamma} & 0 \\ 0 & \sqrt{2\gamma} \end{bmatrix}, \tag{2}$$

where $\mathbf{u}_t = (u_{1_t}, u_{2_t})^\top$ is the control input, $\hat{\xi}_t = (\hat{\xi}_{1_t}, \hat{\xi}_{2_t})^\top$ is the input laser, $\kappa(\alpha) = \alpha\omega_0/L$ is an optomechanical coupling strength, and $\alpha = (\alpha_1, \alpha_2)^\top$.

The output light of the system is subjected to the following equation:

$$d\hat{Y}_t = C\hat{\mathbf{x}}_t dt + D(\mathbf{u}_t dt + d\hat{\xi}_t), \tag{3}$$

$$C = HB^\top, D = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{4}$$

where $\hat{Y}_t = (\hat{\eta}_{1_t}, \hat{\eta}_{2_t})^\top$, and H is a 2-dimentional row vector satisfying $HH^\top = 4$.

The outputs commute with each other, i.e., $[\hat{Y}_t, \hat{Y}_s] = 0, \forall s, t$, thus the sequence of scalars $\mathcal{Y}_t = \{y_s : 0 \leq s \leq t\}$ is obtained. For the observer, the variables can be regarded as classical random variables, which implies the existence of classical conditional expectations $\pi(\hat{\mathbf{x}}_t) := \mathbb{E}(\hat{\mathbf{x}}_t|\mathcal{Y}_t)$. The quantum Kalman filter is the algorithm that updates the estimates $\pi(\hat{\mathbf{x}}_t)$ depending on the measurement outcome y_t:

$$d\pi(\hat{\mathbf{x}}_t) = A\pi(\hat{\mathbf{x}}_t)dt + B\mathbf{u}_t dt + (V_t E^\top + F)d\tilde{y}_t, \tag{5}$$

$$\begin{aligned} \dot{V}_t = (A - FE)V_t + V_t(A - FE)^\top \\ +2V_t E^\top E V_t + BB^\top - 4FF^\top, \end{aligned} \tag{6}$$

$$E = HC, F = -\frac{1}{4}BH^\top, \tag{7}$$

where $d\tilde{y}_t = dy_t - \mathbb{E}(d\hat{Y}_t|\mathcal{Y}_t)$ represents the innovation term showing the difference of dy_t and the estimate of $d\hat{Y}_t$. V_t represents the (symmetrized) error covariance matrix.

To cool the oscillator, it is expected that a feedback controller minimizing the estimated value of the position operator $\hat{q}_t$ and momentum operator $\hat{p}_t$. Because of the linearity of both the dynamics and the output equation, this requirement is satisfied with the LQG control. We can construct an optimal control law $\mathbf{u}_t^*$ that minimizes the following quadratic-type cost function:

$$J = \left\langle \frac{1}{2} \int_0^T (\hat{\mathbf{x}}_t^\top Q \hat{\mathbf{x}}_t + \mathbf{u}_t^\top R \mathbf{u}_t) dt \right\rangle \tag{8}$$

where $Q \geq 0$ and $R > 0$ represents the penalty for the variables and the control input, respectively.

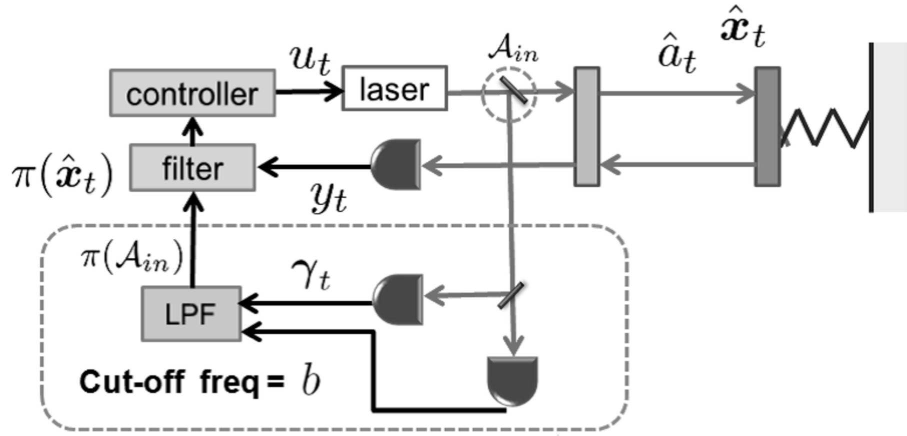

図 3 Schematic of the adaptive quantum Kalman filter and LQG controller.

