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Concerning actions of 3-manifold groups: from topological and arithmetic viewpoints

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Abstract

After reviewing Marc Culler and Peter B. Shalen's construction of essential surfaces contained in 3-dimensional manifolds, we shall present an extension of there construction utilising *character varieties and Bruhat-Tits buildings of higher dimension.* This extended method enables us to construct *essential tribranched surfaces* contained in 3-dimensional manifolds in a systematic way.

This is a report of the author's talk "Concerning actions of 3-manifold groups: from topological and arithmetic viewpoints" given at the RIMS Conference: *Intelligence of Low-dimensional Topology* 2014. In 1983, Marc Culler and Peter B. Shalen proposed a systematic method to construct *essential surfaces* contained in 3-dimensional manifolds in [CS83]. Their method heavily utilised *algebro-geometric ideas*; to sum up, they introduced *geometry of* $SL(2)$*-character varieties* and the *theory of Bruhat-Tits trees* to establish their construction. It should therefore be worth introducing more sophisticated algebraic and arithmetic concepts to extending Culler and Shalen's classical method. In this brief report we shall propose a systematic construction of *essential tribranched surfaces* contained in 3-manifolds, which is regarded as a natural extension of Culler and Shalen's result [CS83] (this is a joint work with Takahiro KITAYAMA, Tokyo Institute of Technology). Refer also to [H14], which deals with the same contents as this report but written in Japanese.

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§ 1 Introduction: Essential surfaces and group actions

Throughout this article all 3-dimensional manifolds are assumed to be compact, connected, irreducible and orientable. As an introduction of this report, we here review the (somewhat mysterious) relation between the *topological* concepts of essential surfaces and *algebraic* actions of 3-manifold groups on trees.

First let us recall the definition of essential surfaces (see [Sh02, Definition 1.5.1] for example).

Definition 1.1 (Essential surfaces). A surface[*1] S contained in a 3-manifold M is said to be *essential* if the following four conditions are fulfilled:

(ES1) (incompressibility) for each connected component S_i of S, the canonical homomorphism $\pi_1(S_i) \to \pi_1(M)$ is injective;

(ES2) (bicollaredness) S is *bicollared*; that is, there exists a homeomorphism h from $S \times [-1, 1]$ onto a neighbourhood of S in M such that
- $h(x, 0) = x$ holds for each x in S; and
- the intersection of $h(S \times [-1, 1])$ and ∂M coincides with $h(\partial S \times [-1, 1])$;

(ES3) (non-boundary-parallel) no connected component of S is boundary-parallel;

(ES4) (nontriviality) S is nonempty and no connected component of S is homeomorphic to a 2-sphere.

Now let S be an essential surface contained in a 3-manifold M and let $\{S_j\}_{j=1}^{r}$ denote the (finite) set of all connected components of S. By cutting M along S, we may decompose M into several components, which we denote by $N_1, \ldots, N_s$. Now let us shrink each component N_i to a point (or a "vertex") and label it v_i. Whenever a connected

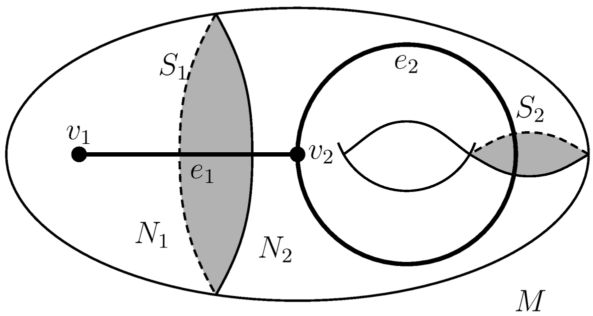


Figure 1. The dual graph associated to an essential surface

component S_j of S separates components N_{i_1} and N_{i_2}, we attach to S_i an "edge" e_j connecting the vertices v_{i_1} and v_{i_2}. As a result we obtain a finite graph, which is called the *dual graph* of M associated to the essential surface S (see Figure 1 above). We finally attach to each vertex v_j of the dual graph the fundamental group $\pi_1(N_i)$ of the corre-

[*1] Here we do not require that S itself is connected.

