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On the state numbers of a virtual knot

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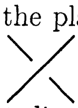
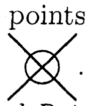
1 Introduction

This note is mainly a summary of our studies for the "state numbers" of a virtual knot obtained in [5] and [6].

In knot theory, there are many "minimal-type" numerical invariants of a knot K. For example, the crossing number $c(K)$ for K is the minimal number of crossings in any diagrams of K, and the unknotting number $u(K)$ is the minimal number of crossing changes in any diagrams of K needed to create a diagram of the unknot. Those invariants measure a certain complexity of the knot.

In [5], we define the *n-state number* $\mathrm{s}_n(K)$ of a virtual knot K. A *state* S of a virtual knot diagram D is a union of circles obtained from D by splicing all real crossings in D. Let $s_n(D)$ be the number of states of D consisting of n circles. The n-state number of a virtual knot K is defined to be the minimal number of $s_n(D)$ for all possible diagram of K. In this note, we show some properties of the n-state numbers of a virtual knot. First, we give upper and lower bounds for the n-state numbers of a virtual knot K for $n = 1, 2, 3$ in terms of the real crossing number of K. Second, we consider a set of virtual knots whose n-state number is equal to i for each non-negative integer i and study the finiteness of the set. Finally, we give lower bounds for the 1-state number $\mathrm{s}_1(K)$ of a virtual knot K in terms of a special value of the Jones polynomial and the Miyazawa polynomial of K.

2 The state numbers fo a virtual knot

A *virtual knot diagram* D is an immersed circle in the plane $\mathbb{R}^2$ whose double points are ordinary crossings, which are called *real crossings*  and *virtual crossings* . A *virtual knot* K is an equivalence class of virtual knot diagrams under *generalized Reidemeister moves* as in Figure 1 (cf. [3]).

A state S of a virtual knot diagram D is a union of circles possibly with virtual crossings obtained from D by splicing all real crossings as in Figure 2. A state S is said to be an *n-state* if S consists of n circles. We denote by $s_n(D)$ the number of n-states of D.

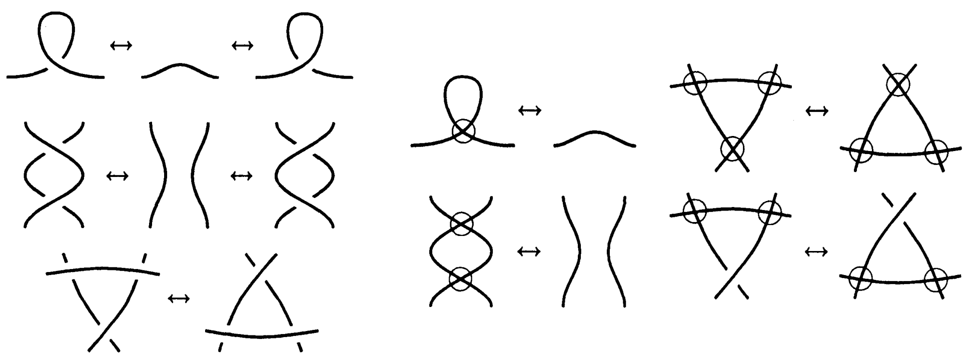


Figure 1:

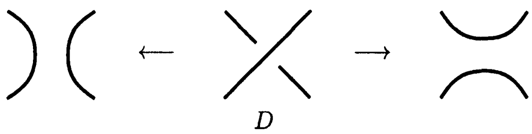


Figure 2:

Example 2.1. Let D be the virtual knot diagram as in Figure 3 (a), which presents the virtual knot labeled by 2.1 in Green's table [1]. By splicing the real crossings in D, we obtained four states as in Figure 3 (b). Three of them are 1-states and the other is a 2-state. Hence we have $s_1(D) = 3$, $s_2(D) = 1$, $s_i(D) = 0$ $(i \geq 3)$.

Let K be a virtual knot. The *n-state number* of K, denoted by $\mathrm{s}_n(K)$, is defined to be the minimal number of $s_n(D)$ for all possible virtual knot diagrams D of K (cf. [5]). For example, we can easily see that $\mathrm{s}_1(K) = 1$ and $\mathrm{s}_i(K) = 0$ $(i \geq 2)$ if K is trivial.