The LQG control theory gives the explicit form of the optimal controller:

$$\mathbf{u}_t^* = -R^{-1}B^\top P_t \pi(\hat{\mathbf{x}}_t) \tag{9}$$

where P is the solution to the following algebric Riccati equation:

$$\dot{P}_t + P_t A + A^\top P_t - P_t B R^{-1} B^\top P_t + Q = 0 \tag{10}$$

We here explain the issue of uncertainty. In reality the input laser field amplitude $\mathcal{A}_{in} = |\mathcal{A}_{in}|(\cos\theta + i\sin\theta)$ is not the value that can be determined beforehand. That is, the phase θ takes a different value in every experiment and thus should be treated as an unknown parameter. As shoun in equation (2), $\mathcal{A}_{in}$ explicitly appears in the A matrix (note that $\alpha = \mathcal{A}_{in}\sqrt{2/\gamma}$). Therefore, we can never carry out precise updates of the estimates, which eventually degrade the total performance of the filtering and control.

3 The adaptive filtering and control

We here describe the hardware sensing scheme. The input laser is divided into two optical paths via a beam splitter, and the new second path is then measured by the *dual homodyne* detector. That is, we measure both the sine and cosine elements of this second laser field, generating the signal $d\gamma_t = \mathcal{A}_{in}dt + d\mathbf{v}_t$, where $\mathbf{v}_t$ is a 2-dimensional standard Gaussian noise with their entries independent with each other. Note that to build the adaptive filter we must introduce *vaccum fluctuations* that subsequently produce the additional noise $\mathbf{v}_t$; this is the unavoidable probabilistic back-action brought by the hardware sensing. The signal γ_t is then processed through a low-pass filter (LPF) with the cut-off angular frequency $b > 0$. The LPF variable $\pi(\mathcal{A}_{in})$ obeys

$$\begin{aligned}\pi(\mathcal{A}_{in}) =& e^{-bt}\mathcal{A}_{in_0} + b\int_0^t e^{-b(t-s)}(\mathcal{A}_{in}ds + d\mathbf{v}_s)\\ =& \mathcal{A}_{in}(1-e^{-bt}) + b\int_0^t e^{-b(t-s)}d\mathbf{v}_s. \end{aligned} \tag{11}$$

We find $\mathbb{E}(\pi(\mathcal{A}_{in})) \to \mathcal{A}_{in}$ as $t \to \infty$, thus $\pi(\mathcal{A}_{in})$ asymptotically gives an unbiased estimate of $\mathcal{A}_{in}$.

The adaptive Kalman filter estimates $\hat{\mathbf{x}}_t$, using the LPF variable $\pi(\mathcal{A}_{in})$ instead of the unknown parameter $\mathcal{A}_{in}$. Writing such an estimater as $\pi^A(\hat{\mathbf{x}}_t)$, the adaptive Kalman filter and LQG controller are given as follows:

$$d\pi^A(\hat{\mathbf{x}}_t) = A^A\pi^A(\hat{\mathbf{x}}_t)dt + B\mathbf{u}_t^A dt + (V_tE^\top + F)d\tilde{y}_t^A, \tag{12}$$

$$\mathbf{u}_t^A = -R^{-1}B^\top P_t^A\pi^A(\hat{\mathbf{x}}_t), \tag{13}$$

where A^A is the A matrix using the LPF variable $\pi(\mathcal{A}_{in})$ instead of the unknown parameter $\mathcal{A}_{in}$, and P^A is the matrix which follows equation (8) using A^A instead of A, and

$$d\tilde{y}_t^A = d\tilde{y}_t + E(\pi(\hat{\mathbf{x}}_t) - \pi^A(\hat{\mathbf{x}}_t))dt$$

which indicates that the adaptive Kalman filter is driven with a difference of the true filter and the adaptive filter.

