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SHARP ASYMPTOTICS FOR THE DAMPED WAVE EQUATIONS

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1 Introduction

This note is concerned with large time behavior of the global solutions to the damped wave equations with a nonlinear convection term:

$$u_{tt}-u_{xx}+u_{t}+\alpha u_{x}+(f(u))_{x}=0, \quad x\in\mathbb{R},\ t>0, \tag{1.1}$$

$$u(x,0)=u_{0}(x), \quad u_{t}(x,0)=u_{1}(x) \tag{1.2}$$

where $|\alpha|<1$ and $f(u)=\frac{\beta}{2}u^{2}+\frac{\gamma}{3!}u^{3}$. The subscripts t and x stand for the partial derivatives with respect to t and x, respectively. In Ueda and Kawashima [8], it was shown that solution of (1.1) and (1.2) tends to a nonlinear diffusion wave defined by

$$\chi(x,t)=\frac{1}{\sqrt{1+t}}\chi_{*}\left(\frac{x-\alpha(1+t)}{\sqrt{1+t}}\right), \quad x\in\mathbb{R}, \quad t\geq 0, \tag{1.3}$$

where

$$\chi_{*}(x)=\frac{\sqrt{\mu}}{\beta}\frac{(e^{\beta M/2\mu}-1)e^{-\frac{x^{2}}{4\mu}}}{\sqrt{\pi}+(e^{\beta M/2\mu}-1)\int_{x/\sqrt{4\mu}}^{\infty}e^{-y^{2}}dy}, \tag{1.4}$$

$$M=\int_{\mathbb{R}}(u_{0}(x)+u_{1}(x))dx, \quad \mu=1-\alpha^{2}. \tag{1.5}$$

By the Hopf-Cole transformation in Hopf [2] and Cole [1], we see that it is a solutions of the Burgers equation

$$\chi_{t}+(\alpha\chi+\frac{\beta}{2}\chi^{2})_{x}=\mu\chi_{xx}, \quad x\in\mathbb{R}, \quad t>0, \tag{1.6}$$

satisfying

$$\int_{\mathbb{R}}\chi(x,0)dx=M. \tag{1.7}$$

We set, for $1 \le p \le \infty$ and $s \ge 1$,

$$\begin{aligned} E_0^{(s,p)} &= \|u_0\|_{W^{s,p}} + \|u_0\|_{L^1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L^1}, \\ E_1^{(s,p)} &= \|u_0\|_{W^{s,p}} + \|u_0\|_{L^1_1} + \|u_1\|_{W^{s-1,p}} + \|u_1\|_{L^1_1}, \\ E_2^{(s,p)} &= E_1^{(s,p)} + E_1^{(2,1)}. \end{aligned}$$

Concerning the convergence rate of the nonlinear diffusion wave $\chi(x,t)$ to the original solution $u(x,t)$, we can infer the following result from the argument given in [8]: For any $\epsilon > 0$, if $u_0 \in W^{1,p} \cap L^1_1$ and $u_1 \in L^p \cap L^1_1$ and $E_0^{(1,p)}$ is small, then we have

$$\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t))\|_{L^p} \le CE_1^{(1,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)+\epsilon} \tag{1.8}$$

for $l = 0, 1$. Here, $W^{s,p}$ denotes the space of functions $u = u(x)$ such that $\partial_x^l u$ are L^p-functions on $\mathbb{R}$ for $0 \le l \le s$, endowed with the norm $\|\cdot\|_{W^{s,p}}$, while $L^1_1(\mathbb{R})$ is subset of $L^1(\mathbb{R})$ whose elements satisfy $\|u\|_{L^1_1} \equiv \int_{\mathbb{R}} |u|(1+|x|)dx < \infty$.