Let D be an oriented virtual knot diagram. We regard D as the image of an immersion of a circle into $\mathbb{R}^2$ with crossing information at each double point. The *Gauss diagram* of D is an oriented circle regarded as the preimage of the immersed circle with chords, each of which connects the preimages of each double point corresponding to a real crossing. A chord is oriented from the preimage of the over-crossing-point to that of the under-crossing-point in the circle, and labeled by the sign of the corresponding real crossing. Figure 4 illustrates an example of a virtual knot diagram and its Gauss diagram.

Two chords of a Gauss diagram G is *linked* if their end-points appear along the circle of G alternately. A chord is *free* if it is not linked with any other chords.

Lemma 2.2 ([5]). *Let D and D' be virtual knot diagrams with the same Gauss diagram by ignoring the orientation of the circle and the orientation and sign of each chord. Then $s_n(D) = s_n(D')$ holds for any natural number n.*

By Lemma 2.2, $s_n(D)$ is determined by its unoriented and unsigned Gauss diagram G. In this sense, we denote $s_n(D)$ by $s_n(G)$.

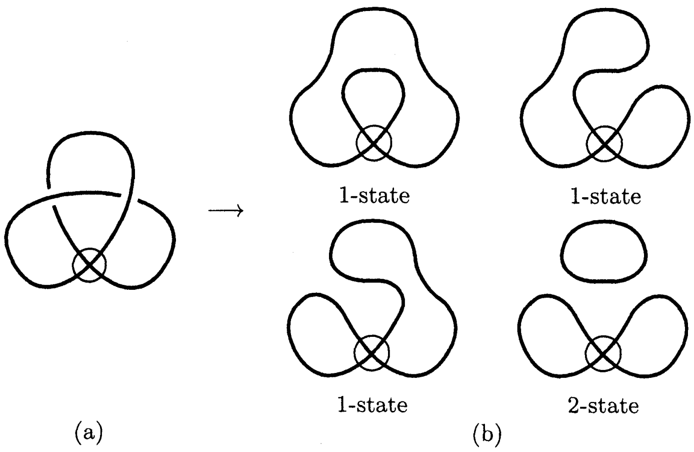


Figure 3:

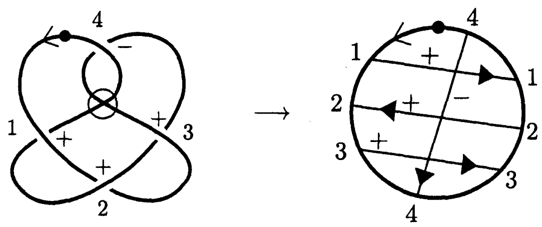


Figure 4:

3 Bounds for 1-,2-, and 3-state numbers

In this section, we give upper bounds and lower bounds for the n-state number of a virtual knot K for $n = 1, 2, 3$. Let $c(K)$ be *the minimal real crossing number* for K, that is, the minimal number of real crossings for all possible virtual knot diagrams of K.

Theorem 3.1 ([5]). *Any virtual knot K satisfies* (1) $1 \leq \mathrm{s}_1(K) \leq \dfrac{2 \cdot 2^{c(K)} + (-1)^{c(K)}}{3}$, (2) $0 \leq \mathrm{s}_2(K) \leq \dfrac{1}{2} \cdot 2^{c(K)}$, *and* (3) $0 \leq \mathrm{s}_3(K) \leq \dfrac{3}{8} \cdot 2^{c(K)}$.

The lower bounds of (2) and (3) in Theorem 3.1 are obvious. For any virtual knot diagram D, we see that there is at least one sequence of virtual knot diagrams $D = D_0, D_1, D_2, \ldots, D_m$ such that D_i is obtained from D_{i-1} by splicing a real crossing in D_{i-1} ($i = 1, 2, \ldots, m$) and D_m has no real crossing. This gives the lower bound of (1) in Theorem 3.1. The upper bounds in Theorem 3.1 are given by the following lemma.

Lemma 3.2 ([5]). *Let G be a Gauss diagram of one circle with r chords. Then we have* (1) $s_1(G) \leq \dfrac{2 \cdot 2^r + (-1)^r}{3}$, (2) $s_2(G) \leq \dfrac{1}{2} \cdot 2^r$, *and* (3) $s_3(G) \leq \dfrac{3}{8} \cdot 2^r$.

We remark that G produces 2^r states. The following examples of Gauss diagrams realizes upper bounds in Lemma 3.2.

