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An SDP approach for some quadratic fractional problems

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Abstract

The nonconvex optimization problem (P) considered in this proposal is to minimize globally a ratio of indefinite quadratic functions over an inequality quadratic constraint. It contains many important applications such as the generalized Rayleigh quotient problem, the total least-squares problem, the trust region method, and many others. Because of the fractional structure, the problem inherits a great degree of difficulty even if one restricts only to homogeneous convex quadratic forms. Recently, we realize that the problem involves not only the traditional fractional programming, but also the fundamental S-lemma, the QPQC problem (quadratically constrained quadratic programming) and the SDP (semi-definite programming) relaxation. It would be important to link all different areas together and come out with fundamental results of real impact. In this article, we characterize the solution structure for the (P) problem by studying the hidden convexity and the SDP reformulation.

Keywords. Fractional programming; nonconvex quadratic optimization; semidefinite programming; S-Lemma.

1 Introduction

In this article, we are interested in the problem of minimizing a ratio of quadratic functions over an admissible set described as follows:

$$\begin{array}{lll} \text{(P)} & \inf & \dfrac{x^TA_1x+2a_1x+c_1(=f_1(x))}{x^TA_2x+2a_2x+c_2(=f_2(x))} \\ & & \text{s.t. } x\in X:=\{x\in \mathbf{R}^n : g(x)\leq 0\}, \end{array}$$

where $g(x)=x^TBx+2b^Tx+d$ is a quadratic and the matrices A_1, A_2, B are symmetric matrices. We denote x^* to be the optimal solution of (P) if it is attained, and λ^* the infimum of the problem, which could be $-\infty$ when it is unbounded below. The problem (P) belongs to the category of *fractional programming.* Many optimization models require to consider efficiency measures such as profit-to-revenue in economics, cost-to-time in transportation, signal-to-noise in electrical engineering, etc. The ratio form leads to the study of fractional programming and the special ratio structure gives additional properties which general nonlinear programming does not have. There are a lot of papers, books, review articles dedicated to the area. The readers can refer to [5, 16, 11, 13] for representative ones. The topics studied therein range from fractional duality, generalized convexity, and computational algorithms.

Among all the developments, the most well-known approach for solving (P) was proposed by Dinkelbach [9] in 1967. His algorithm considers a sequence of subproblems parameterized by λ :

$$(P)_\lambda \qquad f(\lambda) = \inf\{f_1(x) - \lambda f_2(x) : x \in X\}$$

and an iterative scheme was developed to find a value λ_0 such that $f(\lambda_0) = 0$. To ensure the validity of the iteration, X is assumed to be compact and $f_2(x) > 0$ on X. Then, it was shown that $f(\lambda)$ is continuous, concave, strictly decreasing and $\lambda_0 = \lambda^*$. Moreover, (P) and $(P)_{\lambda_0}$ share the same optimal solution set. See [9, 14, 22]. The idea was later generalized to become the Dinkelbach-type algorithm [7, 8] which can find an optimal solution that minimizes the largest of n ratios:

$$\min_{x\in X}\max_{1\le i\le n}\left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_n(x)}{g_n(x)}\right).$$

Different variants of Dinkelbach-type algorithm have been studied such as the interval-type algorithm [4], the dual algorithm [1], the generic algorithm [6], the augmented Lagrange primal-dual method [15] and many others.

Recently, due to the new development on non-convex quadratic optimization and semidefinite programming (SDP), quadratic fractional programming has received much attention. Fang et. al. [10] used a dual approach to minimize the sum of a quadratic function and the ratio of two quadratic functions. Zhang and Hayashi [22] studied a CDT-type quadratic fractional problem subject to two quadratic constraints, one of which is a ball, by an iterative generalized Newton method for finding $f(\lambda_0) = 0$. Beck and Teboulle [3] studied a special case of (P), called the (RQ) problem, over an ellipsoid X:

$$\text{(RQ)} \qquad \left\{\frac{f_1(x)}{f_2(x)} : ||Lx||^2 \le \rho, x \in \mathbb{R}^n\right\}$$

where $L \in \mathbb{R}^{r\times n}$ is a full row rank matrix and $\rho > 0$.

The problem (RQ) arises from fitting data to an overdetermined linear system $Ax \approx b$, where both $A \in \mathbf{R}^{m\times n}$ and $b \in \mathbf{R}^m$ are contaminated by noise. Such a problem is called the Total Least Squares (TLS) problem which was intensively studied and applied in many areas such as signal processing, automatic control, statistics, physics, economics, biology and medicine, etc. Please refer to [2, 3, 12, 19] and the references therein. One of the most important approaches to the (TLS) problem is to find a perturbation matrix $E \in \mathbf{R}^{m\times n}$ and a perturbation vector $r \in \mathbf{R}^m$ such that the sum of squares $||E||^2 + ||r||^2$

is minimized under the consistency constraint $(A+E)x = b+r$. This problem, after some transformations (see [2]), can be recast as

$$\text{(TLS)} \quad \inf_{x\in\mathbf{R}^n} \frac{||Ax-b||^2}{||x||^2+1}.$$

However, the problem (TLS) is in general unstable due to a possible unbounded norm. The issue can be resolved by considering instead a constrained version of (TLS):

$$\text{(RTLS)} \qquad \begin{array}{ll} \inf\limits_{x\in\mathbf{R}^n} & \dfrac{||Ax-b||^2}{||x||^2+1} \\ \text{s.t.} & ||Lx||^2 \leq \rho. \end{array}$$

See [2] for more detail. The problem (RTLS) naturally leads to the following more general (RQ) problem:

$$\text{(RQ)} \qquad \begin{array}{ll} \inf\limits_{x\in\mathbf{R}^n} & \dfrac{x^TA_1x+2a_1x+c_1}{x^TA_2x+2a_2x+c_2} \\ \text{s.t.} & ||Lx||^2 \leq \rho \end{array}$$

which was recently studied in [3, 21].