This observation lead to a natural question whether it is possible to take $\epsilon = 0$ in (1.8) or not. The aim of note is to show that the optimal dacay rate by studying the second asymptotic profile. Indeed, the second asymptotic profile of large time behavior of the solutions is given by

$$V(x,t) = -\kappa d V_*\left(\frac{(x-\alpha(1+t))}{\sqrt{1+t}}\right)(1+t)^{-1}\log(2+t), \quad t \ge 0, \quad x \in \mathbb{R}, \tag{1.9}$$

where

$$V_*(x) = \frac{1}{\sqrt{4\pi}}\partial_x(\eta_*(x)e^{-\frac{x^2}{4}}), \tag{1.10}$$

$$\eta_*(x) = \exp\left(\frac{\beta}{2\mu}\int_{-\infty}^x \chi_*(y)dy\right), \tag{1.11}$$

$$\kappa = \frac{\alpha\beta^2}{4\mu} + \frac{\gamma}{3!}, \quad d = \int_{\mathbb{R}} \frac{1}{\eta_*(y)}\chi_*^3(y)dy. \tag{1.12}$$

Then we have the following result.

Theorem 1.1. *Let $s \ge 2$ and $1 \le p \le \infty$ and assume that $u_0 \in W^{s,p} \cap W^{2,1} \cap L^1_1$ and $u_1 \in W^{s-1,p} \cap W^{1,1} \cap L^1_1$. Let $u(x,t)$ be the global solution of the problem* (1.1) *and* (1.2) *constructed in Proposition 3.1. Then, If $E_0^{(s,p)} + E_0^{(2,1)}$ is small, then we have the following asymptotic relations:*

$$\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t) - V(\cdot,t))\|_{L^p} \le CE_2^{(s,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}. \tag{1.13}$$

for $0 \le l \le s-2$.

Using (1.10), (1.11) and (1.12), we can see that if $M \neq 0$, then $d \neq 0$, $V_*(x) \neq 0$. From (1.9), we have

$$\|\partial_x^l V(\cdot,t)\|_{L^p} = \kappa d \|\partial_x^l V_*(x)\|_{L^p}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}\log(2+t). \tag{1.14}$$

Hence, we see from (1.13) and (1.14) that we can not take $\epsilon = 0$ in (1.8) unless $\kappa M \neq 0$. We remark that the estimate similar to (1.13) was obtained for Burgers equation such as the generalized Burgers equation in Kato [6] and KdV-Burgers in Hayashi and Naumkin [4] and Kaikina and Ruiz-Paredes [5], and Benjamin-Bona-Mathony-Burgers in Hayashi, Kaikina and Naumkin [3].

2 Basic estimates

To state the results, we introduce the modified heat kernel:

$$G_0(x,t) = \frac{1}{\sqrt{4\pi\mu t}} e^{-\frac{(x-\alpha t)^2}{4\mu t}} \tag{2.1}$$

which is the fundamental solution to the linear heat equation $w_t + \alpha w_x = \mu w_{xx}$. We show following two lemmas. The first one is conserned with $L^p - L^q$ estimate for the solution operator $G_0(t)*$. For the proof, see [8].

Lemma 2.1. *Let $1 \leq q \leq p \leq \infty$, and k and l be nonnegative integers. Then we have*

$$\|\partial_x^l \partial_t^k G_0(t) * \phi\|_{L^p} \leq C t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p}+k+l)} \|\phi\|_{L^q}. \tag{2.2}$$

Also, if $\int \phi(x)dx = 0$, then we have

$$\|\partial_x^l \partial_t^k G_0(t) * \phi\|_{L^p} \leq C t^{-\frac{1}{2}(1-\frac{1}{p}+k+l)}(1+t)^{-\frac{1}{2}} \|\phi\|_{L^1_1}. \tag{2.3}$$

The second one is related to the diffusion wave $\chi(x,t)$. The explicit formula of $\chi(x,t)$ is given by (1.3). It is easy to see that

$$|\,\chi(x,t)\,| \leq C\,|\,M\,|\,(1+t)^{-\frac{1}{2}} e^{-(x-\alpha t)^2/(4\mu(1+t))}, \quad x \in \mathbb{R}, \quad t \geq 0. \tag{2.4}$$

Moreover, we get the following (see e.g. [7] and [8]).