Example 3.3. Let F_r be a Gauss diagram as in Figure 5. Then we have $s_1(F_r) = \dfrac{2 \cdot 2^r + (-1)^r}{3}$.

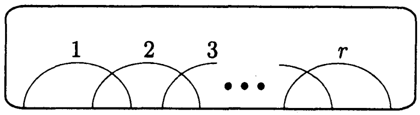
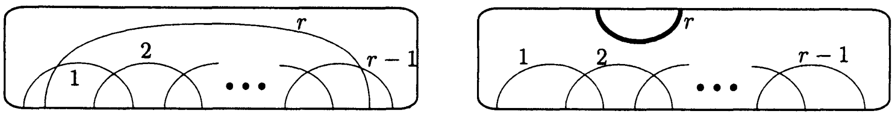


Figure 5:

For $r \leq 3$, there exists a virtual knot K presented by F_r with $\mathrm{s}_1(K) = \dfrac{2 \cdot 2^r + (-1)^r}{3}$. It is an open question whether there exists a virtual knot K presented by F_r realizing $\mathrm{s}_1(K) = \dfrac{2 \cdot 2^r + (-1)^r}{3}$ for $r \geq 4$.

Example 3.4. Let F'_r $(r \geq 3)$ be a Gauss diagram as in Figure 6 (a) and $F_{r-1} + 1$ a Gauss diagram as in Figure 6 (b). A Gauss diagram $F_{r-1} + 1$ is obtained from F_{r-1} by adding a free chord in any place. Then we have $s_2(F'_r) = s_2(F_{r-1} + 1) = \dfrac{1}{2} \cdot 2^r$.



(a) F'_r (b) $F_{r-1} + 1$

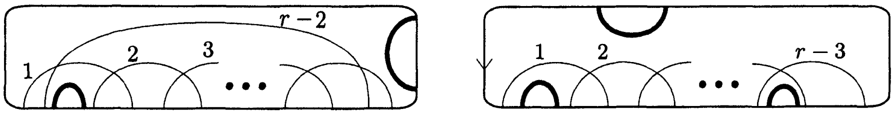
Figure 6:

There exists a virtual knot presented by F'_3 with $\mathrm{s}_2(K) = 4$. It is an open question whether there exists a virtual knot K presented by F'_r realizing $\mathrm{s}_2(K) = \dfrac{1}{2} \cdot 2^r$ for $r \geq 4$.

Example 3.5. A Gauss diagram $F'_{r-2} + 2$ as in Figure 7 (a) is obtained from F'_{r-2} by adding two free chords and a Gauss diagram $F_{r-3} + 3$ as in Figure 7 (b) is obtained from F_{r-3} by adding three free chords in any place. Then we have $s_3(F'_{r-2}+2) = s_3(F_{r-3}+3) = \dfrac{3}{8} \cdot 2^r$.

4 The number of virtual knots with a given state number

Let $\mathcal{S}_n(i)$ be the set of virtual knots with $\mathrm{s}_n(K) = i$ for a non-negative integer i. In this section, we consider the finiteness of $\mathcal{S}_n(i)$.



(a) $F'_{r-2}+2$ (b) $F_{r-3}+3$

Figure 7:

Proposition 4.1 ([5]). (1) *Both of* $\mathcal{S}_1(1)$ *and* $\mathcal{S}_2(0)$ *consist of the trivial knot.* (2) $\mathcal{S}_1(2k)$ *is the empty set for* $k \geq 0$.

It is showed in [5] that $c(K) \leq \mathrm{s}_2(K)$ for any virtual knot K with $c(K) \geq 3$ and $\mathrm{s}_2(K) \leq 1$ for any virtual knot $c(K) \leq 2$.

Theorem 4.2. $\mathcal{S}_2(i)$ *is finite for any non-negative integer* i.

On the other hand, the cases of $n \geq 3$ are different from that of $n = 2$. It is showed that $\mathcal{S}_n(0)$ is a subset of $\mathcal{S}_{n+1}(0)$ ([5]) and $\mathcal{S}_3(0)$ contains infinitely many virtual knots ([6]).

Theorem 4.3. $\mathcal{S}_n(0)$ *is infinite for* $n \geq 3$.

In addition, it is showed in [6] that any non-trivial virtual knot contained in $\mathcal{S}_3(0)$ is *non-classical*, that is, it has no diagram without virtual crossings.

