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(q, t)-hook formula for Birds

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Abstract

We study Okada's conjecture on (q, t)-hook formula of general d-complete posets. Proctor classified d-complete posets into 15 irreducible ones. We try to give a case-by-case proof of Okada's (q, t)-hook formula conjecture using the symmetric functions. Here we give a proof of the conjecture for birds. in which we use Gasper's identity for VWP-series ${}_{12}W_{11}$.

1 Introduction and the main results

Let $\mathbb{N}$ (resp. $\mathbb{Z}$) be the set of nonnegative integers (resp. integers). Throughout this paper we use the standard notation for q-series (see [1, 3, 4, 5]):

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \qquad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

for any integer n. Usually $(a;q)_n$ is called the *q-shifted factorial*, and we frequently use the compact notation:

$$(a_1, a_2, \dots, a_r; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_r;q)_n.$$

The ${}_{r+1}\phi_r$ *basic hypergeometric series* is defined by

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. \tag{1.1}$$

A basic hypergeometric series ${}_{r+1}\phi_r$ is said to be *balanced* if it satisfies $qa_1 \cdots a_{r+1} = b_1 \cdots b_r$ and $z = q$, *well-poised* if it satisfies $qa_1 = a_2 b_1 = \cdots = a_{r+1} b_r$, *very well-poised* if it is well-poised and satisfies $b_1 = a_1^{\frac{1}{2}}$ and $b_2 = -a_1^{\frac{1}{2}}$ (see [3, §2.1]). If ${}_{r+1}\phi_r$ is very well-poised series, we use the notation

$$ {}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_{r+1}\phi_r\left[\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \dots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, qa_1/a_4, \dots, qa_1/a_{r+1} \end{matrix}; q, z\right].$$

Proposition 1.1. Gasper's formula ([2, p.1065, (3.2)], [3, pp.250, Ex.8.15]) reads as follows:

$$\begin{aligned} {}_4\phi_3\left[\begin{matrix} a, b, c, d \\ bq/a, cq/a, dq/a \end{matrix}; q, \frac{q^2}{a^2}\right] &= \frac{(a/d.bq/d, cq/d, abc/d; q)_\infty}{(q/d, ab/d, ac/d, bcq/d; q)_\infty} \\ &\times {}_{12}W_{11}\left(\frac{bc}{d}; \left(\frac{bcq}{ad}\right)^{\frac{1}{2}}, -\left(\frac{bcq}{ad}\right)^{\frac{1}{2}}, q\left(\frac{bc}{d}\right)^{\frac{1}{2}}, -q\left(\frac{bc}{d}\right)^{\frac{1}{2}}, \frac{ab}{d}, \frac{ac}{d}, a, b, c; q, \frac{q}{a}\right), \end{aligned} \tag{1.2}$$

where at least one of a, b, c is of the form q^{-n} $(n = 0, 1, \dots)$.

We use the notatation in [8]. For nonnegative integers n and m we write

$$f(n;m) = f_{q,t}(n;m) = \frac{(t^{m+1};q)_n}{(t^m;q)_n},$$

and

$$F(x) = F(x;q,t) = \frac{(tx;q)_\infty}{(x;q)_\infty},$$

where q and t are parameters and x is a variable (see [8, (5)(6)]). Hereafter we use the convention that $f_{q,t}(n;m) = 0$ for a negative integer $n < 0$.

We use the notation in [7, 12] for partitions. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \dots$ with finitely many λ_i unequal to zero. The length and weight of λ, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero λ_i respectively. When $|\lambda| = N$ we say that λ is a partition of N, and the unique partition of zero is denoted by $\emptyset$. The multiplicity of the part i in the partition λ is denoted by $m_i(\lambda)$. We identify a partition with its diagram (Ferrers graph)

$$D(\lambda) = \{\, (i,j) \in \mathbb{Z}^2 \,:\, 1 \leq j \leq \lambda_i \}. \tag{1.3}$$

The conjugate λ' of λ is the partition obtained by reflecting the diagram of λ in the main diagonal. A partition is said to be *strict* if we have strict inequalities $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ with $r = \ell(\lambda)$. If λ is a strict partition, then its shifted diagram is defined by

$$S(\lambda) = \{\, (i,j) \in \mathbb{Z}^2 \,:\, i \leq j \leq \lambda_i + i - 1 \,\}. \tag{1.4}$$

Hereafter we may use the same symbol λ to represent its diagram (or shifted diagram).

We use standard notation and terminology of [12, Chapter 3] related to posets. We write $x \lessdot y$ if x is covered by y, i.e., $x < y$ and there is no $z \in P$ such that $x < z < y$. A Hasse diagram is a diagram in which one represents each element of P as a vertex in the plane and draws an edge that goes upward from x to y whenever y covers x.

Definition 1.2. ([11], [12, §3.15]) Let P be a poset. A *P-partition* is a map $\pi : P \to \mathbb{N}$ satisfying

$$x \leq y \text{ in } P \quad \Longrightarrow \quad \pi(x) \geq \pi(y) \text{ in } \mathbb{N}. \tag{1.5}$$

Let $\mathscr{A}(P)$ denote the set of P-partitions.

First, we review the definition and some properties of d-complete posets. (See [9, 10].) For $k \geq 3$, we denote by $d_k(1)$ the poset consisting of $2k-2$ elements, called *double-tailed diamond poset*, with the Hasse diagram depicted in Figure 1. The two incomparable elements are called the *sides*, the $k-2$ elements above them are called *neck* elements, and the maximum and minimum elements are called *top* and *bottom* respectively. If $k = 3$ then we call $d_3(1)$ a *diamond*. Let P be a poset. An interval $[w, v] = \{\, x \in P \,:\, w \leq x \leq v \,\}$ is called a *d_k-interval* if it is isomorphic to $d_k(1)$ A *d_k^--interval* ($k \geq 4$) is an interval isomorphic to $d_k(1) - \{\text{top}\}$. A *d_3^--interval* consists of three elements x, y and w such that w is covered by both x and y. A poset P is *d-complete* if it satisfies the following three conditions for every $k \geq 3$:

(D1) If I is a d_k^--interval, then there exists an element v such that v covers the maximal elements of I and $I \cup \{v\}$ is a d_k-interval.

(D2) If $I = [w, v]$ is a d_k-interval and the top v covers u in P, then $u \in I$.

(D3) There are no d_k^--intervals which differ only in the minimal elements.

We quote a proposition due to Proctor [9, Proposition in §3] (also see [8, Proposition 4.1]):

Proposition 1.3. ([9, Proposition in §3]) Let P be a d-complete poset. Suppose that P is connected, i.e., the Hasse digram of P is connected. Then we have

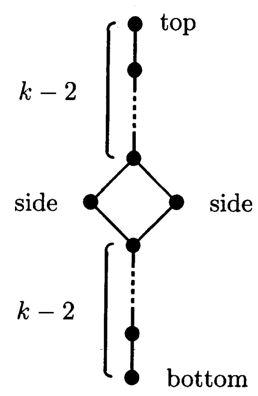

Figure 1: A double-tailed diamond poset $d_k(1)$

(a) P has a unique maximal element v_0.

(b) For each $v \in P$, every saturated chain from v to the maximum element v_0 has the same length.

Hence P admits a rank function $r : P \to \mathbb{N}$ such that $r(x) = r(y) + 1$ if x covers y.

A *rooted tree* is a poset which has a unique maximal element, and is such that each non-maximal element is covered by exactly one other element. Let P be a poset with a unique maximal element. The top tree T of P is the filter (i.e., $x \in T$ and $y \geq x$ implies $y \in T$) of P, whose vertex set consists of all elements $x \in P$ such that every $y \geq x$ is covered by at most one other element of P. T is clearly a rooted tree and an element of T is called *top tree element*. Afterwards we use a particular kind of rooted tree. Let $f \geq 0$ and $h \geq g \geq 0$ be integers. The rooted tree $Y(f; g, h)$ consists of one branch element above which a chain of f elements has been adjoined and below which two non-adjacent chains with g and h elements, respectively.

Let P be a connected d-complete poset with top tree T. An element $x \in P$ is said to be *acyclic* if $x \in T$ and it is not in the neck of any d_k-interval for any $k \geq 3$. An element of P is said to be *cyclic* if it is not acyclic. Let Q be a d-complete poset containing an acyclic element y. Let P be a connected d-complete poset. By Proposition 1.3 (a), let x denote the unique maximal element of P. Then the *slant sum* of Q with P at y, denoted $Q^y\backslash_x P$, is the poset formed by creating a covering relation $x \lessdot y$. A d-complete poset P is *slant irreducible* if it is connected and it cannot be expressed as a slant sum of two non-empty d-complete posets. Suppose that P is a connected d-complete poset with top tree T. An edge $x \lessdot y$ of P is a *slant edge* if $x, y \in T$ and y is acyclic. In [9] Proctor proves P is slant irreducible if and only if it contains no slant edges. Also, P is slant irreducible if and only if every acyclic element is a minimal element of its top tree. ([9, Proposition C of §4]) Given any connected d-complete poset P, first locate all of its slant edges. These may be erased in any order to produce a collection $P_1, P_2, \ldots$ of uniquely determined smaller non-adjacent connected d-complete posets. No new slant edges are created, and so each of $P_1, P_2, \ldots$ are slant irreducible. We say that $P_1, P_2, \ldots$ are the *slant irreducible components* of P. If P is an irreducible component, then its top tree T is of the form $Y(f; g, h)$ for some $f \geq 0$ and $h \geq g \geq 1$ ([9, Theorem of §5]). In the paper he establish the following theorem, which describe the structure of any connected d-complete poset.

Theorem 1.4. (Proctor [9, Theorem in §4]) Let P be a connected d-complete poset. It may be uniquely decomposed into a slant sum of one element posets and irreducible components. The top tree of P is an analogous slant sum of the top trees of the irreducible components.

In §7 of [9] Proctor defines 15 disjoint classes of irreducible components $\mathscr{C}_1, \ldots, \mathscr{C}_{15}$ and have shown that these 15 disjoint classes exhaust the set of all irreducible components. For the list of 15 classes of irreducible d-complete posets see [9, Table 1]. The diagram (1.3) of an ordinary partition λ or the shifted diagram (1.4) of a shifted partition λ is regarded as a poset by defining its order structure as

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2. \tag{1.6}$$

By this order the poset represented by a diagram $P = D(\lambda)$ is called a *shape* with its top tree $T = Y(f; g, h)$ where $f = 0$, $g = \ell(\lambda)$ and $h = \ell(\lambda')$. We use $\mathscr{C}_1$ to express the class of shapes which is a class of irreducible d-complete posets defined in [9].