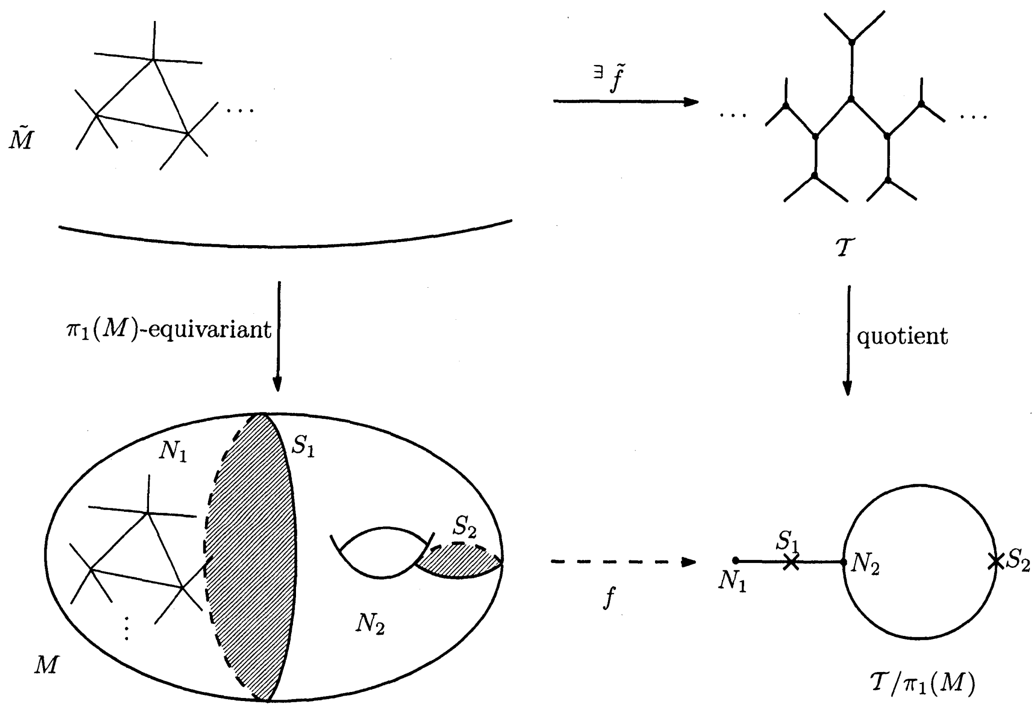


Figure 2. Construction of (essential) surfaces

sponding component N_i (which is called the *vertex group* of v_i), and attach to each edge e_j the fundamental group $\pi_1(S_j)$ of the corresponding component S_j (which is called the *edge group* of e_j). After these procedures we obtain a *graph of groups* $\mathcal{G}_S$ associated to S (see [Se77, Chapitre I., Definition 8] for the definition of graphs of groups). Due to the theory of graphs of groups developed by Hyman Bass and Jean-Pierre Serre (for details refer to [Se77, Chapitre I.]), the graph of groups $\mathcal{G}_S$ is known to be obtained as the *quotient graph* associated to an action of $\pi_1(M)$ on a certain connected and simply connected graph, or namely, on a certain *tree*[*2] $\mathcal{T}$. Each vertex and edge group is recovered (up to conjugation) as the isotropy subgroup of $\pi_1(M)$ (with respect to the action of $\pi_1(M)$ on $\mathcal{T}$) at the corresponding vertex and edge respectively.[*3]

Conversely, given an action of $\pi_1(M)$ on a tree which is *nontrivial*[*4] and *without inversion*,[*5] we may construct an essential surface S contained in M. This is due to results of

[*2] This tree is an analogous object of the *universal covering space* in the theory of covering spaces, and called the *Bass-Serre tree* today. See [Se77, Chapitre I., Section 5.3] for details.

[*3] Moreover we may obtain the *splitting* of $\pi_1(M)$ as the fundamental group $\pi_1(\mathcal{G}_S)$ of the graph of groups $\mathcal{G}_S$.

[*4] An action of a group G on a tree $\mathcal{T}$ is said to be *nontrivial* if the isotropy subgroup G_v does not coincide with the whole group G for each vertex v.

[*5] An action of a group G on a tree $\mathcal{T}$ is said to be *without inversion* if there does not exist an element g of G which fix an edge e but interchange the endpoints of e.

many people including John Robert Stallings [St59, St71], David Bernard Alper Epstein [Ep61] and Friedhelm Waldhausen [Wa67]. We here briefly sketch how to construct an essential surface from a group action on a tree (refer also to [Sh02, Section 2.2.]). First take a triangulation of M and lift it to the universal covering space $\tilde{M}$ of M. Then we obtain a $\pi_1(M)$-equivariant triangulation of $\tilde{M}$. Due to the contractibility of the tree $\mathcal{T}$, we may construct a $\pi_1(M)$-equivariant simplicial map $\tilde{f}\colon \tilde{M} \to \mathcal{T}$ (by using simplicial approximation theorem if necessary). The simplicial map $\tilde{f}$ induces a piecewise-linear map $f\colon M \to \mathcal{T}/\pi_1(M)$ on the quotient, where $\mathcal{T}/\pi_1(M)$ denotes the quotient graph of $\mathcal{T}$ with respect to the action of $\pi_1(M)$ on $\mathcal{T}$. Let E denote the set of the midpoints of all edges of the quotient graph $\mathcal{T}/\pi_1(M)$, and consider the inverse image S' of E under the map $f\colon M \to \mathcal{T}/\pi_1(M)$. The inverse image S' is then a properly embedded surface in M (see Figure 2). While S' itself might not be essential, we may modify S' into an essential surface S by certain "local surgery" (that is, a certain topological operation). We may thus construct an essential surface S in M from the nontrivial action of $\pi_1(M)$ on the tree $\mathcal{T}$.

As a consequence, one observes that there exists a close relation between the concepts of essential surfaces and (nontrivial) actions of 3-dimensional groups on trees:

Essential surfaces	"close relation" $\longleftrightarrow$	Nontrivial actions of $\pi_1(M)$ on trees
(topological concept)		(algebraic group action)

Moreover it is easy to imagine from the arguments above that one might be able to systematically construct essential surfaces contained in 3-manifolds if one could construct *nontrivial actions* of 3-manifold groups on trees in a certain ingenious manner. It was Marc Culler and Peter B. Shalen who realised this vague idea and established a systematic construction of essential surfaces contained in 3-manifolds. In the next section we shall survey their outstanding results of [CS83].

§ 2 Nontrivial actions of 3-manifold groups: Culler-Shalen theory

In their construction of nontrivial actions of $\pi_1(M)$ on trees, Culler and Shalen utilised two highly algebraic (or algebro-geometric) ingredients: namely

- Geometry of $SL(2)$-character varieties; and
- Bruhat-Tits trees associated to the special linear groups of degree two.

We shall see each of them more thoroughly in the following subsections.

§ 2.1 Step 1: Geometry of $SL(2)$-character varieties

Concerning $SL(2)$-character varieties, see also Harada's article [Harada14] in this volume. Let us set

$$\begin{aligned} R(\pi_1(M))_{SL(2)}(\mathbb{C}) &= \mathrm{Hom}(\pi_1(M), SL_2(\mathbb{C})) \\ &= (\text{the set of all } SL_2(\mathbb{C})\text{-representations of } \pi_1(M)). \end{aligned}$$

Using the fact that $\pi_1(M)$ is a group of finite presentation (due to the compactness of M), we may readily verify that $R(\pi_1(M))_{SL(2)}(\mathbb{C})$ admits a structure of an affine algebraic variety over $\mathbb{C}$, which we call the *$SL(2)$-representation variety of* $\pi_1(M)$. Next let $X(\pi_1(M))_{SL(2)}(\mathbb{C})$ denote the set of all *$SL(2)$-characters* of $\pi_1(M)$; more specifically, an element of $X(\pi_1(M))_{SL(2)}(\mathbb{C})$ is the *trace* $\chi_\rho = \mathrm{Tr}(\rho)$ of a certain $SL(2)$-representation $\rho\colon \pi_1(M) \to SL_2(\mathbb{C})$ of $\pi_1(M)$. It is widely known that $X(\pi_1(M))_{SL(2)}(\mathbb{C})$ also admits a structure of an affine algebraic variety over $\mathbb{C}$, which we call the *$SL(2)$-character variety* of $\pi_1(M)$.

