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Weak and Strong Convergence Theorems for Generalized Nonlinear Mappings in Hilbert Spaces
(ヒルベルト空間における非線形写像の弱収束・強収束定理)

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Abstract. In this article, using strongly asymptotically invariant sequences, we first prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. Next, we prove a weak convergence theorem of Mann's type [24] for the mappings. Furthermore, using the idea of mean convergence by Shimizu and Takahashi [25, 26], we prove a strong convergence theorem of Halpern's type [8] for the mappings. The nonlinear ergodic theorem and the strong convergence theorem in this article generalize the Kawasaki and Takahashi nonlinear ergodic theorem [19] and the Hojo and Takahashi strong convergence theorem [11], respectively.

1 Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H. For a mapping $T : C \to H$, we denote by $F(T)$ the set of fixed points of T. Kocourek, Takahashi and Yao [20] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings [7], nonspreading mappings [21, 22] and hybrid mappings [31]. A mapping $T : C \to H$ is said to be *generalized hybrid* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers; see also [1]. We call such a mapping an (α, β)-*generalized hybrid* mapping. Kocourek, Takahashi and Yao [20] and Hojo and Takahashi [11] proved the following nonlinear ergodic and strong convergence theorems for generalized hybrid mappings, respectively.

Theorem 1.1 ([20]). *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H, let T be a generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$ and let P be the metric projection of H onto $F(T)$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to $p \in F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Theorem 1.2 ([11]). *Let C be a non-empty, closed and convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences*

$\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \ldots$, where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where P is the metric projection of H onto $F(T)$.

Very recently, Kawasaki and Takahashi [19] introduced a broader class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. A mapping T from C into H is said to be *widely more generalized hybrid* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0 \end{aligned} \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-*widely more generalized hybrid* mapping; see also [18]. An $(\alpha, \beta, \gamma, \delta, 0, 0, 0)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [20] if $\alpha + \beta = -\gamma - \delta = 1$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. In [19], Kawasaki and Takahashi proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [3] for such new mappings in a Hilbert space. In particular, by using their fixed point theorems, they proved directly Browder and Petryshyn's fixed point theorem [5] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [20] for super generalized hybrid mappings.

In this article, using strongly asymptotically invariant sequences, we first prove a nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space. Next, we prove a weak convergence theorem of Mann's type [24] for the mappings. Furthermore, using the idea of mean convergence by Shimizu and Takahashi [25, 26], we prove a strong convergence theorem of Halpern's type [8] for the mappings. The nonlinear ergodic theorem and the strong convergence theorem in this article generalize the Kawasaki and Takahashi nonlinear ergodic theorem [] and the Hojo and Takahashi strong convergence theorem [11].

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [30], we have that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|y\|^2 - \|x\|^2 \le 2\langle y - x, y \rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore, we know that for $x, y, u, v \in H$

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a non-empty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let C be a non-empty, closed and convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \tag{2.4}$$

for all $x, y \in H$; see [30] for more details. For proving main results in this article, we also need the following lemmas proved in Takahashi and Toyoda [32] and Aoyama, Kimura, Takahashi and Toyoda [2].

Lemma 2.1 ([32]). *Let D be a non-empty, closed and convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Lemma 2.2 ([2]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a *Banach limit* on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [28] for the proof of existence of a Banach limit and its other elementary properties. For $f \in \ell^\infty$, define $\ell_1 : \ell^\infty \to \ell^\infty$ as follows:

$$\ell_1 f(k) = f(1 + k), \quad \forall k \in \mathbb{N}.$$

A sequence $\{\mu_n\}$ of means on ℓ^∞ is said to be *strongly asymptotically invariant* if

$$\|\ell_1^* \mu_n - \mu_n\| \to 0,$$

where ℓ_1^* is the adjoint operator of ℓ_1. See [6] for more details. The following definition which was introduced by Takahashi [27] is crucial in the fixed point theory. Let h be a bounded function of $\mathbb{N}$ into H. Then, for any mean μ on ℓ^∞, there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z \rangle = (\mu)_k \langle h(k), z \rangle, \quad \forall z \in H.$$

Such h_μ is contained in $\overline{\text{co}}\{h(k) : k \in \mathbb{N}\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A. In particular, let T be a mapping of a subset C of a Hilbert space H into itself such that $\{T^k x : k \in \mathbb{N}\}$ is bounded for some $x \in C$. Putting $h(k) = T^k x$ for all $k \in \mathbb{N}$, we have that there exists $z_0 \in H$ such tat

$$\mu_k \langle T^k x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

We denote such z_0 by $T_\mu x$. From Kawasaki and Takahashi [19], we also know the following fixed point theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 2.3 ([19]). *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) or (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ under the conditions (1) and (2).

3 Nonlinear ergodic theorems

In this section, using the technique developed by Takahashi [27], we prove a mean convergence theorem for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following three lemmas. The following lemma was proved by Kawasaki and Takahashi [19].

Lemma 3.1 ([19]). *Let H be a real Hilbert space, let C be a closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0, \; \zeta + \eta \geq 0$ *and* $\alpha + \beta > 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \; \varepsilon + \eta \geq 0$ *and* $\alpha + \gamma > 0$.

Then T is quasi-nonexpansive.

The following two lemmas by Hojo and Takahashi [12] are crucial in the proof of our main theorem in this section.