4 Numerical examples

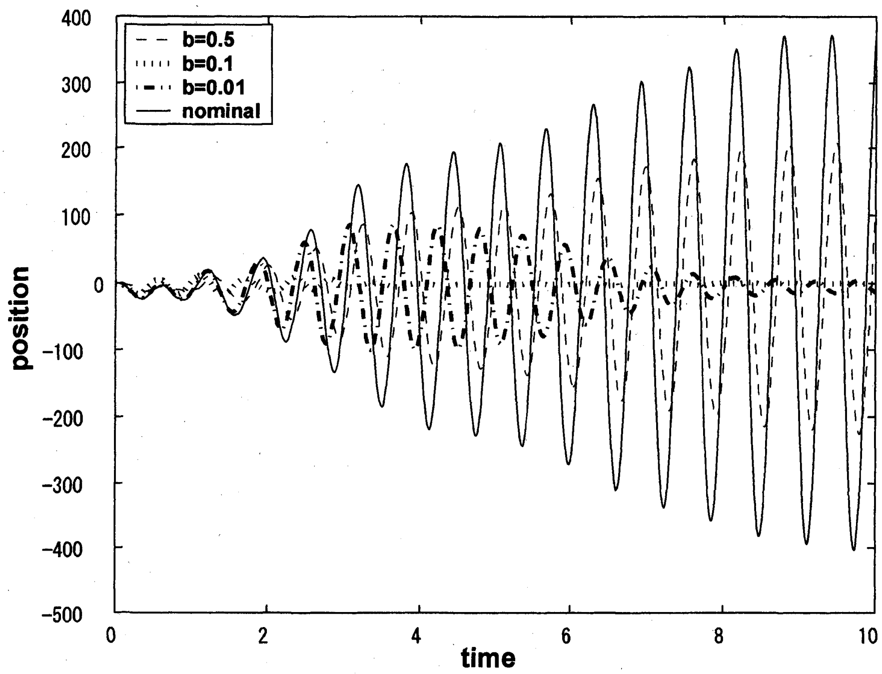

図 4 The difference between the adaptive (or nominal) and the true estimates for $\hat{q}_t$

Let us here see some actual trajectories of the estimates of the adaptive filter and controller. We particularly focus on three types of the filter where the cut-off frequency is chosen as $b = 0.5, b = 0.1$, or $b = 0.01$. The true laser amplitude is $\mathcal{A}_{in} = (1,0)^\top$, which can be used to run the true Kalman filter. For comparison, we also examine a nominal filter that set $\mathcal{A}_{in}$ to be $(0,1)^\top$. Fig.4 shows the difference between the adaptive (or the nominal) and the true estimates of $\hat{q}_t$. We see that the nominal filter fails in the estimation, while the adaptive Kalman filters well estimate, particularly when $b = 0.1$. We can numerically see the optimal cut-off frequency which

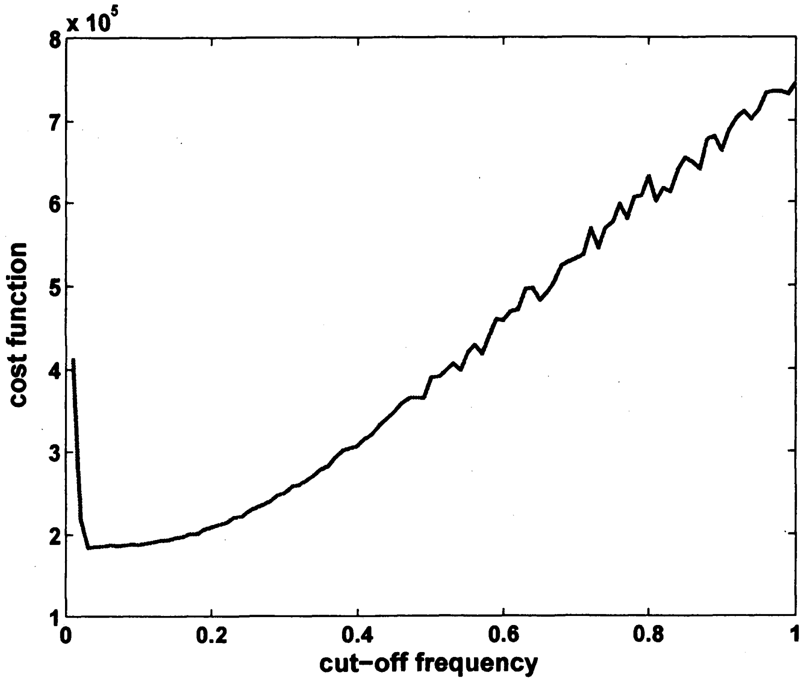

図 5 The cut-off frequency dependence of the cost function

minimizes the cost function J. Fig.5 shows the cut-off frequency dependence of the cost function. The final time T in this simulation is 3. At lower cut-off frequencies, the cost function takes large value. The cost function takes the minimum value at $b = 0.03$, and gradually takes larger value as b get larger. In general, with smaller value of b, we can obtain more precise estimation in the steady state case, but it shows the more slowly convergence to the true filter. In other words, in the early stage of the estimation, the adaptive Kalman filter with small value of b has almost no information of $\mathcal{A}_{in}$, and thus has the same quality as that of the nominal filter. On the other hand, with larger value of b, the adaptive Kalman filter has fast convergence, though it cannot achieve precise estimation in the steady state. Fig.5 implies that when we take a feedback control which requires rapid convergence and accuracy, there exists the optimal value of the cut-off frequency in terms of the trade-off between the convergence and the accuracy.

5 Conclusion

In this paper, for an optomechanical oscillator coupling with a laser field, we have proposed an adaptive filtering and control scheme that estimates both the system variables and the unknown phase parameter of the input field. It was then shown that, by choosing the cut-off angular frequency in the LPF for the adaptive filtering, we can obtain the robust filter and controller against the phase uncertainty.

Though we have assumed that the uncertainty is constant and only appears in the phase, in reality these assumptions may not be satisfied. The robust filtering problem in the more general setting remains open.

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