Another important class $\mathscr{C}_2$ is the set of posets $P = S(\alpha)$ of shifted diagrams for strict partitions α, which is called *shifted shapes* with its top tree $T = Y(f, g, h)$ where $f = g = 1$ and $h = \ell(\alpha)$. Its Hasse diagram is designated by Figure 1 in which the first row has α_1 vertices, the second row α_2 vertices and so on. When depicting these posets as a Hasse diagram, we use the convention that a northwest vertex is larger than another in southeast. Here the larger dots and the heavier edges indicate the top tree. For later use we denote by $P = P_2(\alpha)$ the Shifted shape associated with a strict partition α. If $P = P_2(\alpha)$ is the shifted shape associated with a strict partition α,

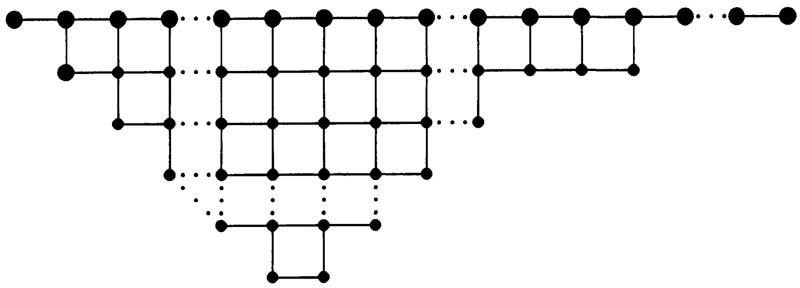

Figure 2: Shifted shapes C_2

then P-partition

$$\pi = (\pi_{ij})_{(i,j)\in S(\alpha)} \tag{1.7}$$

satisfies

$$\pi_{ij} \leq \pi_{i+1,j}, \qquad \pi_{ij} \leq \pi_{i,j+1}, \tag{1.8}$$

whenever the both sides defined. For example, Figure 1 is a P-partition for shifted shape $(8, 5, 2, 1)$.

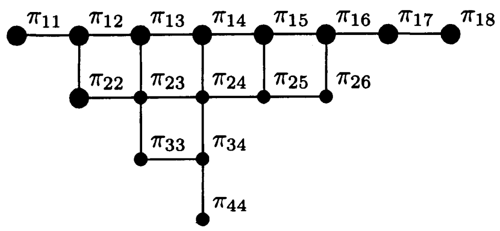

Figure 3: P-partition for shifted shape $(8, 5, 2, 1)$

In this paper we mainly consider only birds $\mathscr{C}_3$ (Figure 1). Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Define the *bird* $P = P_3(\alpha, \beta; f)$ by

$$P = P_{\mathrm{H}} \cup P_{\mathrm{R}} \cup P_{\mathrm{L}} \cup P_{\mathrm{T}}$$

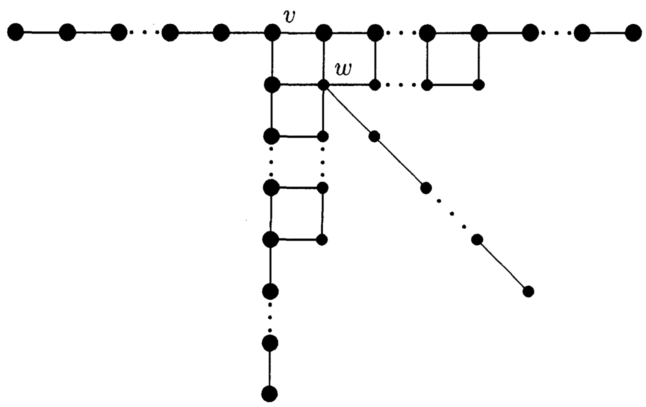

Figure 4: Birds C_3

where

$$\begin{aligned} P_{\mathrm{H}} &= \{ (1,j) \,:\, -f+1 \le j \le 1\}, \\ P_{\mathrm{R}} &= \{ (i,j) \,:\, i \le j \le \alpha_i + i - 1 \ (i=1,2) \}, \\ P_{\mathrm{L}} &= \{ (i,j) \,:\, j \le i \le \beta_j + j - 1 \ (j=1,2) \}, \\ P_{\mathrm{T}} &= \{ (i,i) \,:\, 2 \le i \le f+2\} \end{aligned}$$

as a set and we regard it as a poset by defining its order structure (1.6) if and only if the both of (i_1, j_1) and (i_2, j_2) are in $P_{\mathrm{H}} \cup P_{\mathrm{R}} \cup P_{\mathrm{L}}$ or in P_{T} (see [9, Table 1 and Figure 5.3]). We call P_{H} the *head*, P_{T} the *tail*, P_{R} (resp. P_{L}) the *right* (resp. *left*) *wing* of P. The Hasse diagram of a bird is as in Figure 1. Strictly speaking, we have to impose the condition $\alpha_1 = \alpha_2 + 1$ and $\beta_1 = \beta_2 + 1$ to let P be slant irreducible, but here we don't need this condition. For example, Figure 1 stands for $P = P_3((4,3),(4,2);2)$. We have the chain $[v, v_2]$ (resp. $[w_2, w]$), which is the head (resp. tail) of

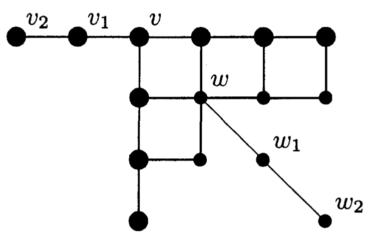

Figure 5: Bird $P = P_3((4,3),(3,2);2)$ and banner $P = P_6((9,6,3,2);2)$

P. Recall that a P-partition π satisfies the condition (1.5). When $P = P_3(\alpha, \beta; f)$, we associate the quadruple $(\sigma, \tau; \rho, \theta)$ with π, where

$$\sigma = (\sigma_{i,j})_{(i,j)\in P_{\mathrm{R}}}, \quad \tau = (\tau_{i,j})_{(j,i)\in P_{\mathrm{L}}}, \quad \rho = (\rho_i)_{i=0,\dots,f}, \quad \theta = (\theta_i)_{i=0,\dots,f}$$

with

$$\begin{aligned} &\sigma_{i,j} = \pi(i,j) \quad \text{for } (i,j) \in P_{\mathrm{R}}, \qquad &\tau_{i,j} = \pi(j,i) \quad \text{for } (i,j) \in P_{\mathrm{L}}, \\ &\rho_{-i+1} = \pi(1,i) \quad \text{for } (1,i) \in P_{\mathrm{H}}, \qquad &\theta_{i-2} = \pi(i,i) \quad \text{for } (i,i) \in P_{\mathrm{T}}. \end{aligned} \tag{1.9}$$

Hence we use the convention that $\rho_0 = \sigma_{11} = \tau_{11}$ and $\theta_0 = \sigma_{22} = \tau_{22}$. We write $\pi = (\sigma, \tau; \rho, \theta)$ hereafter. If $P = P_3((4,3),(4,2);2)$ then π is as the left picture of Figure 1.

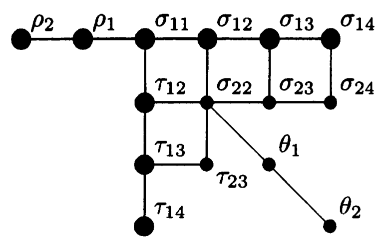

Figure 6: A P-partition

Let P be a connected d-complete poset and T its top tree. Let C be a set, called *a set of colors*, whose cardinality is the same as T. A *coloring* of P a coloring map c of P to the set of colors C. P is said to be *properly colored* if the coloring map c satisfies

(C1) $c(x) \neq c(y)$ if x and y are incomparable,

(C2) $c(x) \neq c(y)$ if x covers y.

It is *simply colored* if, in addition:

(C3) whenever an interval $[w, v]$ is a chain, the colors of the elements $c(x)$ in the interval $[w, v]$ are distinct.

If P is a rooted tree, then it is simply colored by the identity map $P \to P$, i.e. we assign a distinct color to each vertex of P.

Proposition 1.5. ([10, Proposition 8.6]) Let P be a connected d-complete poset and T its top tree. Let C be a set whose cardinality is the same as T. Then a bijection $c : T \to C$ can be uniquely extended to a proper coloring $c : P \to C$ satisfying the following condition:

(C4) If $[w, v]$ is a d_k-interval then $c(w) = c(v)$.

Such a map $c : P \to I$ is called a *d-complete coloring*.

For example, in the both picture of Figure 1 because $[w_2, v_2]$ (resp. $[w_1, v_1]$) is a d_5-interval (resp. d_4-interval), w_2 (resp. w_1, w) and v_2 (resp. v_1, v) have the same color. In Figure **??** v_1 (resp. v_2) and v_3 (resp. v_4) have the same color since $[v_3, v_1]$ (resp. $[v_4, v_2]$ is a d_4-interval, however, the v_1 and v_2 have distinct colors since the both are in the top tree.

Proposition 1.6. (1) If α is a strict partition with length≥ 2, then the top tree of the shifted shape $P = P_2(\alpha)$ is given by

$$T = \{(1, j) \,:\, 1 \leq j \leq \alpha_1\} \cup \{(2, 2)\}, \tag{1.10}$$

and a d-complete coloring $c : P \to \{0, 0', 1, 2, \dots, \alpha_1 - 1\}$ is given by

$$c(i, j) = \begin{cases} j - i & \text{if } i < j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd}, \\ 0' & \text{if } i = j \text{ and } i \text{ is even}. \end{cases} \tag{1.11}$$

Hence we see that P has the top tree $Y(1; 1, \alpha_1 - 1)$.

(2) If α and β are strict partitions with length$= 2$ and $f \geq 1$ then the top tree of the bird $P = P_3(\alpha, \beta; f)$ is given by

$$T = \{(1, j) \,:\, -f + 1 \leq j \leq \alpha_1\} \cup \{(i, 1) \,:\, 1 \leq i \leq \beta_1\}, \tag{1.12}$$

and a d-complete coloring $c : P \to \{-f, \dots, -1, 0, 1, 2, \dots, \alpha_1 - 1\} \cup \{1', 2', \dots, (\beta_1 - 1)'\}$ is given by

$$c(i,j) = \begin{cases} j - i & \text{if } i < j, \text{ i.e., } (i,j) \in P_{\mathrm{R}}, \\ (i-j)' & \text{if } 1 \le j < i, \text{ i.e., } (i,j) \in P_{\mathrm{L}}, \\ j - 1 & \text{if } i = 1 \text{ and } j \le 1, \text{ i.e., } (i,j) \in P_{\mathrm{H}}, \\ -i + 2 & \text{if } i = j \ge 2, \text{ i.e., } (i,j) \in P_{\mathrm{T}}. \end{cases} \tag{1.13}$$

Hence we see that P has the top tree $Y(f; \alpha_1 - 1, \beta_1 - 1)$.

Let P be a connected d-complete poset and $c : P \to C$ a d-complete coloring. Let z_i $(i \in C)$ be indeterminates. For a P-partition $\pi \in \mathscr{A}(P)$, we put

$$z^{\pi} = \prod_{v \in P} z_{c(v)}^{\pi(v)}.$$

As in [8, p.412] we associate a monomial $z[H_P(v)]$ to each $v \in P$, called the *hook monomial*, which is uniquely determined by induction as follows:

(a) If v is not the top of any d_k-interval, then we define

$$z[H_P(v)] = \prod_{w \le v} z_{c(w)}.$$

(b) If v is the top of a d_k-interval $[w, v]$, then we define

$$z[H_P(v)] = \frac{z[H_P(x)] \cdot z[H_P(y)]}{z[H_P(w)]},$$

where x and y are the sides of $[w, v]$.