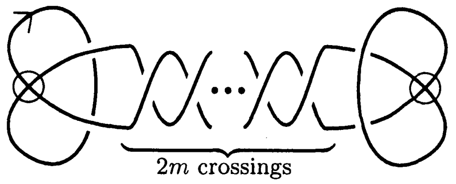
In [5], for any Gauss diagram G of a virtual knot digram D it is showed that $s_1(G) \geq r(G) - f(G)$, where $r(G)$ is the number of chords and $f(G)$ is the number of free chords in G. For any Gauss diagram G whose $s_1(G)$ is an odd prime, we see that there exists at least one free chord in G corresponding to a real crossing in D which can be eliminated by a Reidemeister move I.

Theorem 4.4. (1) $\mathcal{S}_1(i)$ *is finite for any odd prime* i. *In particular,* $\mathcal{S}_1(3)$ *consists of four virtual knots* $2.1, 3.5, 3.6,$ *and* 3.7 *in Green's table.*
(2) $\mathcal{S}_1(9)$ *is infinite.*

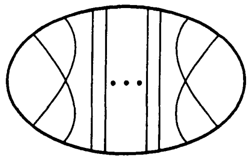
The outline of a proof of Theorem 4.4 (2) is the following. Let D_m be a virtual knot diagram as in Figure 8 (a) and K_m the virtual knot presented by D_m. The Gauss diagram G_m of D_m is illustrated in Figure 8 (b). We see that $s_1(D_m) = 9$ for any non-negative integer m. Thus we have $\mathrm{s}_1(K_m) \leq 9$. We can show that K_m is not equivalent to $K_{m'}$ if $m \neq m'$ by the Miyazawa polynomial. Since $\mathcal{S}_1(i)$ is finite for $i \leq 7$ by Proposition 4.1 (1) and Theorem 4.4 (1), $\{K_m\}_{m \geq 0}$ contains infinitely many virtual knots whose 1-state number is equal to 9.

5 Lower bounds for the 1-state number by polynomial invariants

The Jones polynomial of an ordinary knot is naturally generalized to that of a virtual knot through Kauffman's f-polynomial. Let K be an oriented virtual knot and D an



(a) D_m



(b) G_m

Figure 8:

oriented virtual knot diagram of K with the writhe $w(D)$. Let S be a state of the unoriented D. Let $a(S)$ (resp. $b(S)$) be the number of A-splices (resp. B-splices) to obtain S from D as in Figure 9. We denote by $|S|$ the number of circles in S. Then Kauffman's f-polynomial is defined as

$$f_K(A) = (-A^{-3})^{w(D)} \sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{|S|-1} \in \mathbb{Z}[A, A^{-1}].$$

By substituting $A = t^{-\frac{1}{4}}$, we obtain the Jones polynomial $V_K(t)$ of K.

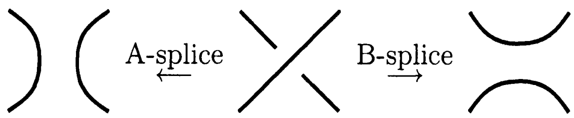


Figure 9:

Proposition 5.1 ([5]). *For any virtual knot K, we have* $\mathrm{s}_1(K) \geq |f_K(\xi)| = |V_K(-1)|$, *where* $\xi = e^{\frac{\pi}{4}i}$.

We review the definition of the Miyazawa polynomial of an oriented virtual knot K ([4]) as the state-sum of *poled states* ([2]).

Let K be an oriented virtual knot and D an oriented virtual knot diagram of K with the writhe $w(D)$. There are two ways of splicing at a real crossing c in D with respect to the orientation of arcs. One of which is coherent, otherwise is non-coherent. For a non-coherent splicing, we set up a pair of poles on spliced arcs as in Figure 10. If c is positive, then A-splicing at c is coherent and B-splicing is non-coherent. By splicing all real crossing of D, we obtained a state with poles. A state with poles is called a poled state.

Let S be a poled state of D and $a(S)$ (resp. $b(S)$) the number of A-splices (resp. B-splices) to obtain S from D. Let $\mathcal{C}(S)$ be the set of circles in a poled state S.