In the context of algebraic geometry, the natural action of $SL_2(\mathbb{C})$ on the set $R(\pi_1(M))_{SL(2)}(\mathbb{C})$ via conjugation $\rho \mapsto \gamma^{-1}\rho\gamma$ canonically extends to the action of the (reductive) algebraic group $SL(2)_{/\mathbb{C}}$ on the representation variety $R(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$, and the character variety $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ is represented as the (geometric invariant theoretical) quotient variety of $R(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ with respect to this action. In particular there exists a natural "quotient" morphism

$$\pi\colon R(\pi_1(M))_{SL(2)_{/\mathbb{C}}} \to X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}.$$

Let γ be an element of $\pi_1(M)$. Then we associate to γ a regular function I_γ defined on $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ satisfying

$$I_\gamma(\chi_\rho) = \mathrm{Tr}\,\rho(\gamma)$$

for every complex point χ_ρ of $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ (the "evaluation-at-γ" function). It is known that the affine coordinate ring of $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ is generated over $\mathbb{C}$ by a finite number of such evaluation functions (see [CS83, Proposition 1.4.1]).

Now let us take an (affine) algebraic curve C contained in $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$. By a standard argument in algebraic geometry (refer to [Mu76] for basic facts on algebraic geometry), one may verify that there exists a "lift" D of C with respect to π; in other words, there exists an algebraic curve D contained in $R(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ such that the restriction of π to D (which we also abbreviate as π) surjects onto C. The canonical projection $\pi\colon D \to C$ extends to their projective completions, and induces a regular morphism

$\tilde{\pi}\colon \tilde{D} \to \tilde{C}$ on the (unique) smooth projective models of (the projective completions of) C and D.

For such D we may consider the *tautological representation*

$$\rho_{\mathrm{taut},D}\colon \pi_1(M) \to SL_2(\mathbb{C}[D])$$

where $\mathbb{C}[D]$ denotes the ring of regular functions on D. This is a certain kind of "universal representations," and characterised by the following property: for every closed point y of D and for every element γ of $\pi_1(M)$, we have

$$\rho_{\mathrm{taut},D}(\gamma)(y) = \rho_y(\gamma)$$

where ρ_y denotes the $SL_2(\mathbb{C})$-representation of $\pi_1(M)$ corresponding to the closed point y regarded as a complex point of $R(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$. Composing it with the canonical maps*[6]

$$\mathbb{C}[D] \hookrightarrow \mathbb{C}(D) \xrightarrow{\sim} \mathbb{C}(\tilde{D})$$

(here we denote by $\mathbb{C}(D)$ and $\mathbb{C}(\tilde{D})$ the fields of rational functions on D and $\tilde{D}$ respectively), we may readily extend $\rho_{\mathrm{taut},D}$ to the *tautological representation associated to* $\tilde{D}$;

$$\rho_{\mathrm{taut},\tilde{D}}\colon \pi_1(M) \to SL_2(\mathbb{C}(\tilde{D})).$$

§ 2.2 Step 2: Bruhat-Tits trees associated to $SL(2)$

The tree on which we let 3-manifold groups act is the *Bruhat-Tits tree* associated to the special linear group $SL(2)$ defined over a discrete valuation field (refer to [Se68, Chapitre I, Section 1] for details on discrete valuation fields). We here recall its precise definition. Let $K = (K, v)$ be a field equipped with a (normalised) discrete valuation $v\colon K^\times \to \mathbb{Z}$ and $\mathcal{O}_v$ its valuation ring. We denote by ϖ_v a uniformiser of $\mathcal{O}_v$ (that is, an element of $\mathcal{O}_v$ satisfying $v(\varpi_v) = 1$). Let V_0 be the standard 2-dimensional K-vector space with basis $\{e_1, e_2\}$. Recall that an *$\mathcal{O}_v$-lattice* L of V_0 is a free $\mathcal{O}_v$-submodule of V_0 which satisfies $L \otimes_{\mathcal{O}_v} K = V_0$. An $\mathcal{O}_v$-lattice L_1 of V_0 is said to be *homothetic* to another $\mathcal{O}_v$-lattice L_2 if there exists an invertible element a of K such that L_1 coincides with aL_2 (as an $\mathcal{O}_v$-submodule of V_0). One readily verifies that the homothety relation is an equivalence relation on the set of all $\mathcal{O}_v$-lattices of V_0.

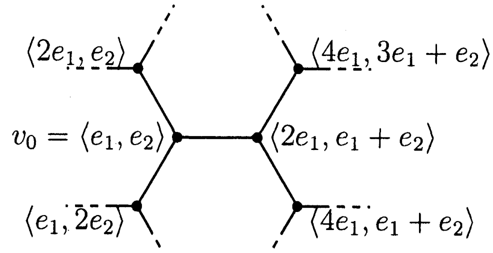
*[6] The second isomorphism is due to the fact that D and $\tilde{D}$ are *birational* to each other.

Definition 2.1 (The Bruhat-Tits tree). Let $V(\mathcal{T}_K)$ be the set of all homothety classes $[L]$ of $\mathcal{O}_v$-lattices of V_0 (the *vertex set* of $\mathcal{T}_K$). We equip $V(\mathcal{T}_K)$ with the following *adjacency relation*:

> two elements v_1 and v_2 of $V(\mathcal{T}_K)$ are *adjacent* if and only if there exist $\mathcal{O}_v$-lattices L_1 and L_2 representing v_1 and v_2 respectively such that $\varpi_v L_1 \subset L_2 \subset L_1$ holds.

Let $E(\mathcal{T}_K)$ denote the set of all adjacent pairs (v_1, v_2) of $V(\mathcal{T}_K)$ (the *edge set* of $\mathcal{T}_K$). Then one may verify that the resulted graph $\mathcal{T}_K = (V(\mathcal{T}_K), E(\mathcal{T}_K))$ is indeed a tree (refer to [Se77, Chapitre II, Théorèm 1]), which we call the *Bruhat-Tits tree* associated to $SL(2)_{/K}$.

The figure on the right side illustrates the Bruhat-Tits tree associated to the special linear group of degree two defined over the 2-adic number field $\mathbb{Q}_2$. Note that the Bruhat-Tits tree $\mathcal{T}_K$ is equipped with the natural action of $SL_2(K)$; the action of an element g of $SL_2(K)$ on the tree $\mathcal{T}_K$ is explicitly defined by $g * v = [gL]$ when an $\mathcal{O}_v$-lattice L of V_0 represents the vertex v. It is known that this action is strongly transitive and without inversion (refer to [Se77, Chapitre II, Section 1]).