Further we denote $z[H_p] = \{ z[H_P(v)] \ : \ v \in P \}$ the set of the hook monomials, and let $F(z[H_p]; q, t)$ denote the product of $F(z[H_P(v)]; q, t)$ over $v \in P$, i.e.,

$$F(z[H_p]; q, t) = \prod_{v \in P} F(z[H_P(v)]; q, t).$$

Let P be a connected d-complete poset with the maximum element v_0, and the rank function $r : P \to \mathbb{N}$. Let T be the top tree of P. Take T as a set of colors and let $c : P \to T$ be the d-complete coloring such that $c(v) = v$ for all $v \in T$. Let $\widehat{P} = P \sqcup \{\widehat{1}\}$ be the extended poset, where $\widehat{1}$ is the new maximum element of $\widehat{P}$ which covers v_0. Then $\widehat{P}$ has its top tree $\widehat{T} = T \sqcup \{\widehat{1}\}$, where $\widehat{c} : \widehat{P} \to \widehat{T}$ with $\widehat{c}(\widehat{1}) = \widehat{1}$.

Definition 1.7. Given a P-partition $\pi \in \mathscr{A}(P)$, let $\widehat{\pi} : \widehat{P} \to \mathbb{N}$ be the extensions of π defined by $\widehat{\pi}(\widehat{1}) = 0$. Define a weight $W_P(\sigma; q, t)$ by putting

$$W_P(\pi; q, t) = \frac{\displaystyle\prod_{\substack{x, y \in \widehat{P} \\ x < y,\ \widehat{c}(x) \sim \widehat{c}(y)}} f(\pi(x) - \pi(y); d(x, y))}{\displaystyle\prod_{\substack{x, y \in P \\ x < y,\ c(x) = c(y)}} f(\sigma(x) - \sigma(y); e(x, y)) f(\sigma(x) - \sigma(y); e(x, y) - 1)}, \tag{1.14}$$

where $\widehat{c}(x) \sim \widehat{c}(y)$ means that $\widehat{c}(x)$ and $\widehat{c}(y)$ are adjacent to each other in T, and

$$d(x, y) = \frac{r(y) - r(x) - 1}{2}, \qquad e(x, y) = \frac{r(y) - r(x)}{2}.$$

Note that if $c(x) \sim c(y)$ then $r(y) - r(x)$ is odd, and if $c(x) = c(y)$ then $r(y) - r(x)$ is even, hence $d(x, y)$ and $e(x, y)$ are nonnegative integers.

Now we quote Okada's (q,t)-hook formula conjecture.

Conjecture 1.8. (Okada [8]) Let P be a connected d-complete poset. Using the notations defined above, we have

$$\sum_{\pi\in\mathscr{A}(P)} W_P(\pi;q,t)z^{\pi} = F\left(z[H_p];q,t\right). \tag{1.15}$$

Okada has proven this conjecture for Shapes and Shifted shapes. The purpose of this paper is to prove his conjecture for birds and banners.

Theorem 1.9. Okada's (q,t)-hook formula conjecture is true for birds and banners.

Given a P-patition $\pi\in\mathscr{A}(P)$ for the shfted shape $P=P_2(\alpha)$ for a strict partition α, we write

$$f_\alpha^{\mathrm{ND}}(\pi;q,t) = \prod_{\substack{(i,j)\in\alpha\\ i<j}}\prod_{m\geq 0}\frac{f(\pi_{i,j}-\pi_{i-m,j-m-1};m)f(\pi_{i,j}-\pi_{i-m-1,j-m};m)}{f(\pi_{i,j}-\pi_{i-m,j-m};m)f(\pi_{i,j}-\pi_{i-m-1,j-m-1};m)}, \tag{1.16}$$

$$f_\alpha^{\mathrm{D}}(\pi;q,t) = \prod_{(i,i)\in\alpha}\prod_{\substack{m\geq 0\\ m \text{ even}}}\frac{f(\pi_{i,i}-\pi_{i-m-1,i-m};m)f(\pi_{i,i}-\pi_{i-m-2,i-m-1};m+1)}{f(\pi_{i,i}-\pi_{i-m,i-m};m)f(\pi_{i,i}-\pi_{i-m-2,i-m-2};m+1)}. \tag{1.17}$$

Here we use the convention that $\pi_{i,j}=0$ if $i\leq 0$ or $j\leq 0$. Further we use the following short notation. Let m and n be positive integers such that $m\leq n$. When $\rho=(\rho_m,\dots,\rho_n)$ and $\theta=(\theta_m,\dots,\theta_n)$ satisfy

$$0\leq\rho_n\leq\cdots\leq\rho_m\leq\theta_m\leq\cdots\leq\theta_n, \tag{1.18}$$

we write

$$\Phi_m^n(\rho,\theta;q,t) = \prod_{i=m+1}^{n}\frac{f(\rho_{i-1}-\rho_i;0)f(\theta_{i-1}-\rho_i;0)f(\theta_i-\rho_{i-1};0)f(\theta_i-\theta_{i-1};0)}{f(\theta_i-\rho_i;i)f(\theta_i-\rho_i;i+1)}. \tag{1.19}$$

Proposition 1.10. (1) Let α be a strict partition of length r and $P=P_2(\alpha)$ the associated shifted shape. If $\pi=(\pi_{ij})_{(i,j)\in\alpha}$ is a P-partition (1.7) satisfying the condition (1.8), then its weight $W_P(\pi;q,t)$ is given by

$$W_P(\pi;q,t) = f_\alpha^{\mathrm{D}}(\pi;q,t)\,f_\alpha^{\mathrm{ND}}(\pi;q,t). \tag{1.20}$$

(2) Let α and β be strict partitions of length 2. Let $f>0$ be a positive integer, and set $P=P_3(\alpha,\beta;f)$ to be the bird associated with α, β and f. If $\pi=(\sigma,\tau;\rho,\theta)$ is a P-partition satisfying the condition (1.9), then its weight $W_P(\pi;q,t)$ is given by

$$\begin{aligned} W_P(\pi;q,t) &= \frac{f(\sigma_{22}-\sigma_{12};0)f(\tau_{22}-\tau_{12};0)f(\rho_f;0)f(\theta_f;f+1)}{f(\sigma_{22}-\sigma_{11};0)f(\sigma_{22}-\sigma_{11};1)} \\ &\quad\times\Phi_0^f(\rho,\theta;q,t)f_\alpha^{\mathrm{ND}}(\sigma;q,t)f_\beta^{\mathrm{ND}}(\tau;q,t). \end{aligned} \tag{1.21}$$

Here we use the convention that $\sigma_{11}=\tau_{11}=\rho_0$ and $\sigma_{22}=\tau_{22}=\theta_0$.

Proposition 1.11. (1) Let α be a strict partition of length r and $P=P_2(\alpha)$ the associated shifted shape. Let n be an integer such that $n\geq\alpha_1$, and let α^c be the strict partition formed by the complement of α in $[n]$, i.e.,

$$\{\alpha_1,\dots,\alpha_r\}\cup\{\alpha_1^c,\dots,\alpha_{n-r}^c\}=[n].$$

We write $y_0=z_{0'}$ (see Proposition 1.6 (1)) hereafter. Then we have

$$F\left(z[H_p];q,t\right) = \prod_{\alpha_i^c<\alpha_j}F\left(\tilde{z}_{\alpha_i^c}^{-1}\tilde{z}_{\alpha_j};q.t\right)\prod_i F\left(\tilde{z}_{\alpha_i};q,t\right)\prod_{i<j}F\left(w\,\tilde{z}_{\alpha_i}\tilde{z}_{\alpha_j};q,t\right), \tag{1.22}$$

where $\begin{cases} w=y_0/z_0 \text{ and } \tilde{z}_i=\prod_{k=0}^{i-1}z_k\ (i=1,\dots,n). & \text{if } r \text{ is odd,}\\ w=z_0/y_0 \text{ and } \tilde{z}_i=y_0\prod_{k=1}^{i-1}z_k\ (i=1,\dots,n). & \text{if } r \text{ is even.}\end{cases}$

(2) Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions of length 2. Let $f > 0$ be a positive integer, and set $P = P_3(\alpha, \beta; f)$ the bird associated with f, α and β. Let m, n be integers such that $m \geq \ell(\alpha)$ and $n \geq \ell(\beta)$, and let α^c (resp. β^c) be the strict partition formed by the complement of α (resp. β) in $[m]$ (resp. $[n]$). We write $y_i = z_{i'}$ for $i = 1, \dots, \beta_1 - 1$ and $x_i = z_{-i}$ for $i = 1, \dots, f$. Further we may write $x_0 = y_0 = z_0$. (See Proposition 1.6 (2)). Then we have

$$F(z[H_p]; q, t) = \prod_{\alpha_i^c < \alpha_j} F\left(\widetilde{z}_{\alpha_i^c}^{-1} \widetilde{z}_{\alpha_j}; q.t\right) \prod_{\beta_i^c < \beta_j} F\left(\widetilde{y}_{\beta_i^c}^{-1} \widetilde{y}_{\beta_j}; q.t\right) \prod_{i=1}^{f} F\left(\widetilde{x}_i; q.t\right)$$
$$\times \prod_{i=1}^{f} F\left(\frac{\widetilde{x}_0^2}{\widetilde{x}_i} \prod_{k,l=1}^{2} \widetilde{y}_l \widetilde{z}_k; q.t\right) \prod_{i,j=1}^{2} F\left(\widetilde{x}_0 \widetilde{y}_{\beta_j} \widetilde{z}_{\alpha_i}; q, t\right) \tag{1.23}$$

where $\widetilde{x}_i = \prod_{k=i}^{f} x_i$ for $i = 0, \dots, f$, $\widetilde{y}_i = \prod_{k=1}^{i-1} y_k$ for $i = 1, \dots, n$, and $\widetilde{z}_i = \prod_{k=1}^{i-1} z_k$ for $i = 1, \dots, m$.