Lemma 2.2. *Let k, l and m be nonnegative integers. If $|M| < 1$, then, for $1 \leq p \leq \infty$, the estimate*

$$\|(\partial_t + \alpha\partial_x)^m \partial_x^l \partial_t^k \chi(\cdot,t)\|_{L^p} \leq C\,|\,M\,|\,(1+t)^{-\frac{1}{2}(1-\frac{1}{p}+k+l+2m)} \tag{2.5}$$

holds.

For the latter sake, we introduce η defined by

$$\eta(x,t) \equiv \eta_* \left(\frac{x-\alpha(1+t)}{\sqrt{1+t}}\right) = \exp(\frac{\beta}{2\mu}\int_{-\infty}^{x} \chi(y,t)dy). \tag{2.6}$$

We easily have

$$\min\{1, e^{\frac{\beta M}{2\mu}}\} \le \eta(x,t) \le \max\{1, e^{\frac{\beta M}{2\mu}}\}, \tag{2.7}$$

$$\min\{1, e^{-\frac{\beta M}{2\mu}}\} \le \frac{1}{\eta(x,t)} \le \max\{1, e^{-\frac{\beta M}{2\mu}}\}. \tag{2.8}$$

Moreover, we get the following corollary (see [6]).

Corollary 2.3. *Let l be a positive integer. If $|M| \le 1$, then we have*

$$\|\partial_x^l \eta(\cdot,t)\|_{L^p} + \|\partial_x^l \frac{1}{\eta}(\cdot,t)\|_{L^p} \le C|M|(1+t)^{-\frac{1}{2}(l-\frac{1}{p})}. \tag{2.9}$$

Next, we deal with the following linearized equation which coresponds to (3.6), (3.7) below:

$$z_t + (\alpha z + \beta \chi z)_x = \mu z_{xx}, \quad x \in \mathbb{R},\ t > \tau, \tag{2.10}$$

$$z(x,\tau) = z_0(x). \tag{2.11}$$

The explicit representation formula (2.12), and the decay estimate (2.14) and (2.15) below play a crucial role in our analysis. For the proof of Lemma 2.4, Lemma 2.5 and Lemma 2.6, see [6].

Lemma 2.4. *If we set*

$$U[w](x,t,\tau) = \int_{\mathbb{R}} \partial_x(\eta(x,t)G_0(x-y,t-\tau))\frac{1}{\eta(y,\tau)}\int_{-\infty}^{y} w(\xi)d\xi dy, \quad x \in \mathbb{R},\ 0 \le \tau < t, \tag{2.12}$$

then the solutions for (2.10) *and* (2.11) *is given by*

$$z(x,t) = U[z_0](x,t,\tau), \quad x \in \mathbb{R},\ t > \tau. \tag{2.13}$$

Lemma 2.5. *Let $1 \le p \le \infty$ and l be a nonnegative integer. Assume that $| M | \le 1$, $z_0 \in L_1^1(\mathbb{R})$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then, the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^p} \le Ct^{-\frac{1}{2}(2-\frac{1}{p}+l)}\|z_0\|_{L_1^1}, \quad t > 0 \tag{2.14}$$

holds.

Lemma 2.6. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Assume that $| M | \leq 1$, $z_0 \in W^{l,p}(\mathbb{R}) \cap L^1(\mathbb{R})$. Then the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-\frac{1}{p}+l)}(\|z_0\|_{L^1} + \|z_0\|_{W^{l,p}}) \tag{2.15}$$

holds.

From Lemma 2.5 and Lemma 2.6, we get the following uniform estimate.

Corollary 2.7. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Assume that $| M | < 1$, $z_0 \in L_1^1(\mathbb{R}) \cap W^{l,p}$ and $\int_{\mathbb{R}} z_0(x)dx = 0$. Then, the estimate*

$$\|\partial_x^l U[z_0](\cdot,t,0)\|_{L^p} \leq C(1+t)^{-(2-\frac{1}{p}+l)/2}(\|z_0\|_{L_1^1} + \|z_0\|_{W^{l,p}}), \quad t > 0 \tag{2.16}$$

holds.