Bruhat-Tits tree associated to $SL(2)_{/\mathbb{Q}_2}$

We continue to use the notation used in the previous subsection. In order to relate $SL(2)$-representations of the 3-manifold group $\pi_1(M)$ to the theory of Bruhat-Tits trees, let us consider a closed point x of $\tilde{C}$ and take a lift y of x with respect to $\tilde{\pi}$ (more specifically, y is a closed point of $\tilde{D}$ satisfying $\tilde{\pi}(y) = x$). By elementary facts on algebraic curves, each closed point y of $\tilde{D}$ endow the rational function field $\mathbb{C}(\tilde{D})$ of $\tilde{D}$ with a discrete valuation[*7] v_y, and thus we may consider the *Bruhat-Tits tree* $\mathcal{T}_y = \mathcal{T}_{K_y}$ associated to the special linear group $SL(2)$ defined over the discrete valuation field $K_y = (\mathbb{C}(\tilde{D}), v_y)$. Composing the tautological representation $\rho_{\mathrm{taut}, \tilde{D}}$ with the canonical action of $SL_2(K_y)$ on the Bruhat-Tits tree $\mathcal{T}_y$, we obtain the *action of* $\pi_1(M)$

$$\pi_1(M) \xrightarrow{\rho_{\mathrm{taut}, \tilde{D}}} SL_2(K_y) \xrightarrow{\text{canonical}} \mathrm{Aut}(\mathcal{T}_y)$$

on the Bruhat-Tits tree *associated to the closed point* y *of* $\tilde{D}$.

[*7] The discrete valuation v_y is nothing but the "order-at-y" function; namely, the evaluation of v_y at a rational function f defined on $\tilde{D}$ is the order of the Laurent expansion of f around y.

§ 2.3 Step 3: Construction of nontrivial actions

A closed point x (resp. y) of $\tilde{C}$ (resp. $\tilde{D}$) is said to be an *ideal point* if it is contained in the "boundary" of the projective completion of C (resp. D). Or in other words, ideal points are points added in the procedure of the projective completion. The key to the construction of nontrivial actions of $\pi_1(M)$ on the Bruhat-Tits trees is the following theorem (denoted as the "Fundamental Theorem" in [CS83, Theorem 2.2.1]).

Theorem 2.2 (Culler, Shalen). *Let x be a closed point of $\tilde{C}$ and y a lift of x to $\tilde{D}$ with respect to $\tilde{\pi}$. Consider the Bruhat-Tits tree $\mathcal{T}_y$ associated to the closed point y of $\tilde{D}$ and let γ be an element of $\pi_1(M)$. Then the following statements are equivalent:*

(1) *the evaluation-at-γ function I_γ is regular at x;*
(2) *under the action of $\pi_1(M)$ on $\mathcal{T}_y$ (defined as in the previous subsection), there exists a vertex of $\mathcal{T}_y$ which is stabilised by γ.*

In particular the action of $\pi_1(M)$ on the Bruhat-Tits tree $\mathcal{T}_{\tilde{y}}$ is nontrivial if $\tilde{x}$ is an ideal *point of $\tilde{C}$ and $\tilde{y}$ is a lift of $\tilde{x}$.*

We may prove Theorem 2.2 by a certain easy and tautological argument based upon the fact that the affine coordinate ring of $X(\pi_1(M))_{SL(2)/\mathbb{C}}$ is generated by evaluation functions $\{I_\gamma\}_{\gamma\in\pi_1(M)}$. We omit the details of the proof, and the readers are referred to [CS83, Theorem 2.2.1] or [Sh02, Property 5.4.2 and Property 5.4.4] for details.

By virtue of Theorem 2.2, we obtain a nontrivial action (without inversion) of $\pi_1(M)$ on a tree, and thus we may construct an essential surface contained in M as explained in the previous section. This is an outline of Culler and Shalen's construction of essential surfaces in [CS83].

§ 3 Extending towards representations of higher dimension

The content of this section is a joint work with Takahiro Kitayama, Tokyo Institute of Technology. The details shall appear in [HK14].

As we have already seen in the previous section, Culler and Shalen's construction of essential surfaces is based upon the deep study of $SL(2)$*-representations of* $\pi_1(M)$. Therefore it is natural for us to consider the following naive question:

Naive Question:

Cannot we play the same game for representations of $\pi_1(M)$ *of higher dimension*??

Or equivalently, are there any extension or any generalisation of Culler-Shalen theory for representations of $\pi_1(M)$ *of higher dimension*? The main result of [HK14] presents an *affirmative* answer to this naive question.

§ 3.1 (Essential) Tribranched surfaces

The main difference between the main result of [HK14] from that of Culler and Shalen in [CS83] is that the former one provides not (usual) essential surfaces but essential *tribranched* surfaces in general. In order to introduce the notion of essential tribranched surfaces, we first give the definition of *tribranched surfaces.*

Definition 3.1 (Tribranched surfaces). A closed subspace Σ of M is called a *tribranched surface* if the following conditions are fulfilled:

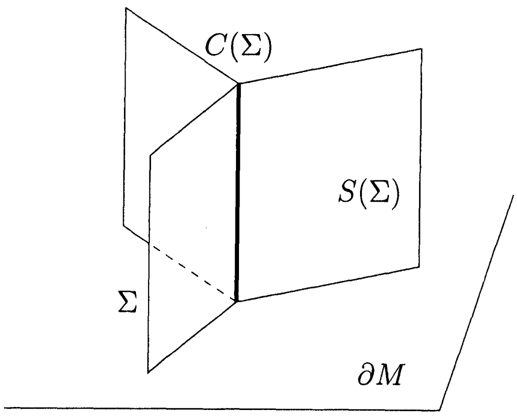
(TBS0) the pair (M, Σ) is locally homeomorphic to $(\mathbb{C} \times [0, \infty), Y \times [0, \infty))$, where Y is a closed subspace of $\mathbb{C}$ defined as

$$Y = \left\{ re^{\sqrt{-1}\theta} \;\middle|\; r \in [0, \infty), \quad \theta = 0, \pm\frac{2}{3}\pi \right\};$$

(TBS1) the intersection of Σ and a sufficiently small tubular neighbourhood of $C(\Sigma)$ in M is homeomorphic to $Y \times C(\Sigma)$;

(TBS2) each connected component of $S(\Sigma)$ is orientable.