2 Macdonald polynomials

We follow the notation and terminology of [7] for the symmetric functions. If λ and μ are partitions then $\mu \subseteq \lambda$ if μ is contained in λ, i.e., $\mu_i \leq \lambda_i$ for all $i \geq 1$. If $\mu \subseteq \lambda$ then the skew-diagram λ/μ denotes the set-theoretic difference between λ and μ, i.e., those squares of λ not contained in μ. The skew diagram λ/μ is a vertical r-strip if $|\lambda - \mu| = |\lambda| - |\mu| = r$ and if, for all $i \geq 1$, $\lambda_i \geq \mu_i$ is at most one, i.e., each row of $\lambda - \mu$ contains at most one square. The set of all vertical r-strips is denoted by $\mathscr{V}_r$ and the set of all vertical strips by $\mathscr{V} = \biguplus_{r=0}^{\infty} \mathscr{V}_r$. The skew diagram λ/μ is a horizontal r-strip if $|\lambda - \mu| = r$ and if, for all $i \geq 1$, $\lambda'_i - \mu'_i$ is at most one, i.e., each column of $\lambda - \mu$ contains at most one square. For two partitions λ and μ, we write $\lambda \succ \mu$ if $\lambda \supset \mu$ and λ/μ is a horizontal strip. Note that λ/μ is a horizontal strip if and only if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$. The set of all horizontal r-strips is denoted by $\mathscr{H}_r$ and the set of all horizontal strips by $\mathscr{H}$. Let $s = (i, j)$ be a square in the diagram of λ, and let $a(s)$ and $l(s)$ be the arm-length and leg-length of s, given by

$$a(s) = \lambda_i - j, \qquad l(s) = \lambda'_j - i$$

Then we define the rational functions let

$$b_\lambda(s) = b_\lambda(s; q, t) := \begin{cases} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}, & \text{if } s \in \lambda, \\ 1, & \text{otherwise,} \end{cases}$$

and [6, (3.6)] [7, VI.7 (6.19), VI.7 Ex.4]

$$b_\lambda(q, t) := \prod_{s \in \lambda} b_\lambda(s; q, t) = \prod_{i \geq 1} \prod_{m \geq 0} \frac{f_{q,t}(\lambda_i - \lambda_{i+m+1}; m)}{f_{q,t}(\lambda_i - \lambda_{i+m}; m)}, \tag{2.1}$$

$$b_\lambda^{\mathrm{el}}(q, t) := \prod_{\substack{s \in \lambda \\ l(s) \text{ even}}} b_\lambda(s; q, t) = \prod_{i \geq 1} \prod_{\substack{m \geq 0 \\ m \text{ even}}} \frac{f_{q,t}(\lambda_i - \lambda_{i+m+1}; m)}{f_{q,t}(\lambda_i - \lambda_{i+m}; m)}, \tag{2.2}$$

$$b_\lambda^{\mathrm{oa}}(q, t) := \prod_{\substack{s \in \lambda \\ a(s) \text{ odd}}} b_\lambda(s; q, t). \tag{2.3}$$

If $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ are two sequences of independent indeterminates, then we write

$$\Pi(x; y; q, t) = \prod_{i,j} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} = \prod_{i,j} F(x_i y_j; q, t). \tag{2.4}$$

Let $\mathfrak{S}_n$ denote the symmetric group, acting on $x = (x_1, \dots, x_n)$ by permuting the x_i, and let $\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ and Λ denote the ring of symmetric polynomials in n independent variables and the ring of symmetric polynomials in countably many variables, respectively. For $\lambda = (\lambda_1, \dots, \lambda_n)$ a partition of at most n parts the monomial symmetric function m_λ is defined as

$$m_\lambda(x) = \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations α of λ, and $x = (x_1, \dots, x_n)$. For $\ell(\lambda) > n$ we set $m_\lambda(x) = 0$. The monomial symmetric functions $m_\lambda(x)$ for $\ell(\lambda) \leq n$ form a $\mathbb{Z}$-basis of Λ_n. For r a nonnegative integer the power sums p_r are given by $p_0 = 1$ and $p_r = m_{(r)}$ for $r > 1$. More generally the power-sum products are defined as $p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x)\cdots$ for an arbitrary partition $\lambda = (\lambda_1, \lambda_2, \dots)$. Define the Macdonald scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on the ring of symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda\mu} z_\lambda \prod_i \prod_{i=1}^{n} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

with $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ and $m_i = m_i(\lambda)$. If we denote the ring of symmetric functions in Λ_n variables over the field $\mathbb{F} = \mathbb{Q}(q,t)$ of rational functions in q and t by $\Lambda_{n,\mathbb{F}}$, then the Macdonald polynomial $P_\lambda(x) = P_\lambda(x;q,t)$ is the unique symmetric polynomial in $\Lambda_{n,\mathbb{F}}$ such that [VI (4.7)]Mac:

$$P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda\mu}(q,t) m_\mu(x)$$

with $u_{\lambda\lambda} = 1$ and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \qquad \text{if } \lambda \neq \mu.$$

The Macdonald polynomials $P_\lambda(x;q,t)$ with $\ell(\lambda) \leq n$ form an $\mathbb{F}$-basis of $\Lambda_{n,\mathbb{F}}$. If $\ell(\lambda) > n$ then $P_\lambda(x;q,t) = 0$. $P_\lambda(x;q,t)$ is called *Macdonald's P-function*. Since $P_\lambda(x_1, \dots, x_n, 0; q, t) = P_\lambda(x_1, \dots, x_n; q, t)$ one can extend the Macdonald polynomials to symmetric functions containing an infinite number of independent variables $x = (x_1, x_2, \dots)$, to obtain a basis of $\mathbb{F} = \Lambda \otimes \mathbb{F}$. A second Macdonald symmetric function, called *Macdonald's Q-function*, is defined as

$$Q_\lambda(x;q,t) = b_\lambda(q,t) P_\lambda(x;q,t). \tag{2.5}$$

The normalization of the Macdonald inner product is then $\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda\mu}$ for all λ, μ, which is equivalent to

$$\sum_\lambda P_\lambda(x;q,t) Q_\lambda(y;q,t) = \Pi(x;y;q,t). \tag{2.6}$$

(See [7, VI.4, (4.13)].) Let $g_r(x;q,t) := Q_{(r)}(x;q,t)$, or equivalently, [7, VI.2, (2.8)]

$$\prod_{i=1}^{\infty} \frac{(tx_i y; q)_\infty}{(x_i y; q)_\infty} = \sum_{r=0}^{\infty} g_r(x;q,t) y^r.$$

Then the *Pieri coefficients* $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ are given by [7, VI.6, (6.24)]

$$P_\mu(x;q,t) g_r(x;q,t) = \sum_{\substack{\lambda \\ \lambda - \mu \in \mathscr{H}_r}} \phi_{\lambda/\mu}(q,t) P_\lambda(x;q,t),$$

$$Q_\mu(x;q,t) g_r(x;q,t) = \sum_{\substack{\lambda \\ \lambda - \mu \in \mathscr{H}_r}} \psi_{\lambda/\mu}(q,t) Q_\lambda(x;q,t).$$

Another direct expressions for $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ is given in [7, VI.6, Ex.2] as

$$\phi_{\lambda/\mu}(q,t) = \prod_{1 \leq i \leq j \leq \ell(\lambda)} \frac{f(\lambda_i - \mu_j, j-i) f(\mu_i - \lambda_{j+1}, j-i)}{f(\lambda_i - \lambda_j, j-i) f(\mu_i - \mu_{j+1}, j-i)}, \tag{2.7}$$

$$\psi_{\lambda/\mu}(q,t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f(\lambda_i - \mu_j, j-i) f(\mu_i - \lambda_{j+1}, j-i)}{f(\mu_i - \mu_j, j-i) f(\lambda_i - \lambda_{j+1}, j-i)}. \tag{2.8}$$

Here we use these expressions to rewrite Okada's (q,t)-hook formula conjectures by the Pieri coefficients. For any three partitions λ, μ, ν let $f^{\lambda}_{\mu\nu}$ be the coefficient P_λ in the product $P_\mu P_\nu$: [7, VI (7.1')]:

$$P_\mu(x;q,t)P_\mu(x;q,t)=\sum_\lambda f^{\lambda}_{\mu\nu}P_\lambda(x;q,t) \tag{2.9}$$

Now let λ,μ be partitions and define $Q_{\lambda/\mu}\in\Lambda_{\mathbb{F}}$ by

$$Q_{\lambda/\mu}(x;q,t)=\sum_\nu f^{\lambda}_{\mu\nu}Q_\nu(x;q,t). \tag{2.10}$$

Then $Q_{\lambda/\mu}(x;q,t)=0$ unless $\lambda\supset\mu$, and $Q_{\lambda/\mu}$ is homogeneous of degree $|\lambda|-|\mu|$, which is called *Macdonald's skew Q-function.* We define *Macdonald's skew P-function* $P_{\lambda/\mu}$ as

$$Q_{\lambda/\mu}(x;q,t)=\frac{b_\lambda(q,t)}{b_\lambda(q,t)}P_{\lambda/\mu}(x;q,t). \tag{2.11}$$

holds. Let T be a tableau of shape $\lambda-\mu$ and weight ν, thought as a sequence of partitions $(\lambda^{(0)},\dots,\lambda^{(r)})$ such that

$$\mu=\lambda^{(0)}\subset\lambda^{(1)}\subset\cdots\subset\lambda^{(r)}=\lambda$$

and such that each $\lambda^{(i)}-\lambda^{(i-1)}$ is a horizontal strip. Let

$$\phi_T(q,t)=\prod_{i=1}^{r}\phi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t),$$

$$\psi_T(q,t)=\prod_{i=1}^{r}\psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t).$$

Then we have [7, VI, (7.13), (7.13')]

$$Q_{\lambda/\mu}(x;q,t)=\sum_T\phi_T(q,t)x^T,$$

$$P_{\lambda/\mu}(x;q,t)=\sum_T\psi_T(q,t)x^T,$$

summed over tableaux T of shape $\lambda-\mu$, where $x^T=\prod_{i=1}^{r}x_i^{|\lambda^{(i)}-\lambda^{(i-1)}|}$. It also holds [7, VI.7, (7.9) (7.9')]

$$Q_\lambda(x,z;q,t)=\sum_\mu Q_{\lambda/\mu}(x,z;q,t)Q_\mu(x,z;q,t), \tag{2.12}$$

$$P_\lambda(x,z;q,t)=\sum_\mu P_{\lambda/\mu}(x,z;q,t)P_\mu(x,z;q,t), \tag{2.13}$$

where the sums on the right are over partitions $\mu\subset\lambda$. The following lemma has appeared in the proof of [13, Proposition 2.2] (also see [7, I.5, Ex.26] and [14, Proposition 5.1]).

Lemma 2.1. Let μ and ν be partitions, and $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$ are independent indeterminates.

$$\sum_\lambda Q_{\lambda/\mu}(x;q,t)P_{\lambda/\nu}(y;q,t)=\Pi(x;y;q,t)\sum_\tau Q_{\nu/\tau}(x;q,t)P_{\mu/\tau}(y;q,t) \tag{2.14}$$

In [13] Vuletić has presented so-called a generalized MacMahon's formula. The following theorem gives a generalized form of [13, Proposition 2.2], which we use in the proof of Okada's conjecture.