In order to prove Proposition 3.5, we prepare the following lemma.

Lemma 2.8. *Let $1 \leq p \leq \infty$ and l be a nonnegative integer. Suppose $| M | \leq 1$. Then the estimates*

$$\|\partial_x^l U[\partial_x w](x,t,\tau)\|_{L^p} \leq C(t-\tau)^{-(2-\frac{1}{p}+l)/2}\|\frac{1}{\eta}w(\cdot,\tau)\|_{L^1}, \tag{2.17}$$

$$\|\partial_x^l U[\partial_x w](x,t,\tau)\|_{L^p} \leq C(t-\tau)^{-1/2}\sum_{m=0}^{l}(1+t)^{-(l-m)/2}\|\partial_x^m(\frac{1}{\eta}w)(\cdot,\tau)\|_{L^p} \tag{2.18}$$

hold.

Proof. From (2.12), we have

$$\begin{aligned}&\partial_x^l U[\partial_x w](x,t,\tau)\\ = &\sum_{n=0}^{l+1-n} {}_lC_n \partial_x^{l+1-n}\eta(x,t)\int_{\mathbb{R}}\partial_x^n G_0(x-y,t-\tau)\frac{1}{\eta(y,\tau)}w(y,\tau)dy.\end{aligned} \tag{2.19}$$

From Corollary 2.3, it follows that

$$\|\partial_x^l U[\partial_x w(\tau)](x,t,\tau)\|_{L^p} \leq C\sum_{n=0}^{l+1}(1+t)^{-(l+1-n)/2}\|\partial_x^n I(\cdot,t,\tau)\|_{L^p}, \tag{2.20}$$

where we put

$$I(x,t,\tau) = \int_{\mathbb{R}} G_0(x-y,t-\tau)\frac{1}{\eta(y,\tau)}w(y,\tau)dy. \tag{2.21}$$

First, we shall prove (2.17). From Lemma 2.1, we have

$$\|\partial_x^n I(\cdot,t,\tau)\|_{L^p} \ \leq \ C(t-\tau)^{-(1-\frac{1}{p}+n)/2}\|\frac{1}{\eta}w(\cdot,\tau)\|_{L^1}. \tag{2.22}$$

Therefore, by (2.20) and (2.22), we obtain (2.17).

Next, we shall prove (2.18). From Lemma 2.1, we have

$$\|I(\cdot,t,\tau)\|_{L^p} \le C\|w(\cdot,\tau)\|_{L^p}. \tag{2.23}$$

In the following, let $1 \le n \le l+1$. From (2.2) and (2.9), it follows that

$$\|\partial_x^n I(\cdot,t)\|_{L^p} \le C(t-\tau)^{-\frac{1}{2}}\|\partial_x^{n-1}(\frac{1}{\eta}w)(\cdot,\tau)\|_{L^p}. \tag{2.24}$$

Therefore, by (2.20), (2.23) and (2.24), we obtain (2.18) This completes the proof. □

3 Proof of Theorem 1.1

We prepare the following two propositions concerning the decay rate and the asymptotic profile of the solution to the problem (1.1) and (1.2). All proposition stated below were proved mainly in [8].

Proposition 3.1. *Let $s \ge 2$, $1 \le p \le \infty$. Assume that $u_0 \in W^{s,p} \cap L^1$, $u_1 \in W^{s-1,p} \cap L^1$ and $E_0^{(s,p)}$ is small. Then the initial value problem for* (1.1) *and* (1.2) *has a unique global solution $u(x,t)$ with*

$$u(x,t) \in \begin{cases} C([0,\infty); W^{s,p} \cap L^1), & 1 \le p < \infty, \\ L^\infty((0,\infty); W^{s,\infty}) \cap C([0,\infty); L^1), & p = \infty. \end{cases}$$

Moreover, the solution satisfies

$$\|u(\cdot,t)\|_{L^1} \le CE_0^{(s,p)}, \tag{3.1}$$

$$\|\partial_x^l\partial_t^k u(\cdot,t)\|_{L^p} \le CE_0^{(s,p)}(1+t)^{-\frac{1}{2}(1-\frac{1}{p}+k+l)} \tag{3.2}$$

for $k = 0,1,2$ and $0 \le k+l \le s$.