In the definition above we denote by $C(\Sigma)$ the closed subset of Σ which corresponds to the subset $\{0\} \times [0, \infty)$ of $\mathbb{C} \times [0, \infty)$ under the identification in (TBS0), by $S(\Sigma)$ the complement of a sufficiently small tubular neighbourhood of $C(\Sigma)$ in Σ, and by $M(\Sigma)$ the complement of a sufficiently small tubular neighbourhood of Σ in M. We call the set $C(\Sigma)$ the *branch set* of Σ. The figure on the right side shows how a tribranched surface locally looks like.



A tribranched Surface Σ (local picture)

The *essentiality* of tribranched surfaces is defined similarly to the classical concept of incompressibility (namely the condition (ES1) of Definition 1.1). The precise definition of essential tribranched surfaces is given as follows.

Definition 3.2 (Essential tribranched surfaces). A tribranched surface Σ in M is said to be *essential* if it has following properties:

(ETBS1) for each component N of $M(\Sigma)$, the natural functorial homomorphism $\pi_1(N) \to \pi_1(M)$ is not surjective;

(ETBS2) for connected components C, S and N of $C(\Sigma)$, $S(\Sigma)$ and $M(\Sigma)$ respectively, the natural functorial homomorphisms $\pi_1(C) \to \pi_1(S)$ and $\pi_1(S) \to \pi_1(N)$ are injective (if they exist);

(ETBS3) there does not exist a connected component of Σ which is contained in a ball in M or a collar of ∂M.

The notion of (essential) tribranched surfaces is a certain extension of the notion of usual (essential) surfaces; indeed usual (essential) surfaces may be regarded as (essential) tribranched surfaces with the *empty* branch sets. (Essential) Tribranched surfaces seem, however, to have mysterious and strange-looking properties which have not been observed for usual (essential) surfaces. We shall propose several problems and questions concerning (essential) tribranched surfaces in [PLDT14, Section 10].

§ 3.2 Utilising Bruhat-Tits buildings of higher dimension

In order to pursue an extension of Culler and Shalen's method, we should consider counterparts of the following algebraic ingredients (used in [CS83]) for higher-dimensional representations of $\pi_1(M)$:

- the $SL(2)$-character variety $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$; and
- the Bruhat-Tits tree $\mathcal{T}_{\tilde{y}}$.

For the former one it is not difficult at all to generalise the definitions of both the representation variety $R(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ and the character variety $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ to *$SL(n)$-representations of $\pi_1(M)$*; indeed all one need to do is to replace $SL(2)_{/\mathbb{C}}$ (or $SL_2(\mathbb{C})$) with $SL(n)_{/\mathbb{C}}$ (or $SL_n(\mathbb{C})$) everywhere in the definitions of $R(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ and $X(\pi_1(M))_{SL(2)_{/\mathbb{C}}}$ in Section 2.1. We may thus define the *$SL(n)$-representation variety* $R(\pi_1(M))_{SL(n)_{/\mathbb{C}}}$ and the *$SL(n)$-character variety* $X(\pi_1(M))_{SL(n)_{/\mathbb{C}}}$ of $\pi_1(M)$, and obtain a canonical "quotient" morphism

$$\pi\colon R(\pi_1(M))_{SL(n)_{/\mathbb{C}}} \to X(\pi_1)_{SL(n)_{/\mathbb{C}}}.$$

Similarly to the $SL(2)$-character variety, the affine coordinate ring of the $SL(n)$-character variety $X(\pi_1(M))_{SL(n)_{/\mathbb{C}}}$ is known to be generated over $\mathbb{C}$ by a finite number of "evaluation-at-γ" functions I_γ, due to the classical result of Claudio Procesi [Pr76].

For the latter one —the Bruhat-Tits trees— there exist no trees which admit natural actions of the special linear groups $SL(n)$ of higher degree, but due to the vast theory of François Bruhat and Jacques Tits [BT72, BT84] on Euclidean buildings associated to reductive groups defined over local fields, there exists a *higher-dimensional (contractible) simplicial complex* equipped with the natural action of $SL(n)$. It is called the *Bruhat-Tits building* associated to $SL(n)$.

The general definition of the Bruhat-Tits buildings is rather complicated, which we shall omit. Limiting ourselves to the *special linear group* $SL(n)$, however, we may write down the explicit construction of the Bruhat-Tits building associated to $SL(n)$ as follows (refer also to [Ga97, Chapter 19]). As in Section 2.2, let $K = (K, v)$ be a discrete valuation field and $\mathcal{O}_v$ its valuation ring. Let us denote by V_0 the standard n-dimensional K-vector space. We also consider *$\mathcal{O}_v$-lattices* of V_0 and the *homothety relation* among them.

Definition 3.3 (The Bruhat-Tits building associated $SL(n)$). Consider an abstract simplicial complex $\mathcal{B}_{n,K}$ of dimension $n-1$ constructed in the following manner;

- the vertex set of $\mathcal{B}_{n,K}$ is defined as the set of all homothety classes of $\mathcal{O}_v$-lattices in V_0; and
- (distinct) k-vertices $v_0, v_1, \ldots, v_k$ of $\mathcal{B}_{n,K}$ form a k-simplex if and only if there exist $\mathcal{O}_v$-lattices $L_0, L_1, \ldots, L_k$ of V_0 representing $v_0, v_1, \ldots, v_k$ respectively such that

$$\varpi_v L_0 \subset L_k \subset L_{k-1} \subset \cdots \subset L_1 \subset L_0$$

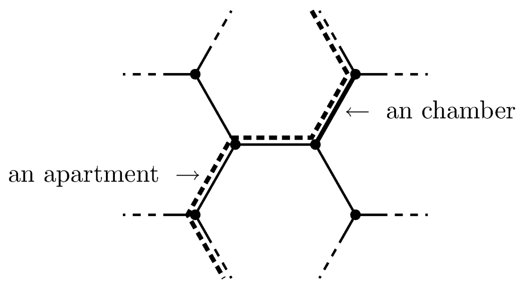
holds (here ϖ_v denotes a uniformiser of K).

One may verify that the (abstract) simplicial complex $\mathcal{B}_{n,K}$ satisfies all the axioms of (thick) Euclidean buildings (à la Jacques Tits), which we call *the Bruhat-Tits building associated to $SL(n)_{/K}$*.