Theorem 2.2. Fix a positive integer T and two partitions μ^0 and μ^T. Let $x^0,\dots,x^{T-1}$, $y^1,\dots,y^T$ be sets of variables. Then we have

$$\sum_{(\lambda^1,\mu^1,\lambda^2,\dots,\lambda^T)}\prod_{i=1}^{T} Q_{\lambda^i/\mu^{i-1}}(x^{i-1};q,t)P_{\lambda^i/\mu^i}(y^i;q,t)$$
$$=\prod_{0\le i<j\le T}\Pi(x^i,y^j;q,t)\sum_{\nu}Q_{\mu^T/\nu}(x^0,\dots,x^{T-1};q,t)P_{\mu^0/\nu}(y^1,\dots,y^T;q,t) \tag{2.15}$$

where the sum runs over $(2T-1)$-tuples $(\lambda^1,\mu^1,\lambda^2,\dots,\mu^{T-1},\lambda^T)$ of partitions satisfying

$$\mu^0\subset\lambda^1\supset\mu^1\subset\lambda^2\supset\mu^2\subset\cdots\supset\mu^{T-1}\subset\lambda^T\supset\mu^T. \tag{2.16}$$

We define $P^{\delta}_{[\lambda,\mu]}(x;q,t)$ and $Q^{\delta}_{[\lambda,\mu]}(x;q,t)$ for a pair (λ,μ) of partitions, a set $x=(x_1,x_2,\dots)$ of independent variables and $\delta=\pm1$ by

$$P^{\delta}_{[\lambda,\mu]}(x;q,t)=\begin{cases}P_{\lambda/\mu}(x;q,t) & \text{if } \delta=+1,\\ Q_{\mu/\lambda}(x;q,t) & \text{if } \delta=-1,\end{cases}\quad Q^{\delta}_{[\lambda,\mu]}(q,t)=\begin{cases}Q_{\lambda/\mu}(x;q,t) & \text{if } \delta=+1,\\ P_{\mu/\lambda}(x;q,t) & \text{if } \delta=-1.\end{cases}$$

Here we assume $\lambda\supset\mu$ if $\delta=+1$, and $\lambda\subset\mu$ if $\delta=-1$.

Corollary 2.3. Let n be a positive integer, and $\epsilon=(\epsilon_1,\dots,\epsilon_n)$ a sequence of ±1, Fix a positive integer T and two partitions λ^0 and λ^n. Let $x^1,\dots,x^n$ be sets of variables. Then we have

$$\sum_{(\lambda^1,\lambda^2,\dots,\lambda^{n-1})}\prod_{i=1}^{n}P^{\epsilon_i}_{[\lambda^{i-1},\lambda^i]}(x^i;q,t)$$
$$=\prod_{\substack{i<j\\(\epsilon_i,\epsilon_j)=(-1,+1)}}\Pi(x^i;x^j;q,t)\sum_{\nu}Q_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i=-1};q,t)P_{\lambda^0/\nu}(\{x^i\}_{\epsilon_i=+1};q,t), \tag{2.17}$$

$$\sum_{(\lambda^1,\lambda^2,\dots,\lambda^{n-1})}\prod_{i=1}^{n}Q^{\epsilon_i}_{[\lambda^{i-1},\lambda^i]}(x^i;q,t)$$
$$=\prod_{\substack{i<j\\(\epsilon_i,\epsilon_j)=(-1,+1)}}\Pi(x^i;x^j;q,t)\sum_{\nu}P_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i=-1};q,t)Q_{\lambda^0/\nu}(\{x^i\}_{\epsilon_i=+1};q,t), \tag{2.18}$$

where the sum runs over $(n-1)$-tuples $(\lambda^1,\lambda^2,\dots,\lambda^{n-1})$ of partitions satisfying

$$\begin{cases}\lambda^{i-1}\supset\lambda^i & \text{if } \epsilon_i=+1,\\ \lambda^{i-1}\subset\lambda^i & \text{if } \epsilon_i=-1.\end{cases} \tag{2.19}$$

Theorem 2.4. (Warnaar [15, Proposition 1.3, (1.17)])

$$\sum_{\lambda}w^{r(\lambda)}b^{\mathrm{oa}}_{\lambda}(q,t)P_{\lambda}(x;q,t)=\prod_{i\ge1}\frac{(1+wx_i)(qtx_i^2;q^2)_\infty}{(x_i^2;q^2)_\infty}\prod_{i<j}\frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}, \tag{2.20}$$

where $r(\lambda)$ is the number of rows of odd length.

Applying $w_{q,t}$ [7, VI.2, (2.14)] to the both sides of (2.20), we obtain

Corollary 2.5.

$$\sum_{\lambda}w^{r(\lambda')}b^{\mathrm{el}}_{\lambda}(q,t)P_{\lambda}(x;q,t)=\prod_{i\ge1}\frac{(twx_i;q)_\infty}{(wx_i;q)_\infty}\prod_{i<j}\frac{(tx_ix_j;q)_\infty}{(x_ix_j;q)_\infty}. \tag{2.21}$$

From (2.21), we easily obtain

$$\sum_{\lambda} w^{\frac{|\lambda|+r(\lambda')}{2}} b^{\mathrm{el}}_{\lambda}(q,t)P_{\lambda}(x;q,t) = \prod_{i\geq 1} \frac{(twx_i;q)_\infty}{(wx_i;q)_\infty} \prod_{i<j} \frac{(twx_ix_j;q)_\infty}{(wx_ix_j;q)_\infty}, \tag{2.22}$$

and

$$\sum_{\lambda} w^{\frac{|\lambda|-r(\lambda')}{2}} b^{\mathrm{el}}_{\lambda}(q,t)P_{\lambda}(x;q,t) = \prod_{i\geq 1} \frac{(tx_i;q)_\infty}{(x_i;q)_\infty} \prod_{i<j} \frac{(twx_ix_j;q)_\infty}{(wx_ix_j;q)_\infty}. \tag{2.23}$$

3 (q,t)-hook formula and Macdonald polynomials

We define $\phi^{\delta}_{[\lambda,\mu]}(q,t)$ and $\psi^{\delta}_{[\lambda,\mu]}(q,t)$ for a pair (λ,μ) of partitions and $\delta=\pm 1$ by

$$\phi^{\delta}_{[\lambda,\mu]}(q,t) = \begin{cases} \phi_{\lambda/\mu}(q,t) & \text{if } \delta=+1, \\ \psi_{\mu/\lambda}(q,t) & \text{if } \delta=-1, \end{cases} \qquad \psi^{\delta}_{[\lambda,\mu]}(q,t) = \begin{cases} \psi_{\lambda/\mu}(q,t) & \text{if } \delta=+1, \\ \phi_{\mu/\lambda}(q,t) & \text{if } \delta=-1. \end{cases}$$

Here we assume $\lambda \succ \mu$ if $\delta=+1$, and $\lambda \prec \mu$ if $\delta=-1$. We also write

$$|\lambda-\mu|_\delta = \begin{cases} |\lambda-\mu| & \text{if } \delta=+1, \\ |\mu-\lambda| & \text{if } \delta=-1. \end{cases}$$

Let n be a positive integer. Let $\epsilon=(\epsilon_1,\dots,\epsilon_n)$ be a sequence of ± 1. Let $(\lambda^0,\lambda^1,\dots,\lambda^n)$ be an $(n+1)$-tuple of partitions such that $\lambda^{i-1}\succ\lambda^i$ if $\epsilon=+1$, and $\lambda^{i-1}\prec\lambda^i$ if $\epsilon=-1$. Then we write

$$\phi^{\epsilon}_{[\lambda^0,\lambda^1,\dots,\lambda^n]}(q,t) = \prod_{i=1}^{n} \phi^{\epsilon_i}_{[\lambda^{i-1},\lambda^i]}(q,t), \qquad \psi^{\epsilon}_{[\lambda^0,\lambda^1,\dots,\lambda^n]}(q,t) = \prod_{i=1}^{n} \psi^{\epsilon_i}_{[\lambda^{i-1},\lambda^i]}(q,t).$$

Let α be a strict partition, and let n be an integer such that $n\geq\alpha_1$. Define a sequence $\epsilon=\epsilon_n(\alpha)=(\epsilon_1,\dots,\epsilon_n)$ of ± 1 by putting

$$\epsilon_k(\alpha) = \begin{cases} +1 & \text{if } k \text{ is a part of } \alpha, \\ -1 & \text{if } k \text{ is not a part of } \alpha. \end{cases}$$

For example, if $\alpha=(8,5,2,1)$ and $n=10$, then we have $\epsilon=(++--+--+--)$. Let $\pi\in\mathscr{A}(P)$ a P-partition for the the shifted shape $P=P_2(\alpha)$. For each integer $k=0,\dots,n$ we define the kth trace $\pi[k]$ to be the sequence $(\dots,\pi_{2,k+2},\pi_{1,k+1})$ obtained by reading the kth diagonal from SE to NW. Here we use the convention that $\pi[k]=\emptyset$ if $k\geq\alpha_1$. For example, if π is the P-partition of shifted shape $\alpha=(8,5,2,1)$ in Figure 1, then we have $\pi[0]=(\pi_{44},\pi_{33},\pi_{22},\pi_{11})$, $\pi[1]=(\pi_{34},\pi_{23},\pi_{12})$, $\pi[2]=(\pi_{24},\pi_{13})$, $\pi[3]=(\pi_{25},\pi_{14})$, $\pi[4]=(\pi_{26},\pi_{15})$, $\pi[5]=(\pi_{16})$, $\pi[6]=(\pi_{17})$, $\pi[7]=(\pi_{18})$, $\pi[8]=\pi[9]=\pi[10]=\emptyset$, and

$$\pi[0]\succ\pi[1]\succ\pi[2]\prec\pi[3]\prec\pi[4]\succ\pi[5]\prec\pi[6]\prec\pi[7]\succ\pi[8]\prec\pi[9]\prec\pi[10].$$

By direct computation one can easily check

$$\begin{aligned} W_P(\pi;q,t) &= b^{\mathrm{el}}_{\pi[0]}(q,t)\psi^{\epsilon(\alpha)}_{[\pi[0],\dots,\pi[10]]}(q,t) = b^{\mathrm{el}}_{\pi[0]}\psi_{\pi[0]/\pi[1]}\psi_{\pi[1]/\pi[2]}\phi_{\pi[3]/\pi[2]} \\ &\quad\times \phi_{\pi[4]/\pi[3]}\psi_{\pi[4]/\pi[5]}\phi_{\pi[6]/\pi[5]}\phi_{\pi[7]/\pi[6]}\psi_{\pi[7]/\pi[8]}\phi_{\pi[9]/\pi[8]}\phi_{\pi[10]/\pi[9]}. \end{aligned}$$

In the following we write

$$\widehat{\Phi}^n_m(\rho,\theta;q,t) = \frac{f(\rho_n,0)f(\theta_n,n+1)}{f(\rho_m,0)(\theta_m,m+1)}\,\Phi^n_m(\rho,\theta;q,t),$$

$$\widetilde{\Phi}^n_m(\widetilde{x};\rho,\theta;q,t) = \widehat{\Phi}^n_m(\rho,\theta;q,t)\prod_{i=m+1}^{n} \widetilde{x}_i^{\rho_i+\theta_i-\rho_{i-1}-\theta_{i-1}}$$

in short, where $\rho = (\rho_m, \ldots, \rho_n)$ and $\theta = (\theta_m, \ldots, \theta_n)$ satisfy (1.18), and $\widetilde{x} = (\widetilde{x}_m, \ldots, \widetilde{x}_n)$ are indeterminates. For example, if $\pi = (\sigma, \tau; f)$ is the P-partition of the bird $P = P_3(\alpha, \beta; f)$ for $\alpha = (4,3)$, $\beta = (4,2)$ and $f = 2$ (see Figure 1) and satisfies (1.9), then we have

$$W_P(\pi; q, t) = \widehat{\Phi}_0^2(\rho, \theta; q, t)\psi^{\epsilon(\alpha)}_{[\sigma[0], \ldots, \sigma[4]]}(q,t)\phi^{\epsilon(\beta)}_{[\tau[0], \ldots, \tau[4]]}(q,t).$$

Proposition 3.1. (1) Let $P = P_2(\alpha)$ be the shifted shape associated with a strict partition α such that $\ell(\alpha) = r$, and let n be an integer such that $n \geq \alpha_1$. If $\pi \in \mathscr{A}(P)$ is a P-partition satisfying the condition (1.8), then we have

$$W_P(\pi; q, t) = b^{\mathrm{el}}_{\pi[0]}(q,t)\psi^{\epsilon(\alpha)}_{[\pi[0], \ldots, \pi[n]]}(q,t) = \frac{b^{\mathrm{el}}_{\pi[0]}(q,t)}{b_{\pi[0]}(q,t)}\phi^{\epsilon(\alpha)}_{[\pi[0], \ldots, \pi[n]]}(q,t) \tag{3.1}$$

and

$$z^\pi = w^{\frac{|\pi[0]| - r(\pi[0]')}{2}} \prod_{i=1}^{n} \widetilde{z}_i^{\,\epsilon_i(\alpha)\,|\pi[i-1]-\pi[i]|_{\epsilon_i(\alpha)}}, \tag{3.2}$$

where w and $\widetilde{z}_i$ $(1 \leq i \leq n)$ are as in Proposition 1.11 (1).