Proposition 3.2. *Let $s \ge 2$, $1 \le p \le \infty$ and assume that $u_0 \in W^{s,p} \cap L^1_1$ and $u_1 \in W^{s-1,p} \cap L^1_1$. Let $u(x,t)$ be the global solution of the problem for (1.1) and (1.2) constructed in Proposition 3.1. For any $\epsilon > 0$, if $E_0^{(s,p)}$ is small, then we have*

$$\|u(\cdot,t) - \chi(\cdot,t)\|_{L^1} \le CE_1^{(s,p)}(1+t)^{-\frac{1}{2}+\epsilon}, \tag{3.3}$$

$$\|\partial_x^l\partial_t^k(u-\chi)(\cdot,t)\|_{L^p} \le CE_1^{(s,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+k+l)+\epsilon} \tag{3.4}$$

for $k = 0,1,2$ and $0 \le k+l \le s$.

Corollary 3.3. *Assume the same condition as Proposition 3.2. For any $\epsilon > 0$, if $E_0^{(s,p)}$ is small, then we have*

$$\|\partial_x^l(\partial_t + \alpha\partial_x)(u-\chi)(\cdot,t)\|_{L^p} \le CE_1^{(s,p)}(1+t)^{-\frac{1}{2}(4-\frac{1}{p}+l)+\epsilon} \tag{3.5}$$

for $0 \le l \le s-2$.

PROOF. We get from (1.1) and (1.6)

$$(\partial_t + \alpha\partial_x)(u-\chi) = (-\partial_t^2 + \partial_x^2)(u-\chi) - \frac{\beta}{2}(u^2-\chi^2)_x - (\frac{\gamma}{3!}u^3)_x \\ -(\partial_t - \alpha\partial_x)(\partial_t + \alpha\partial_x)\chi.$$

Hence we derive (3.5) from Proposition 3.2, Proposition 3.1, Lemma 2.2 and (1.6). This completes the proof. □

In order to prove our result, we introduce the following auxiliary problem:

(3.6) $$v_t + (\alpha v + \beta\chi v)_x - \mu v_{xx} = -\kappa(\chi^3)_x, \quad x \in \mathbb{R},\ t > 0,$$

(3.7) $$v(x,0) = 0.$$

Here κ is defined by (1.12). We show the asymptotics of the solution of the problem (3.6) and (3.7). For the proof of Proposition 3.4, see [6].

Proposition 3.4. *Assume that* $| M | \le 1$. *Then the estimate*

(3.8) $$\|\partial_x^l(v(\cdot,t) - V(\cdot,t))\|_{L^p(\mathbb{R})} \le C \mid M \mid^3 (1+t)^{-(2-\frac{1}{p}+l)/2}$$

holds.

To prove Theorem 1.1, it is sufficient to show Proposition 3.5 below by virtue of Proposition 3.4.

Proposition 3.5. *Let* $s \ge 2$, $1 \le p \le \infty$ *and assume that* $u_0 \in W^{s,p} \cap W^{2,1} \cap L_1^1$ *and* $u_1 \in W^{s-1,p} \cap W^{1,1} \cap L_1^1$. *Let u(x,t) be the global solution of the problem* (1.1) *and* (1.2) *constructed in Proposition* 3.2. *If* $E_0^{(s,p)} + E_0^{(2,1)}$ *is small, then we have the following asymptotic relations:*

(3.9) $$\|\partial_x^l(u(\cdot,t) - \chi(\cdot,t) - v(\cdot,t))\|_{L^p} \le CE_2^{(s,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}$$

for $0 \le l \le s-2$.