We note that the number of poles on $C \in \mathcal{C}(S)$ is always even. We reduce the number of poles on C by the following moves: A pole can slide along C and pass a virtual crossing. If there exist two successive poles on the same side of C, then they are canceled. See Figures 11 (a) and (b). Let $\tilde{C}$ be the circle with poles obtained from C after reducing poles on C

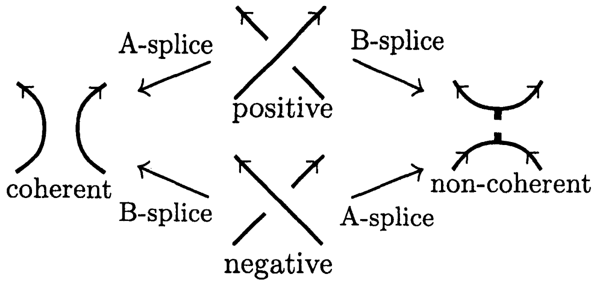


Figure 10:

as possible. Then the poles on $\tilde{C}$ stand on the left and right side of $\tilde{C}$ alternately as in Figure 11 (c).

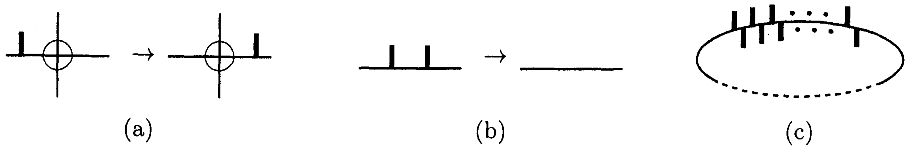


Figure 11:

Let $\lambda(C)$ be the half of the number of poles on $\tilde{C}$ and $c_i(S)$ the number of circles in S with $\lambda(C) = i$. Then the Miyazawa polynomial $R_K(A, \vec{x}) \in \mathbb{Z}[A, A^{-1}, x_1, x_2, \ldots]$ of K is defined by

$$R_K(A, \vec{x}) = (-A^3)^{-w(D)} \sum_S A^{a(S)-b(S)}(-A^2 - A^{-2})^{c_0(S)} x_1^{c_1(S)} x_2^{c_2(S)} \cdots .$$

Example 5.2. Let D be the virtual knot diagram as indicated in Figure 12, which presents the virtual knot labeled by 4.81 in Green's table. Since the number of real crossings of D is equal to four, D has 16 poled states. Each poled state S is assigned by $A^{a(S)-b(S)}(-A^2 - A^{-2})^{c_0(S)} x_1^{c_1(S)} x_2^{c_2(S)} \cdots$ according to the information of the number of A- or B-splices, the number of poles on each circles in S as in Figure 12, where $d = -A^2 - A^{-2}$. Then the Miyazawa polynomial of K is given by

$$R_K(A, \vec{x}) = (-A^2 - A^{-2})(A^8 - A^4 + 1) + (A^6 - A^2 + A^{-2})x_1 + A^{10}x_3 + A^8 x_1 x_2.$$

We describe the Miyazawa polynomial $R_K(A, \vec{x})$ of K as

$$R_K(A, \vec{x}) = F_0(A) + \sum_I F_I(A) x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k},$$

where I is a finite sequence of integers such that $I = (c_1, c_2, \ldots, c_k)$ with $c_i \geq 0$ ($i = 1, 2, \ldots, k-1$) and $c_k > 0$. It is known that $F_0(A)$ is divisible by $-A^2 - A^{-2}$ ([4]). Let $\widetilde{F_0}(A) = F_0(A)/(-A^2 - A^{-2})$.