(2) Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that $\ell(\alpha) = \ell(\beta) = 2$. Let $f > 0$ be a positive integer, and set $P = P_3(\alpha, \beta; f)$ the bird associated with α, β and f. Let m (resp. n) be a positive integer such that $m \geq \alpha_1$ (resp. $n \geq \beta_1$). If $\pi = (\sigma, \tau; \rho, \theta)$ is a P-partition satisfying the condition (1.9), then we have

$$W_P(\pi; q, t) = \widehat{\Phi}_0^f(\rho, \theta; q, t)\, \psi^{\epsilon(\alpha)}_{[\sigma[0], \ldots, \sigma[m]]}(q,t)\, \phi^{\epsilon(\beta)}_{[\tau[0], \ldots, \tau[n]]}(q,t) \tag{3.3}$$

and

$$z^\pi = \widetilde{x}_0^{\rho_0 + \theta_0} \prod_{i=1}^{m} \widetilde{z}_i^{\,\epsilon_i(\alpha)\,|\sigma[i-1]-\sigma[i]|_{\epsilon_i(\alpha)}} \prod_{i=1}^{n} \widetilde{y}_i^{\,\epsilon_i(\beta)\,|\tau[i-1]-\tau[i]|_{\epsilon_i(\beta)}} \prod_{i=1}^{f} \widetilde{x}_i^{\rho_i + \theta_i - \rho_{i-1} - \theta_{i-1}}, \tag{3.4}$$

where $\widetilde{x}_i$ $(0 \leq i \leq f)$, $\widetilde{y}_i$ $(1 \leq i \leq n)$ and $\widetilde{z}_i$ $(1 \leq i \leq m)$ are as in Proposition 1.11 (2).

Proof. (1) From (1.16) and (2.8) we have

$$f^{\mathrm{ND}}_\alpha(\pi; q, t) = \begin{cases} \prod_{1 \leq i \leq j} \frac{f(\pi[1]_i - \pi[0]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i)} \prod_{i=2}^{n} \psi^{\epsilon_i(\alpha)}_{[\pi[i-1], \pi[i]]}(q,t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} \frac{f(\pi[1]_i - \pi[0]_j; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i)} \prod_{i=2}^{n} \psi^{\epsilon_i(\alpha)}_{[\pi[i-1], \pi[i]]}(q,t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}$$

Similarly, from (1.17) and (2.2) we have

$$f^{\mathrm{D}}_\alpha(\pi; q, t) = \begin{cases} \prod_{1 \leq i \leq j} \frac{f(\pi[0]_i - \pi[1]_j; j-i)}{f(\pi[0]_i - \pi[0]_{j+1}; j-i)} b^{\mathrm{el}}_{\pi[0]}(q,t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} \frac{f(\pi[0]_i - \pi[1]_{j+1}; j-i)}{f(\pi[0]_i - \pi[0]_{j+1}; j-i)} b^{\mathrm{el}}_{\pi[0]}(q,t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}$$

Hence we obtain (3.1) from (1.20) since

$$\psi^{\epsilon_1(\alpha)}_{[\pi[0], \pi[1]]}(q,t) = \begin{cases} \prod_{1 \leq i \leq j} \frac{f(\pi[0]_i - \pi[1]_j; j-i) f(\pi[1]_i - \pi[0]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i) f(\pi[0]_i - \pi[0]_{j+1}; j-i)} & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} \frac{f(\pi[1]_i - \pi[0]_j; j-i) f(\pi[0]_i - \pi[1]_{j+1}; j-i)}{f(\pi[1]_i - \pi[1]_j; j-i) f(\pi[0]_i - \pi[0]_{j+1}; j-i)} & \text{if } \epsilon_1(\alpha) = -. \end{cases}$$

Meanwhile, (3.2) can be easily obtained from

$$z^\pi = w^{\pi_{r-1,r-1} + \pi_{r-3,r-3} + \cdots} \prod_{i=1}^{n} \widetilde{z}_i^{\,|\pi[i-1]| - |\pi[i]|}.$$

(2) As in (1) we have

$$f_\alpha^{\mathrm{ND}}(\sigma;q,t) = \begin{cases} f(\sigma_{12}-\sigma_{11};0)\prod_{i=2}^n \psi^{\epsilon_i(\alpha)}_{[\sigma[i-1],\sigma[i]]}(q,t) & \text{if } \epsilon_1(\alpha)=+, \\ \frac{f(\sigma_{23}-\sigma_{22};0)f(\sigma_{23}-\sigma_{11};1)f(\sigma_{12}-\sigma_{11};0)}{f(\sigma_{23}-\sigma_{12};1)}\prod_{i=2}^n \psi^{\epsilon_i(\alpha)}_{[\sigma[i-1],\sigma[i]]}(q,t) & \text{if } \epsilon_1(\alpha)=-. \end{cases}$$

From (1.16) and (2.7) we have

$$f_\beta^{\mathrm{ND}}(\tau;q,t) = \begin{cases} f(\tau_{12}-\tau_{11};0)\prod_{i=2}^n \phi^{\epsilon_i(\beta)}_{[\tau[i-1],\tau[i]]}(q,t) & \text{if } \epsilon_1(\beta)=+, \\ \frac{f(\tau_{23}-\tau_{22};0)f(\tau_{23}-\tau_{11};1)f(\tau_{12}-\tau_{11};0)}{f(\tau_{23}-\tau_{12};0)}\prod_{i=2}^n \phi^{\epsilon_i(\beta)}_{[\tau[i-1],\tau[i]]}(q,t) & \text{if } \epsilon_1(\beta)=+. \end{cases}$$

Hence, if we use (2.7) or (2.8), then we obtain (3.3) from (1.21). On the other hand, (3.4) is easily obtained from

$$z^\pi = z_0^{\sigma_{11}+\sigma_{22}}\prod_{i=1}^f x_i^{\rho_i+\theta_i}\prod_{i=1}^m \widetilde{z}_i^{\,\epsilon_i(\alpha)\,|\sigma[i-1]-\sigma[i]|_{\epsilon_i(\alpha)}}\prod_{i=1}^n \widetilde{y}_i^{\,\epsilon_i(\beta)\,|\tau[i-1]-\tau[i]|_{\epsilon_i(\beta)}}$$

using $z_0^{\sigma_{11}+\sigma_{22}}\prod_{i=0}^f x_i^{\rho_i+\theta_i} = (z_0\widetilde{x}_1)^{\rho_0+\theta_0}\prod_{i=1}^f \widetilde{x}_i^{\,\rho_i+\theta_i-\rho_{i-1}-\theta_{i-1}}$, where we use the convention $\sigma_{11}=\rho_0$ and $\sigma_{22}=\theta_0$.

□

Theorem 3.2. (1) Let $P=P_2(\alpha)$ be the shifted shape associated with a strict partition α of length r. Let n be an integer such that $n\geq\alpha_1$, and let α^c be the strict partition formed by the complement of α in $[n]$. Then we have

$$\sum_{\pi\in\mathscr{A}(P)} W_P(\pi;q,t)\,z^\pi = \prod_{\alpha_k^c<\alpha_l} F\left(\widetilde{z}_{\alpha_k^c}^{-1}\widetilde{z}_{\alpha_l}\right)\sum_\lambda w^{\frac{|\lambda|-r(\lambda')}{2}} b_\lambda^{\mathrm{el}}(q,t)P_\lambda\left(\widetilde{z}_{\alpha_1}\dots,\widetilde{z}_{\alpha_r};q,t\right), \quad (3.5)$$

where w and $\widetilde{z}_i$ $(i=1,\dots,n)$ are as in Proposition 1.11 (1).

(2) Let $\alpha=(\alpha_1,\alpha_2)$ and $\beta=(\beta_1,\beta_2)$ be strict partitions such that $\ell(\alpha)=\ell(\beta)=2$. Let $f>0$ be a positive integer, and set $P=P_3(\alpha,\beta;f)$ to be the bird associated with α, β and f. Let m (resp. n) be a positive integer such that $m\geq\alpha_1$ (resp. $n\geq\beta_1$). If $\pi=(\sigma,\tau;\rho,\theta)$ is a P-partition satisfying the condition (1.9), then we have

$$\sum_{\pi\in\mathscr{A}(P)} W_P(\pi;q,t)z^\pi = \prod_{\alpha_i^c<\alpha_j} F\left(\widetilde{z}_{\alpha_i^c}^{-1}\widetilde{z}_{\alpha_j};q.t\right)\prod_{\beta_i^c<\beta_j} F\left(\widetilde{y}_{\beta_i^c}^{-1}\widetilde{y}_{\beta_j};q.t\right)$$
$$\times\sum_{(\rho,\theta)}\widetilde{\Phi}_0^f(\widetilde{x};\rho,\theta;q,t)P_{(\theta_0,\rho_0)}(\widetilde{x}_0\widetilde{z}_{\alpha_1},\widetilde{x}_0\widetilde{z}_{\alpha_2};q,t)Q_{(\theta_0,\rho_0)}(\widetilde{y}_{\beta_1},\widetilde{z}_{\beta_2};q,t). \quad (3.6)$$

where the sum on the right-hand side is taken over all pairs (ρ,θ) with $\rho=(\rho_0,\dots,\rho_f)$ and $\theta=(\theta_0,\dots,\theta_f)$ satisfying

$$0\leq\rho_f\leq\dots\leq\rho_0\leq\theta_0\leq\dots\leq\theta_f. \quad (3.7)$$

Here $\widetilde{x}_i$ $(0\leq i\leq f)$, y_i $(1\leq i\leq n)$ and z_i $(1\leq i\leq m)$ are as in Proposition 1.11 (2).