PROOF. From (2.6), we obtain

$$\kappa\chi^3 = \left(\frac{\alpha\beta^2}{4\mu} + \frac{\gamma}{3!}\right)\chi^3 = 2\alpha(\beta\chi\chi_x - \mu\chi_{xx}) + \frac{\gamma}{3!}\chi^3 + \eta(\frac{1}{\eta}g_1(\chi))_x,$$

where

(3.10) $$g_1(\chi) = 2\alpha\mu\chi_x - \frac{\alpha\beta}{2}\chi^2.$$

We put

(3.11) $$w(x,t) = u(x,t) + u_t(x,t) - \chi(x,t) - v(x,t).$$

Then $w(x,t)$ satisfies

$$w_t + (\alpha w + \beta\chi w)_x - \mu w_{xx} = (g(u,\chi))_x, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.12}$$

$$w(x,0) = w_0(x), \tag{3.13}$$

where we have set $w_0(x) = u_0(x) + u_1(x) - \chi_0(x)$ and

$$g(u,\chi) = \eta(\frac{1}{\eta}g_1(\chi))_x + g_2(u,\chi) \tag{3.14}$$

$$\begin{aligned} g_2(u,\chi) &= \alpha(\partial_t + \alpha\partial_x)(u-\chi) + \beta\chi(u_t + \alpha\chi_x) - \mu(u_t + \alpha\chi_x)_x \\ &\quad -\{\frac{\beta}{2}(u-\chi)^2 + \frac{\gamma}{3!}(u^3 - \chi^3)\}. \end{aligned} \tag{3.15}$$

Since $u_0(x), u_1(x), \chi(x,0) \in W^{s-1,p} \cap W^{1,1} \cap L^1_1$, we have $w_0(x) \in W^{s-1,p} \cap W^{1,1} \cap L^1_1$. Bisides by (1.5) and (1.7),

$$\int_{\mathbb{R}} w_0(x)dx = 0. \tag{3.16}$$

To prove (3.9), it is sufficent to show the decay estimate (3.17) below by Proposition 3.1, Lemma 2.2, Proposition 3.4 and (1.14).

$$\|\partial_x^l w(\cdot,t)\|_{L^p} \le CE_2^{(s,p)} t^{-\frac{1}{2}(2-\frac{1}{p}+l)}. \tag{3.17}$$

For nonnegative integer m and $1 \le q \le \infty$, we get

$$\begin{aligned} \|\partial_x^m(\frac{1}{\eta}g_1(\chi))(\cdot,t)\|_{L^q} &\le C\sum_{k=0}^{m}(1+t)^{-\frac{m-k}{2}}\|\partial_x^k g_1(\cdot,t)\|_{L^q} \\ &\le C|M|(1+t)^{-\frac{1}{2}(2-\frac{1}{q}+m)}. \end{aligned} \tag{3.18}$$

We shall show that for $0 \le m \le s-2$

$$\|\frac{1}{\eta}g_2(\cdot,t)\|_{L^1} \le CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon}, \tag{3.19}$$

$$\|\partial_x^m(\frac{1}{\eta}g_2)(\cdot,t)\|_{L^p} \le CE_1^{(s,p)}(1+t)^{-\frac{1}{2}(4-\frac{1}{p}+m)+\epsilon}. \tag{3.20}$$

We shall prove only (3.19), since we can prove (3.20) in a similar way. From Lemma 2.2, Proposition 3.1, Proposition 3.2 and Corollarry 3.3, we have

$$\begin{aligned} \|\chi(\cdot,t)(u_t + \alpha\chi_x)(\cdot,t)\|_{L^1} &\le \|\chi(\cdot,t)\|_{L^\infty}(\|\partial_t(u-\chi)(\cdot,t)\|_{L^1} + \|(\partial_t + \alpha\partial_x)\chi(\cdot,t)\|_{L^1}) \\ &\le CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon}, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \|(u_t + \alpha\chi_x)_x(\cdot,t)\|_{L^1} &\le \|\partial_x\partial_t(u-\chi)(\cdot,t)\|_{L^1} + \|\partial_x(\partial_t + \alpha\partial_x)\chi(\cdot,t)\|_{L^1} \\ &\le CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon}, \end{aligned} \tag{3.22}$$