Lemma 3.2 ([12]). *Let C be a non-empty, closed and convex subset of a real Hilbert space H. Let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that it satisfies the following condition (1) or (2):*

(1) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \gamma > 0, \; \varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0, \; \zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

Let $\{\mu_\nu\}$ be a srongly asymptotically invariant net of means on ℓ^∞. For any $x \in C$, define $S_{\mu_\nu}x$ as follows:

$$\langle S_{\mu_\nu}x, y\rangle = (\mu_\nu)_k\langle T^k x, y\rangle, \quad \forall y \in H.$$

Then $\lim_\nu \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0$. In addition, if C is bounded, then

$$\lim_\nu \sup_{x\in C} \|S_{\mu_\nu}x - TS_{\mu_\nu}x\| = 0.$$

Lemma 3.3 ([12]). *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$ *and* $\alpha+\gamma+\varepsilon+\eta > 0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$ *and* $\alpha+\beta+\zeta+\eta > 0$.

If $x_\nu \rightharpoonup z$ and $x_\nu - Tx_\nu \to 0$, then $z \in F(T)$.

Now we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space which was proved by Hojo and Takahashi [12].

Theorem 3.4 ([12]). *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that T satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma > 0$, $\varepsilon+\eta \geq 0$ *and* $\zeta+\eta \geq 0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$, $\zeta+\eta \geq 0$ *and* $\varepsilon+\eta \geq 0$.

Let $\{\mu_\nu\}$ be a srongly asymptotically invariant net of means on ℓ^∞ and let P be the metric projection of H onto $F(T)$. Then for any $x \in C$, the net $\{S_{\mu_\nu}x\}$ converges weakly to a fixed point p of T, where $p = \lim_{n\to\infty} PT^n x$.

Using Theorem 3.4, we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space which was proved by Kawasaki and Takahashi [19].

Theorem 3.5 ([19]). *Let H be a real Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$, $\zeta+\eta \geq 0$ *and* $\alpha+\beta > 0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, $\varepsilon+\eta \geq 0$ *and* $\alpha+\gamma > 0$.

Then for any $x \in C$ the Cesàro means

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to a fixed point p of T and $p = \lim_{n\to\infty} PT^n x$, where P is the metric projection of H onto $F(T)$.

Proof. For any $f = (x_0, x_1, x_2, \dots) \in \ell^\infty$, define

$$\mu_n(f) = \frac{1}{n}\sum_{k=0}^{n-1} x_k, \quad \forall n \in \mathbb{N}.$$

Then $\{\mu_n : n \in \mathbb{N}\}$ is an asymptotically invariant sequence of means on ℓ^∞; see [28, p.78]. Furthermore, we have that for any $x \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x.$$

Therefore, we have the desired result from Theorem 3.4. □

4 Weak convergence theorems of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [24] for widely more generalized hybrid mappings in a Hilbert space. Let C be a non-empty, closed and convex subset of a Hilbert space H. Then we know from Lemma 3.1 that an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping T from C into itself with $F(T) \neq \emptyset$ which satisfies the condition (1) or (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0,\ \ \alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0,\ \ \alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$,

is quasi-nonexpansive. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [17]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.2) that

$$\begin{aligned} \|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 = 0 \end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex. Using Lemma 3.1 and the technique developed by Ibaraki and Takahashi [14, 15], we can prove the following weak convergence theorem.

Theorem 4.1 ([9]). *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$- widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0,\ \ \alpha + \gamma > 0,\ \ \varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0,\ \ \alpha + \beta > 0,\ \ \zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

Let P be the mertic projection of H onto $F(T)$. Let $\{\mu_n\}$ be a srongly asymptotically invariant sequence of means on ℓ^∞. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)T_{\mu_n} x_n, \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges weakly to* $v \in F(T)$, *where* $v = \lim_{n\to\infty} Px_n$.

Using Theorem 4.1, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

Theorem 4.2 ([9]). *Let* H *be a Hilbert space and let* C *be a non-empty, closed and convex subset of* H. *Let* $T : C \to C$ *be a generalized hybrid mapping with* $F(T) \neq \emptyset$. *Let* $\{\mu_n\}$ *be a srongly asymptotically invariant sequence of means on* ℓ^∞. *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 \leq \alpha_n \leq 1$ *and* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Suppose that* $\{x_n\}$ *is the sequence generated by* $x_1 = x \in C$ *and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges weakly to an element* $v \in F(T)$.

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - Ty\|^2 + (1-\beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that an (α, β)-generalized hybrid mapping is an $(\alpha, 1-\alpha, -\beta, -(1-\beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.1. Therefore, we have the desired result from Theorem 4.1. □

5 Strong Convergence Theorems

In this section, using the idea of mean convergence by Shimizu and Takahashi [25] and [26], we prove the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 5.1 ([9]). *Let* C *be a nonempty, closed and convex subset of a real Hilbert space* H. *Let* T *be an* $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$*-widely more generalized hybrid mapping of* C *into itself which satisfies the following condition* (1) *or* (2):

(1) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \gamma > 0, \quad \varepsilon + \eta \geq 0$ *and* $\zeta + \eta \geq 0$;
(2) $\alpha + \beta + \gamma + \delta \geq 0, \quad \alpha + \beta > 0, \quad \zeta + \eta \geq 0$ *and* $\varepsilon + \eta \geq 0$.

Let $\{\mu_n\}$ *be a srongly asymptotically invariant sequence of means on* ℓ^∞. *Let* $u \in C$ *and define sequences* $\{x_n\}$ *and* $\{z_n\}$ *in* C *as follows:* $x_1 = x \in C$ *and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = T_{\mu_n} x_n \end{cases}$$

for all $n = 1, 2, \ldots$, *where* $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$. *If* $F(T) \neq \emptyset$, *then* $\{x_n\}$ *and* $\{z_n\}$ *converge strongly to* Pu, *where* P *is the metric projection of* H *onto* $F(T)$.

Using Theorem 5.1, as in the proof of Theorem 4.2, we can show the result (Theorem 1.2) in Introduction which was obtained by Hojo and Takahashi [11].

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