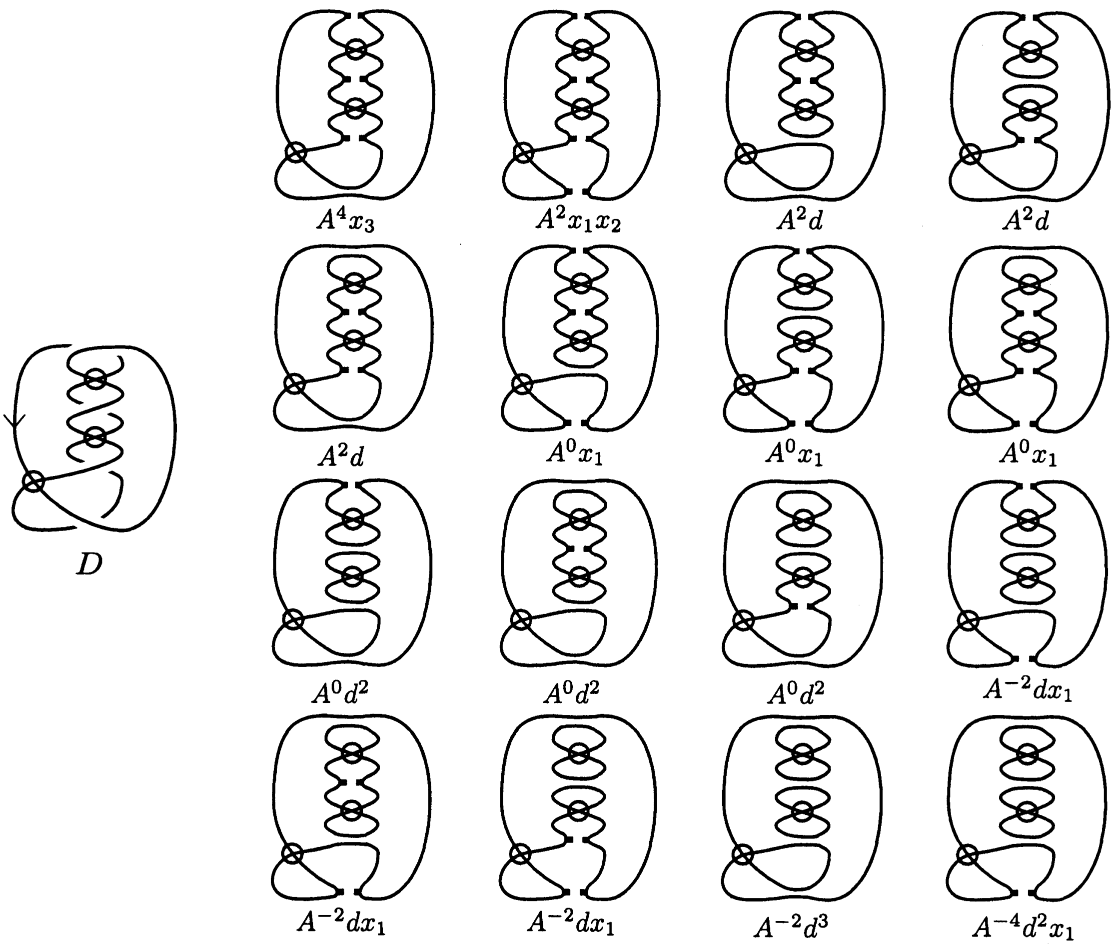


Figure 12:

Theorem 5.3 ([5]). *For any virtual knot K, we have* $\mathrm{s}_1(K) \geq \left|\widetilde{F}_0(\xi)\right| + \sum_{k=1}^{\infty} |F_{I_k}(\xi)| \geq |f_A(\xi)|$, *where* $I_k = (\underbrace{0,0,\ldots,0}_{k-1},1)$ *and* $\xi = e^{\frac{\pi}{4}i}$.

Example 5.4. Let K be the virtual knot 4.81 in Example 5.2. We see that there are seven 1-states obtained from D as in Figure 12. Hence we have $\mathrm{s}_1(K) \leq 7$. By direct calculation, we have $f_K(A) = A^8 - A^4 - A^2 + 1 + A^{-2}$, and hence $|f_K(\xi)| = \sqrt{13}$. By Proposition 5.1, $\sqrt{13} < 4 \leq \mathrm{s}_1(K) \leq 7$.

Since $R_K(A,\vec{x}) = (-A^2 - A^{-2})(A^8 - A^4 + 1) + (A^6 - A^2 + A^{-2})x_1 + A^{10}x_3 + A^8x_1x_2$, we have $\widetilde{F}_0(A) = A^8 - A^4 + 1$, $F_{(1)} = A^6 - A^2 + A^{-2}$, and $F_{(0,0,1)}(A) = A^{10}$. Then we see that

$$\left|\widetilde{F}_0(\xi)\right| + \sum_{k=1}^{\infty} |F_{I_k}(\xi)| = \left|\widetilde{F}_0(\xi)\right| + |F_{(1)}(\xi)| + |F_{(0,0,1)}(\xi)| = 7.$$

Therefore we have $s_1(K) = 7$.

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