$$\begin{aligned}\|(u-\chi)^2(\cdot,t)\|_{L^1} &\le \|(u-\chi)(\cdot,t)\|_{L^\infty}\|(u-\chi)(\cdot,t)\|_{L^1}\\ &\le CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon}, \end{aligned} \tag{3.23}$$

$$\begin{aligned}\|(u^3-\chi^3)(\cdot,t)\|_{L^1} &\le C(\|u^2(\cdot,t)\|_{L^\infty}+\|\chi^2(\cdot,t)\|_{L^\infty})\|(u-\chi)(\cdot,t)\|_{L^1}\\ &\le CE_1^{(2,1)}(1+t)^{-\frac{3}{2}+\epsilon}, \end{aligned} \tag{3.24}$$

Summing up these estimate, we obtain (3.19) from (3.15) and Corollary 3.3. Applying the Duhamel principle for the problem (3.12) and (3.13), we have from Lemma 2.4

$$w(x,t)=U[w_0](x,t,0)+\int_0^t U[\partial_x g(u,\chi)(\tau)](x,t,\tau)d\tau, \quad x\in\mathbb{R},\ t>0. \tag{3.25}$$

For $l\le s-2$, we have from (3.25), Corollary 2.7 and Lemma 2.8

$$\begin{aligned}\|\partial_x^l w(\cdot,t)\|_{L^p} &\le CE_2^{(s,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}\\ &+ C\sum_{m=0}^{l+1}\|\partial_x^{l+1-m}\eta(\cdot,t)\|_{L^\infty}\int_0^{t/2}\|\partial_x^{m+1}G_0(t-\tau)*(\frac{1}{\eta}g_1(\chi))(\cdot,\tau)\|_{L^p}d\tau\\ &+ C\int_0^{t/2}(t-\tau)^{-\frac{1}{2}(2-\frac{1}{p}+l)}\|\frac{1}{\eta}g_2(\cdot,\tau)\|_{L^1}d\tau\\ &+ C\sum_{m=0}^{l}\int_{t/2}^{t}(t-\tau)^{-\frac{1}{2}}(1+t)^{-(l-m)/2}\|\partial_x^m(\frac{1}{\eta}g)(\cdot,\tau)\|_{L^p}d\tau\\ &\equiv I_1+I_2+I_3+I_4. \end{aligned} \tag{3.26}$$

First we evaluate I_2. We have from Corollary 2.3 and Lemma 2.1

$$\begin{aligned} I_2 &\le C\sum_{m=0}^{l+1}(1+t)^{-\frac{l+1-m}{2}}\int_0^{t/2}(t-\tau)^{-\frac{1}{2}(2-\frac{1}{p}+m)}\|\frac{1}{\eta}g_1(\chi)(\cdot,,\tau)\|_{L^1}d\tau\\ &\le C\mid M\mid t^{-\frac{1}{2}(3-\frac{1}{p}+l)}\int_0^{t/2}(1+\tau)^{-\frac{1}{2}}d\tau\\ &\le C\mid M\mid t^{-\frac{1}{2}(2-\frac{1}{p}+l)}. \end{aligned} \tag{3.27}$$

Next we evaluate I_3. From (3.19), we have

$$\begin{aligned} I_3 &\le CE_1^{(2,1)}t^{-\frac{1}{2}(2-\frac{1}{p}+l)}\int_0^{t/2}(1+\tau)^{-\frac{5}{4}}d\tau\\ &\le CE_1^{(2,1)}t^{-\frac{1}{2}(2-\frac{1}{p}+l)}. \end{aligned} \tag{3.28}$$

Finally we evaluate I_4. From (3.20), it follows that

$$\begin{aligned} I_4 &\le CE_1^{(s,p)}\int_{t/2}^t(t-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}(3-\frac{1}{p}+l)}d\tau\\ &\le CE_1^{(s,p)}(1+t)^{-\frac{1}{2}(2-\frac{1}{p}+l)}. \end{aligned} \tag{3.29}$$

Summing up these estimates, we obtain (3.17). This completes the proof. □

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