Proof. (1) Since

$$\psi_{\pi[i-1]/\pi[i]}(q,t)\widetilde{z}_i^{\,|\pi[i-1]-\pi[i]|} = P_{\pi[i-1]/\pi[i]}(\widetilde{z}_i;q,t),$$
$$\phi_{\pi[i]/\pi[i-1]}(q,t)\widetilde{z}_i^{\,-|\pi[i-1]-\pi[i]|} = Q_{\pi[i]/\pi[i-1]}(\widetilde{z}_i^{-1};q,t)$$

(see [7, VI.7, (7.14)(7.14')]), we can use (2.17) to take the sum of the product of (3.1) and (3.2), then we obtain

$$\sum_{\pi} W_P(\pi;q,t)z^{\pi} = \prod_{\alpha_k^c<\alpha_l} F\left(\widetilde{z}_{\alpha_k^c}^{-1}\widetilde{z}_{\alpha_l}\right) \sum_{\pi[0]} b^{\mathrm{el}}_{\pi[0]}(q,t)w^{(|\pi[0]|-r(\pi[0]'))/2}P_{\pi[0]}(\widetilde{z}_{\alpha_1},\dots,\widetilde{z}_{\alpha_r};q,t),$$

where the sum on the right-hand side runs over all partitions $\pi[0]$.

(2) Again, using (2.17) to take the sum of the product of (3.3) and (3.4), we obtain

$$\sum_{\pi} W_P(\pi;q,t)z^{\pi} = \prod_{\alpha_k^c<\alpha_l} F\left(\widetilde{z}_{\alpha_k^c}^{-1}\widetilde{z}_{\alpha_l}\right) \prod_{\beta_k^c<\beta_l} F\left(\widetilde{y}_{\beta_k^c}^{-1}\widetilde{y}_{\beta_l}\right)\widetilde{x}_0^{\rho_0+\theta_0}$$
$$\times \sum_{(\rho,\theta)} \widetilde{\Phi}_0^f(\rho,\theta)\widetilde{x}_0^{\rho_0+\theta_0}P_{\sigma[0]}(\widetilde{z}_{\alpha_1},\widetilde{z}_{\alpha_2};q,t)Q_{\tau[0]}(\widetilde{y}_{\beta_1},\widetilde{y}_{\beta_2};q,t),$$

where the sum on the right-hand side runs over all pairs (ρ,θ) satisfying (3.7) with $\sigma[0]=\tau[0]=(\theta_0,\rho_0)$. Finally we use $\widetilde{x}_0^{\rho_0+\theta_0}P_{(\theta_0,\rho_0)}(\widetilde{z}_{\alpha_1},\widetilde{z}_{\alpha_2};q,t)=P_{(\theta_0,\rho_0)}(\widetilde{x}_0\widetilde{z}_{\alpha_1},\widetilde{x}_0\widetilde{z}_{\alpha_2};q,t)$. □

If we apply Warner's formula (2.23) to (3.5) we can obtain the (q,t)-hook formula (1.22) for shifted shapes. This gives another proof of [8, Proposition 4.5 (b)]. Now we look at the right-hand side of the conjectured identies in the cases of birds. From Proposition 1.11 we can derive the following theorem.

Theorem 3.3. Let $\alpha=(\alpha_1,\alpha_2)$ and $\beta=(\beta_1,\beta_2)$ be strict partitions of length 2. Let $f>0$ be a positive integer, and set $P=P_3(\alpha,\beta;f)$ the bird associated with f, α and β. Let m,n be integers such that $m\geq\ell(\alpha)$ and $n\geq\ell(\beta)$, and let α^c (resp. β^c) be the strict partition formed by the complement of α (resp. β) in $[m]$ (resp. $[n]$). Then we have

$$F(z[H_p];q,t) = \prod_{\alpha_i^c<\alpha_j} F\left(\widetilde{z}_{\alpha_i^c}^{-1}\widetilde{z}_{\alpha_j};q.t\right) \prod_{\beta_i^c<\beta_j} F\left(\widetilde{y}_{\beta_i^c}^{-1}\widetilde{y}_{\beta_j};q.t\right)$$
$$\times \sum_{\substack{\lambda\\ \ell(\lambda)\leq 2}} \sum_{l=0}^{\lambda_2} \sum_{k_1,\dots,k_f\geq 0} \sum_{\substack{l_1,\dots,l_f\geq 0\\ l_1+\cdots+l_f=l}} \prod_{i=1}^{f} f(k_i,0)f(l_i,0)\widetilde{x}_i^{k_i-l_i}$$
$$\times \frac{b_{\lambda-l\cdot 1^2}(q,t)}{b_\lambda(q,t)}P_\lambda(\widetilde{x}_1\widetilde{z}_{\alpha_1},\widetilde{x}_1\widetilde{z}_{\alpha_2};q,t)Q_\lambda(\widetilde{y}_{\beta_1},\widetilde{y}_{\beta_2};q,t) \tag{3.8}$$

where $\widetilde{x}_i$ $(1\leq i\leq f$, $\widetilde{y}_i$ $(1\leq i\leq n)$ and $\widetilde{z}_i$ $(1\leq i\leq m)$ are as in Proposition 1.11 (2).

Proof. From (2.6) we have

$$\prod_{i,j=1}^{2} F\left(\widetilde{x}_1\widetilde{y}_{\beta_j}\widetilde{z}_{\alpha_i};q,t\right) = \sum_{\mu} P_\mu(\widetilde{x}_1\widetilde{z}_{\alpha_1},\widetilde{x}_1\widetilde{z}_{\alpha_2})Q_\mu(\widetilde{y}_{\beta_1},\widetilde{y}_{\beta_2}).$$

By the binomials theorem we have

$$\prod_{i=1}^{f} F(\widetilde{x}_i;q.t) = \sum_{k_1,\dots,k_f\geq 0}\prod_{i=1}^{f} f(k_i;0)\widetilde{x}_i^{k_i},$$

$$\prod_{i=1}^{f} F\left(\frac{\widetilde{x}_1^2}{\widetilde{x}_i}\prod_{k,l=1}^{2}\widetilde{y}_l\widetilde{z}_k;q.t\right) = \sum_{l_1,\dots,l_f\geq 0}\prod_{i=1}^{f} f(l_i;0)\widetilde{x}_i^{-l_i}\left(\widetilde{x}_1^2\prod_{k,l=1}^{2}\widetilde{y}_l\widetilde{z}_k\right)^{l_1+\cdots+l_f}.$$

By [7, VI.4, (4.17)] and (2.5) we obtain

$$(\widetilde{x}_1^2\widetilde{z}_1\widetilde{z}_2)^l\, P_\mu(\widetilde{x}_1\widetilde{z}_{\alpha_1}, \widetilde{x}_1\widetilde{z}_{\alpha_2}) = P_{\mu+l\cdot 1^2}(\widetilde{x}_1\widetilde{z}_{\alpha_1}, \widetilde{x}_1\widetilde{z}_{\alpha_2})$$
$$(\widetilde{y}_1\widetilde{y}_2)^l\, Q_\mu(\widetilde{y}_{\beta_1}, \widetilde{y}_{\beta_2}) = \frac{b_\mu(q,t)}{b_{\mu+l\cdot 1^2}(q,t)}\, Q_{\mu+l\cdot 1^2}(\widetilde{y}_{\beta_1}, \widetilde{y}_{\beta_2}).$$

From (1.23) we obtain

$$F(z[H_p];q,t) = \prod_{\alpha_i^c<\alpha_j} F\left(\widetilde{z}_{\alpha_i^c}^{-1}\widetilde{z}_{\alpha_j};q.t\right) \prod_{\beta_i^c<\beta_j} F\left(\widetilde{y}_{\beta_i^c}^{-1}\widetilde{y}_{\beta_j};q.t\right)$$
$$\times \sum_{l\geq 0}\sum_{\substack{\mu\\ \ell(\mu)\leq 2}}\sum_{k_1,\dots,k_f\geq 0}\sum_{\substack{l_1,\dots,l_f\geq 0\\ l_1+\cdots+l_f=l}} \prod_{i=1}^{f} f(k_i,0)f(l_i,0)\widetilde{x}_i^{k_i-l_i}$$
$$\times \frac{b_\mu(q,t)}{b_{\mu+l\cdot 1^2}(q,t)} P_{\mu+l\cdot 1^2}(\widetilde{x}_1\widetilde{z}_{\alpha_1}, \widetilde{x}_1\widetilde{z}_{\alpha_2};q,t) Q_{\mu+l\cdot 1^2}(\widetilde{y}_{\beta_1}, \widetilde{y}_{\beta_2};q,t).$$

This immediatey impies (3.8). □

4 Proof by Gasper's formula

Now we are in position to prove Okada's conjecture for Birds and Banners, i.e., Theorem 1.9. We use the fact that Macdonald's polynomials are the basis of $\Lambda_{\mathbb{F}}$. (cf. [6]). To prove the birds case, we fix integers ρ_0 and θ_0 such that $\theta_0 \geq \rho_0 \geq 0$, and nonnegative integers $r_1, \dots, r_f$. If we compare the coefficient of $\prod_{i=1}^f \widetilde{x}_i^{r_i} \cdot P_\lambda(\widetilde{x}_1\widetilde{z}_{\alpha_1}, \widetilde{x}_1\widetilde{z}_{\alpha_2};q,t)Q_\lambda(\widetilde{y}_{\beta_1}, \widetilde{y}_{\beta_2};q,t)$ in (3.6) and (3.8), the following identity must hold:

$$\sum_{\substack{(\rho_1,\dots,\rho_f)\\ 0\leq\rho_f\leq\cdots\leq\rho_1\leq\rho_0}} \widehat{\Phi}_0^f(\rho,\theta;q,t) = \sum_{l=0}^{\rho_0}\sum_{\substack{l_1,\dots,l_f\geq 0\\ l_1+\cdots+l_f=l}} \frac{b_{(\theta_0-l,\rho_0-l)}(q,t)}{b_{(\theta_0,\rho_0)}(q,t)} \prod_{i=1}^{f} f(l_i;0)f(l_i+r_i;0),$$

where $(\theta_1,\dots,\theta_f)$ is determined from θ_0 and $(\rho_1,\dots,\rho_f)$ by using the equations $\theta_i = \rho_{i-1}+\theta_{i-1}+r_i-\rho_i$ for $i=1,\dots,f$. Since (2.1) implies

$$b_{(\theta_0,\rho_0)} = f(\theta_0-\rho_0;0)\frac{f(\theta_0;1)}{f(\theta_0-\rho_0;1)} f(\rho_0;0),$$

we obtain

$$\frac{b_{(\theta_0-l,\rho_0-l)}(q,t)}{b_{(\theta_0,\rho_0)}(q,t)} = \frac{f(\rho_0-l;0)f(\theta_0-l;1)}{f(\rho_0;0)f(\theta_0;1)}.$$

Hence it is enough to prove

$$\sum_{\substack{(\rho_1,\dots,\rho_f)\\ 0\leq\rho_f\leq\cdots\leq\rho_1\leq\rho_0}} \widehat{\Phi}_0^f(\rho,\theta;q,t) = \sum_{l=0}^{\rho_0}\sum_{\substack{l_1,\dots,l_f\geq 0\\ l_1+\cdots+l_f=l}} \frac{f(\rho_0-l;0)f(\theta_0-l;1)}{f(\rho_0;0)f(\theta_0;1)} \prod_{i=1}^{f} f(l_i;0)f(l_i+r_i;0). \quad (4.1)$$

In fact a more general formula holds. If we prove the following theorem, then the proof of (4.1) are done.