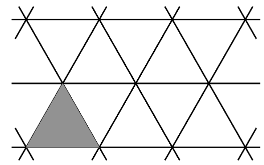
We cite [AB08] and [Ga97] as basic references on (Euclidean) buildings. Combinatorially the (Euclidean) building $\mathcal{B}_{n,K}$ consists of the $(n-1)$-dimensional (real) Euclidean spaces (called *apartments* of $\mathcal{B}_{n,K}$) tessellated by equilateral $(n-1)$-dimensional simplices (called *chambers* of $\mathcal{B}_{n,K}$), which are glued together along (several) chambers.*8 The right figure of Table 1 illustrates an apartment and a chamber of the Bruhat-Tits buildings associated to the special linear group of degree three. Due to the contractibility of every apartment of $\mathcal{B}_{n,K}$ and the axioms of buildings, we readily verify that the Bruhat-Tits building $\mathcal{B}_{n,K}$ is *contractible*. Note that the Bruhat-Tits building $\mathcal{B}_{2,K}$ associated to the special linear

*8 The Bruhat-Tits building $\mathcal{B}_{n,K}$ is said to be *Euclidean* since each apartment of $\mathcal{B}_{n,K}$ is the Euclidean space.

group $SL(2)_{/K}$ of degree 2 is none other than the Bruhat-Tits tree $\mathcal{T}_K$ associated to $SL(2)_{/K}$ by definition. The left figure of Table 1 illustrates how the Bruhat-Tits trees are regarded as Euclidean buildings.



The Bruhat-Tits tree as an Euclidean building



An apartment and a chamber of
the Bruhat-Tits building associated to $SL(3)$

Table 1. The Bruhat-Tits buildings

As in the case for Bruhat-Tits trees, the Bruhat-Tits building $\mathcal{B}_{n,K}$ is equipped with the natural action of $SL_n(K)$; namely an element g of $SL_n(K)$ acts on a vertex v of $\mathcal{B}_{n,K}$ by $g * v = [gL]$ when an $\mathcal{O}_v$-lattice L of V_0 represents v. This action is known to be strongly transitive and *type-preserving.*[*9]

Now let us take an (affine) algebraic curve C contained in $X(\pi_1(M))_{SL(n)/\mathbb{C}}$ and take a "lift" D of C with respect to π. The restriction of the natural quotient map π to D extends to a regular morphism $\tilde{\pi} \colon \tilde{D} \to \tilde{C}$ on the smooth projective models of (the projective completions of) C and D, as in Section 2.1. Let x be a closed point of $\tilde{C}$ and y its lift to $\tilde{D}$ with respect to $\tilde{\pi}$. Let v_y denote the discrete valuation of the field of rational functions $\mathbb{C}(\tilde{D})$ of $\tilde{D}$ associated to the closed point y and let us regard $\mathbb{C}(\tilde{D})$ as a discrete valuation field $K_y = (\mathbb{C}(\tilde{D}), v_y)$. Then by composing the tautological representation

$$\rho_{\mathrm{taut},\tilde{D}} \colon \pi_1(M) \to SL_n(\mathbb{C}(\tilde{D})) = SL_n(K_y)$$

with the natural action of $SL_n(K_y)$ on the Bruhat-Tits building $\mathcal{B}_{n,K_y}$, we obtain the *action of* $\pi_1(M)$

$$\pi_1(M) \xrightarrow{\rho_{\mathrm{taut},\tilde{D}}} SL_n(K_y) \xrightarrow{\text{canonical}} \mathrm{Aut}(\mathcal{B}_y)$$

on the Bruhat-Tits building $\mathcal{B}_y = \mathcal{B}_{n,K_y}$ *associated to the closed point y of $\tilde{D}$.*

Similarly to the proof of Theorem 2.2, we may verify the following statement:

[*9] The fact that the natural action of $SL_n(K)$ on the Bruhat-Tits building $\mathcal{B}_{n,K}$ is *type-preserving* corresponds to the fact that the natural action of $SL_2(K)$ on the Bruhat-Tits tree $\mathcal{T}_K$ is *without inversion*. For the precise definition of type-preserving actions, refer to [Ga97, Section 5.2] for example.

Theorem 3.4 (H.-Kitayama). *Let x be a closed point of $\tilde{C}$ and y a lift of x to $\tilde{D}$ with respect to $\tilde{\pi}$. Consider the Bruhat-Tits building $\mathcal{B}_y$ associated to the closed point y of $\tilde{D}$ and let γ be an element of $\pi_1(M)$. Then there exists a vertex of $\mathcal{B}_y$ which is stabilised by γ under the action of $\pi_1(M)$ on $\mathcal{B}_y$ (defined as above) if the evaluation-at-γ function I_γ is regular at x.*
In particular the action of $\pi_1(M)$ on the Bruhat-Tits building $\mathcal{B}_{\tilde{y}}$ is nontrivial if $\tilde{x}$ is an ideal *point of $\tilde{C}$ and $\tilde{y}$ is a lift of $\tilde{x}$.*

We remark that the converse of Theorem 3.4 does not hold in general for higher-dimensional representations, unlike Theorem 2.2.

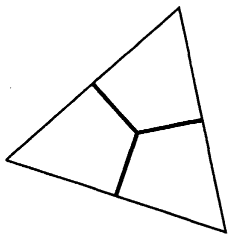
§ 3.3 Construction of tribranched surfaces

We briefly summarise how to construct a tribranched surface from a nontrivial (and type-preserving) action of $\pi_1(M)$ on the Bruhat-Tits building $\mathcal{B}_{\tilde{y}}$ associated to an ideal point of the curve $\tilde{D}$ obtained as in Theorem 3.4. The construction is similar to the method of Stallings et al. which we have already explained in Section 1.

Let n be a natural number greater than or equal to three, and assume that the boundary of M is not empty when n is strictly greater than three. Let us take a triangulation of M and lift it to the universal covering space $\tilde{M}$ of M. Therefore $\tilde{M}$ is equipped with the $\pi_1(M)$-triangulation. When n is strictly greater than three, we may take a *spine*[*10] $\tilde{K}$ of $\tilde{M}$ because of the non-empty assumption on the boundary of M. Due to the contractibility of the Bruhat-Tits building $\mathcal{B}_{\tilde{y}}$, we may construct a $\pi_1(M)$-equivariant simplicial map from $\tilde{M}$ (resp. from the 2-dimensional subcomplex $\tilde{K}$ of $\tilde{M}$ if n is strictly greater than three) to the 2-*skeleton* $\mathcal{B}_{\tilde{y}}^{(2)}$ of $\mathcal{B}_{\tilde{y}}$. Composing it with a retraction $\tilde{M} \to \tilde{K}$ when n is strictly greater than three, we finally obtain a $\pi_1(M)$-equivalent piecewise-linear map

$$\tilde{f} \colon \tilde{M} \to \mathcal{B}_{\tilde{y}}^{(2)},$$

which induces a piecewise-linear map $f \colon M \to \mathcal{B}_{\tilde{y}}^{(2)}/\pi_1(M)$ on the quotient complexes.



Since the target $\mathcal{B}_{\tilde{y}}^{(2)}/\pi_1(M)$ of f is now a simplicial complex *of dimension two*, it is absurd to consider the "set of all midpoints of edges." We thus consider for a 2-dimensional simplicial complex Δ the 1-dimensional subcomplex $Y(\Delta)$ of the barycentric subdivision of Δ consisting of the barycentres of all 1- and 2-simplices and all the edges connecting them. The figure on the left side shows $Y(\Delta)$ when Δ consists of a single 2-simplex. We

[*10] A *spine* of a topological manifold M is a locally tamely embedded subcomplex K of M so that K is a strong deformation retract of M and the natural inclusion $K \hookrightarrow M$ is a simple homotopy equivalence.

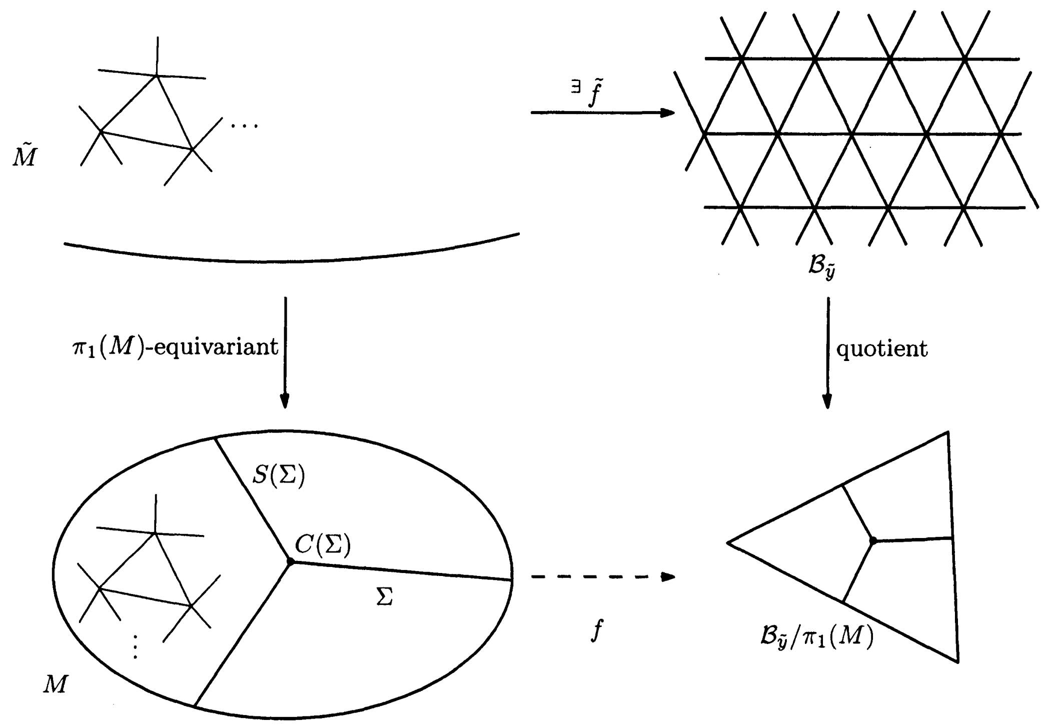


Figure 3. Construction of (essential) tribranched surfaces

deal with $Y(\mathcal{B}_{\tilde{y}}^{(2)}/\pi_1(M))$ as a counterpart of the "set of all midpoints of edges" E of the quotient graph $\mathcal{T}/\pi_1(M)$ in Section 1, and consider the inverse image Σ' of $Y(\mathcal{B}_{\tilde{y}}^{(2)}/\pi_1(M))$ under the map $f\colon M \to \mathcal{B}_{\tilde{y}}^{(2)}/\pi_1(M)$ (see Figure 3). We readily verify that Σ' is a properly embedded tribranched surface in M. As in Section 1, we may modify Σ' into an *essential* tribranched surface Σ by certain "local surgery." For details refer to [HK14, Section 4.2].

Consequently we obtain the following theorem, which is the main result of [HK14]:

Theorem 3.5 (H.-Kitayama). *Let n be a natural number greater than or equal to three, and let M be a 3-manifold. Furthermore assume that the boundary of M is not empty if n is strictly greater than three. Then associated to each ideal point $\tilde{x}$ of the $SL(n)$-character variety $X(\pi_1(M))_{SL(n)/\mathbb{C}}$, we may construct an essential tribranched surface $\Sigma_{\tilde{x}}$ contained in M.*

Conversely, we may also associate to an essential tribranched surface Σ a 2-*complex of groups* $\mathcal{G}_\Sigma$ in a way similar to the construction of the *graph of groups* $\mathcal{G}_S$ associated to an essential surface S (the details shall appear in [HK14]; refer also to the final part of [PLDT14, Section 10]). These results reveal a close relation between the concepts of essential tribranched surfaces and (nontrivial) actions of 3-dimensional groups on certain simply-connected 2-dimensional complexes:

Essential tribranched surfaces	$\overset{\text{"close relation"}}{\longleftrightarrow}$	Nontrivial actions of $\pi_1(M)$ on certain simply-connected 2-dimensional complexes
(topological concept)		(algebraic group action)

§ 4 Several problems from the arithmetic viewpoint

In this final section we propose several problems and questions concerning actions of 3-manifold groups from the *arithmetic viewpoint.* The readers are referred to [PLDT14, Section 10] for problems and questions from the *topological viewpoint.*

§ 4.1 From the arithmetic topological viewpoint

Arithmetic topology is a research field where one pursues analogous theory and analogous phenomena between *low-dimensional topology* and *number theory.* Recently there has been drastic progress in this research field by virtue of great efforts of many people including Masanori Morishita and his colleagues. See also Mizusawa's article [Mi14] in this volume for another aspect of the progress of arithmetic topology.