Theorem 4.1. Let m and n be nonegative integers. Let k_0, ρ_0, θ_0 be integers such that $0 \leq k_0 \leq$

$\rho_0 \le \theta_0$, and let $\gamma_1, \ldots, \gamma_n$ be nonnegative integers. Then we have

$$\sum_{\substack{(\rho_1,\ldots,\rho_n) \\ k_0 \le \rho_n \le \cdots \le \rho_1 \le \rho_0}} f(\rho_n - k_0; 0) f(\theta_n - k_0; m+n)$$

$$\times \prod_{i=1}^{n} \frac{f(\rho_{i-1} - \rho_i; 0) f(\theta_{i-1} - \rho_i; i+m-1) f(\theta_i - \rho_{i-1}; i+m-1) f(\theta_i - \theta_{i-1}; 0)}{f(\theta_i - \rho_i; i+m-1) f(\theta_i - \rho_i; i+m)}$$

$$= \sum_{\substack{k_1,\ldots,k_n \ge 0 \\ k_1+\cdots+k_n \le \rho_0 - \rho_{m+1}}} f(\rho_0 - \sum_{i=0}^{n} k_i; 0) f(\theta_0 - \sum_{i=0}^{n} k_i; m) \prod_{i=1}^{n} f(k_i; 0) f(k_i + \gamma_i; 0), \tag{4.2}$$

where the sum on the left-hand side runs over all n-tuples $(\rho_1, \ldots, \rho_n)$ of nonnegative integers such that $k_0 \le \rho_n \le \cdots \le \rho_1 \le \rho_0$, the sum on the right-hand side runs over all n-tuples $(k_1, \ldots, k_n)$ of nonnegative integers which satisfy $k_1 + \cdots + k_n \le \rho_0 - \rho_{m+1}$, and θ_i is determined from ρ_i, ρ_{ii-1} and θ_{i-1} by $\theta_i = \gamma_i + \theta_{i-1} + \rho_{i-1} - \rho_i$ for $i = 1, \ldots, n$.

Before we prove this theorem, we need the following lemma which is a special case (i.e., $n = 1$) of this theorem.

Lemma 4.2. Let m be a nonnegative integer. Let k_0, ρ_0 and θ_0 be integers such that $0 \le k_0 \le \rho_0 \le \theta_0$, and let γ be a nonnegative integer. Then we have

$$\sum_{\rho=k_0}^{\rho_0} f(\rho - k_0; 0) f(\theta - k_0; m+1) \frac{f(\rho_0 - \rho; 0) f(\theta_0 - \rho; m) f(\theta - \rho_0; m) f(\theta - \theta_0; 0)}{f(\theta - \rho; m) f(\theta - \rho; m+1)}$$

$$= \sum_{k=0}^{\rho_0 - k_0} f(\rho_0 - k_0 - k; 0) f(\theta_0 - k_0 - k; m) f(k; 0) f(k+\gamma; 0), \tag{4.3}$$

where $\theta = \gamma + \rho_0 + \theta_0 - \rho$.

Proof. Set S_1 to be the left-hand side of (4.3). If one puts $k = \rho_0 - \rho$, then $\rho = \rho_0 - k$ and $\theta = k + \gamma + \theta_0$. Hence one obtains

$$S_1 = \sum_{k=0}^{\rho_0 - k_0} f(\rho_0 - k_0 - k; 0) f(k + \gamma + \theta_0 - k_0; m+1)$$
$$\times \frac{f(k; 0) f(k + \gamma + \theta_0 - \rho_0; m) f(k + \theta_0 - \rho_0; m) f(k + \gamma; 0)}{f(2k + \gamma + \theta_0 - \rho_0; m) f(2k + \gamma + \theta_0 - \rho_0; m+1)}.$$

If we use

$$(\alpha; q)_{2k} = (\alpha^{\frac{1}{2}}; q)_k (-\alpha^{\frac{1}{2}}; q)_k (\alpha^{\frac{1}{2}} q^{\frac{1}{2}}; q)_k (-\alpha^{\frac{1}{2}} q^{\frac{1}{2}}; q)_k,$$

then the factors in the denominator are written as $f(2k+\gamma+\theta_0-\rho_0; m) = f(\gamma+\theta_0-\rho_0; m) \frac{\left(t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}; q\right)_k}{\left(t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}, -t^{\frac{m}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}; q\right)_k}$ and $f(2k + \gamma + \theta_0 - \rho_0; m+1) = f(\gamma + \theta_0 - \rho_0; m+1) \times \frac{\left(t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, -t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0}{2}}, t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m+2}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}; q\right)_k}{\left(t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+1}{2}}, t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}, -t^{\frac{m+1}{2}} q^{\frac{\gamma+\theta_0-\rho_0+2}{2}}; q\right)_k}$. Meanwhile, the factors in the numerator are $f(\rho_0 - k_0 - k; 0) = f(\rho_0 - k_0; 0) \frac{(q^{-\rho_0+k_0}; q)_k}{(t^{-1} q^{-\rho_0+k_0+1}; q)_k} \left(\frac{q}{t}\right)^k$, $f(k + \gamma + \theta_0 - k_0; m+1) = f(\gamma + \theta_0 - k_0; m+1) \frac{(t^{m+2} q^{\gamma+\theta_0-k_0}; q)_k}{(t^{m+1} q^{\gamma+\theta_0-k_0+1}; q)_k}$, $f(k + \gamma + \theta_0 - \rho_0; m) = f(\gamma + \theta_0 - \rho_0; m) \frac{(t^{m+1} q^{\gamma+\theta_0-\rho_0}; q)_k}{(t^m q^{\gamma+\theta_0-\rho_0+1}; q)_k}$, $f(k+\theta_0-\rho_0; m) = f(\theta_0-\rho_0; m) \frac{(t^{m+1} q^{\theta_0-\rho_0}; q)_k}{(t^m q^{\theta_0-\rho_0+1}; q)_k}$, $f(k+\gamma; 0) = f(k+\gamma; 0) \frac{(tq^\gamma; q)_k}{(q^\gamma; q)_k}$. Hence, substituting these factors, we obtain

$$S_1 = C \cdot {}_{12}W_{11}\Big(bc/d; (bcq/ad)^{\frac{1}{2}}, -(bcq/ad)^{\frac{1}{2}}, q(bc/d)^{\frac{1}{2}}, -q(bc/d)^{\frac{1}{2}},$$
$$ab/d, ac/d, a, b, c; q, q/a\Big),$$

where $a = t$, $b = tq^\gamma$, $c = q^{-\rho_0+k_0}$, $d = t^{-m}q^{-\theta_0+k_0}$ and

$$C = \frac{f(\rho_0 - k_0; 0)f(\gamma + \theta_0 - k_0; m+1)f(\theta_0 - \rho_0; m)f(\gamma; 0)}{f(\gamma + \theta_0 - \rho_0, m+1)}.$$

On the other hand, Set S_2 to be the right-hand side of (4.3). If we use $f(\rho_0 - k_0 - k; 0) = f(\rho_0 - k_0; 0)\frac{(q^{-\rho_0+k_0};q)_k}{(t^{-1}q^{-\rho_0+k_0+1};q)_k}\left(\frac{q}{t}\right)^k$, $f(\theta_0 - k_0 - k; m) = f(\theta_0 - k_0; m)\frac{(t^{-m}q^{-\theta_0+k_0};q)_k}{(t^{-m-1}q^{-\theta_0+k_0+1};q)_k}\left(\frac{q}{t}\right)^k$ and $f(k+\gamma; 0) = f(k+\gamma; 0)\frac{(tq^\gamma;q)_k}{(q^\gamma;q)_k}$, then we obtain

$$S_2 = f(\rho_0 - k_0; 0)f(\theta_0 - k_0; m)f(\gamma; 0)\,{}_4\phi_3\left[\begin{matrix} q^{-\rho_0+k_0}, t^{-m}q^{-\theta_0+k_0}, t, tq^\gamma \\ t^{-1}q^{-\rho_0+k_0+1}, t^{-m-1}q^{-\theta_0+k_0+1}, q^{\gamma+1}\end{matrix}; q, \frac{q^2}{t^2}\right].$$

Hence Gasper's formula (1.2) proves that $S_1 = S_2$. The details are left to the reader. This completes our proof. □

Proof of Theorem 4.1. We proceed by induction on n. If $n = 1$, then (4.2) is nothing but (4.3). Let $n \geq 2$ and assume (4.2) is true for $n-1$. If we set S to be the left-hand side of (4.2), then we have

$$\begin{aligned} S = &\sum_{\rho_1=k_0}^{\rho_0} \frac{f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_1 - \theta_0; 0)}{f(\theta_1 - \rho_1; m)f(\theta_1 - \rho_1; m+1)} \\ &\times \sum_{\substack{(\rho_2,\dots,\rho_n) \\ k_0 \leq \rho_n \leq \dots \leq \rho_2 \leq \rho_1}} f(\rho_n - k_0; 0)f(\theta_n - k_0; m+n) \\ &\times \prod_{i=2}^{n} \frac{f(\rho_{i-1} - \rho_i; 0)f(\theta_{i-1} - \rho_i; i+m-1)f(\theta_i - \rho_{i-1}; i+m-1)f(\theta_i - \theta_{i-1}; 0)}{f(\theta_i - \rho_i; i+m-1)f(\theta_i - \rho_i; i+m)}. \end{aligned}$$

We can use our induction hypothesis to obtain

$$\begin{aligned} S = &\sum_{\substack{k_2,\dots,k_n \geq 0 \\ k_2+\dots+k_n \leq \rho_0 - k_0}} \prod_{i=2}^{n} f(k_i; 0)f(k_i + \gamma_i; 0) \\ &\times \sum_{\rho_1 = k_0 + \sum_{i=2}^n k_i}^{\rho_0} f(\rho_1 - k_0 - \sum_{i=2}^{n} k_i; 0)f(\theta_1 - k_0 - \sum_{i=2}^{n} k_i; m+1) \\ &\times \frac{f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_1 - \theta_0; 0)}{f(\theta_1 - \rho_1; m)f(\theta_1 - \rho_1; m+1)}. \end{aligned}$$

If we use (4.3) again, then we obtain

$$\begin{aligned} S = &\sum_{\substack{k_2,\dots,k_n \geq 0 \\ k_2+\dots+k_n \leq \rho_0 - k_0}} \prod_{i=2}^{n} f(k_i; 0)f(k_i + \gamma_i; 0) \\ &\times \sum_{0 \leq k_1 \leq \rho_0 - k_0 - \sum_{i=2}^n k_i} f(\rho_0 - \sum_{i=0}^{n} k_i, 0)f(\theta_0 - \sum_{i=0}^{n} k_i, m)f(k_1, 0)f(k_1 + \gamma_1, 0) \end{aligned}$$

which equals the right-hand side of (4.2). This completes our proof. □

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