Table 2 is a part of the (what is called) "dictionary" of arithmetic topology, which indicates correspondences between analogous objects in low-dimensional topology and number theory. According to this dictionary, the counterparts of the $SL(2)$-character varieties of 3-manifold groups (which are certain kinds of "moduli spaces" of representations of 3-manifold groups) are the *universal deformation spaces* of the absolute Galois groups, which were first introduced by Barry Mazur in [Ma89] and have played crucial roles in recent arithmetic geometry.*[11] Moreover it has been widely known that there exist very interesting (rigid analytic) curves in the universal deformation space of (certain) residual Galois representations of the absolute Galois group $G_{\mathbb{Q}}$ of the rational number field $\mathbb{Q}$: namely curves of *families of p-adic modular forms* (Hida family, Coleman family, and ultimately the "eigencurve").

Question 4.1. *For such a curve of a family of p-adic modular forms, could we play the same* (*or at least analogous*) *game as Culler and Shalen's method?*

Note, however, that there exist several crucial difficulties in this setting. Firstly, curves of families of p-adic modular forms are not algebraic curves but *rigid analytic* curves,

*[11] For example, the deep study of the universal deformation spaces of Galois representations is one of the keys of Andrew Wiles's solution of Fermat's Last Theorem.

Low-dimensional topology	Number theory
3-manifold M connected, oriented, closed	number field k i.e., $k/\mathbb{Q}$: finite extension
link $L = \{K_1, \ldots, K_r\}$	a finite set of finite places $S = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$
link group $\pi_1(V_L)$ $V_L = M \setminus L$: link complement	Galois group $G_S = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ $\mathbb{Q}_S/\mathbb{Q}$: maximal extension unramified outside S
$\rho\colon \pi_1(V_L) \to SL_2(\mathbb{C})$	$\rho\colon G_S \to GL_2(\mathcal{O})$ Galois representation
$\boxed{\text{character variety}}$ $X(\pi_1(V_L))_{SL(2)/\mathbb{C}} = R(\pi_1(V_L))_{SL(2)/\mathbb{C}} /\!/ SL(2)_{/\mathbb{C}}$ ※ affine algebraic variety	$\boxed{\text{universal deformation space}}$ $\mathcal{X}_2 = \mathrm{Spf}\, \mathcal{R}_\rho$ $\mathcal{R}_\rho$: the universal deformation ring ※ affine formal scheme (rigid analytic)
curve $\subset X_2$ through the monodromy representation of the hyperbolic structure of M (when M: hyperbolic) [Thurston's theorem]	a p-adic family of modular forms (Hida family, the eigencurve) [Masanori Morishita, Sachiko Ohtani, etc...]

Table 2. The "dictionary" of arithmetic topology

and thus we could not apply results on (classical) birational geometry of algebraic curves to them and need to heavily utilise rigid analytic geometry. Secondly, although we may interpret a point of rigid analytic curves as a valuation of a certain Banach algebra, it is not a *discrete* valuation in general. Hence the associated Bruhat-Tits building no longer admits a structure as a *simplicial complex.* Nevertheless there exists a possibility for us to obtain a certain action of the *absolute Galois group on a (combinatorially complicated) object*, and it might be interesting to study such a mysterious action of the absolute Galois representation.

§ 4.2 From the zeta functional viewpoint

As we have also seen in Section 1, we may associate to an essential surface S contained in a 3-manifold M a *graph of groups* $\mathcal{G}_S$ in a canonical manner. Moreover the underlying graph of $\mathcal{G}_S$ is a *finite* graph due to the compactness of M. For a general finite graph $\mathcal{G}$, we may consider the *zeta function of the graph* $\mathcal{G}$ (or the *Ihara zeta function of* $\mathcal{G}$) $\zeta_{\mathcal{G}}(u)$, which has its origin in Ihara's thorough study of a p-adic analogue of the Selberg zeta functions in [Iha66] and has been developed by many people including Toshikazu Sunada, Ki-ichiro Hashimoto, Hyman Bass and Audrey Terras.

Under the background above let us consider the zeta function of the graph (of groups) $\mathcal{G}_S$ associated to an essential surface S, which seems to contain certain information of the essential surface S. Moreover if the essential surface S is associated to an ideal point of (a curve C contained in) the $SL(2)$-character variety $X(\pi_1(M))_{SL(2)}$, we may obtain another zeta function; namely the *Hasse-Weil zeta function associated to* $X(\pi_1(M))_{SL(2)}$ (or associated to C). See also Harada's article [Harada14] in this volume for the Hasse-Weil zeta functions of $SL(2)$-character varieties. As Ihara has already observed in [Iha68], the zeta functions of certain graphs might have close relations with the Hasse-Weil zeta functions of certain algebraic curves, and thus it might be interesting to investigate the relation between the zeta functions of the graphs associated to essential surfaces and the Hasse-Weil zeta functions of character varieties.

We finally give one remark. When we consider the zeta function of $\mathcal{G}_S$, we ignore the information of vertex and edge groups of $\mathcal{G}_S$, and the underlying graph (that is, the dual graph of S) might become too simple. For example, let us consider the knot complement V_K of an arbitrary knot K. In this case V_K contains the Seifert surface S_K of K which is essential. The graph of groups $\mathcal{G}_{S_K}$ associated to the Seifert surface S_K then consists of a single vertex with the trivial vertex group and a single edge representing S_K.*12 Therefore the zeta function $\zeta_{\mathcal{G}_{S_K}}(u)$ reduces to an uninteresting function $(1-u)^{-1}$ because $\mathcal{G}_S$ admits only one prime cycle of length one.*13 This observation implies that we should impose certain constraints on essential surfaces (or on 3-manifolds themselves) when we consider the zeta functions of the graphs associated to them, or more ultimately, we should try to construct more sophisticated zeta-function-like invariants of graphs *of groups* which reflect the information of their vertex and edge groups.

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*12 According to the theory of graphs of groups, this implies the classical fact that the fundamental group of a knot complement is represented as a HNN extension of that of its Seifert surface.

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