The (RQ) problem is a very special type of (P) with a convex homogeneous quadratic form. This problem could be unbounded below (although the admissible set is bounded). Even if the infimum of the problem is finite, it may not be attainable. The research in [2, 3, 21] was then devoted to find conditions under which the (RQ) problem has an attainable infimum (which is called the "*attainment problem*" in literature) and to develop a SDP reformulation (semi-definite programming) for solving it. Very surprisingly, with the help of a powerful *alternative theorem* - the S-lemma, it was proved in [21] that the (RQ) problem has an attainable infimum *if and only if* the following (SDP)

$$\max\left\{\lambda : \begin{pmatrix} A_1 & a_1 \\ a_1^T & c_1 \end{pmatrix} - \lambda \begin{pmatrix} A_2 & a_2 \\ a_2^T & c_2 \end{pmatrix} + \eta \begin{pmatrix} L^TL & 0 \\ 0 & -\rho \end{pmatrix} \succeq 0, \eta \geq 0\right\} \tag{1}$$

has a unique solution and the SDP reformulation is tight. This is a very strong result with an important implication. The (RQ) problem is certainly non-convex, but it can be solved via a convex SDP reformulation. In other words, the (RQ) problem inherits a so-called "*hidden convexity*" in its problem structure.

Unfortunately, we immediately found that the above result can not hold even when the homogeneous constraint $||Lx||^2 \leq \rho$ is slightly changed to have an extra linear term. Here is the counter example we construct.

$$\inf_{x\in\mathbf{R}^3}\left\{\frac{x_1^2+1}{x_2^2+1} : g_1(x) = x_1^2 + 2x_3 - 1 \leq 0\right\}. \tag{2}$$

Notice that the constraint function $g_1(x) = x_1^2 + 2x_3 - 1$ is non-homogeneous with a linear term $2x_3$ and the objective function in (2) is strictly positive. Letting x_1 be fixed to 0 and x_2 go to infinity, we observe that the infimum of 2 is indeed 0, which certainly can not be attained by a set of positive values.

On the other hand, the example (2) with the following data

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$a_1 = a_2 = 0,\; c_1 = c_2 = 1, d_1 = -1$ gives the corresponding (SDP) reformulation:

$$\max \left\{ \lambda : \begin{pmatrix} 1+\eta & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & \eta & 1-\lambda-\eta \end{pmatrix} \succeq 0, \; \eta \geq 0 \right\}.$$

It can be checked that this (SDP) reformulation has a unique solution at $(\lambda, \eta) = (0, 0)$, but the infimum of the example (2) is not attained.

This interesting example leads us to consider the attainment problem for the general (P), to discuss the related SDP reformulation and to explore any possible hidden convexity.

2 Preliminaries

For most studies in fractional programming, it is often assumed that $f_2(x) > 0, \forall x \in X$. Namely, (P) is well-defined. Moreover, the feasible set X is assumed to be compact so that problem (P) is bounded from below and the optimal value is attained. In general, a well-defined problem (P) is not necessarily bounded from below and can not be always attained. The following two lemmas, generalizing some basic results in fractional programming, characterize completely the boundedness and the attainment properties of Problem (P) from observing the parametric function $f(\lambda)$ defined in $(P)_\lambda$.

Lemma 1 (The boundedness problem) *Suppose that problem (P) is well defined. It is bounded below if and only if there exists a* $\bar{\lambda} \in \mathbb{R}$ *such that* $f(\bar{\lambda}) \geq 0$. *Furthermore, if* $\lambda^* > -\infty$, *then*

$$\lambda^* = \max_{f(\lambda) \geq 0} \lambda.$$

Proof. Suppose $\lambda^* > -\infty$. Then $\dfrac{f_1(x)}{f_2(x)} \geq \lambda^*$ for all $x \in X$. Since problem (P) is well-defined, $f_2(x) > 0, \forall x \in X$ and thus $f_1(x) - \lambda^* f_2(x) \geq 0, \forall x \in X$. In other words,

$$f(\lambda^*) = \inf_{x \in X} \left\{ f_1(x) - \lambda^* f_2(x) \right\} \geq 0.$$

Conversely, if there exists a $\bar{\lambda} \in \mathbb{R}$ such that

$$f(\bar{\lambda}) = \inf_{x \in X} \left\{ f_1(x) - \bar{\lambda} f_2(x) \right\} \geq 0,$$

then $f_1(x) - \bar{\lambda} f_2(x) \geq 0, \forall x \in X$, which is equivalent to $\dfrac{f_1(x)}{f_2(x)} \geq \bar{\lambda}, \forall x \in X$. It implies that $\lambda^* \geq \bar{\lambda}$, so problem (P) is bounded below. The result $\lambda^* = \max_{f(\lambda) \geq 0} \lambda$ also follows immediately. □

Example 1 below shows that it is possible for a bounded problem (P) to have $f(\lambda^*) > 0$.

Example 1 *It is easy to check*

$$\lambda^* = \inf_{x \in \mathbb{R}^3} \left\{ \frac{x_1^2 + 1}{x_2^2 + 1} : g(x) = x_1^2 + 2x_3 - 1 \leq 0 \right\} = 0$$

by letting $x_1 = 0$ and x_2 going to infinity. Solving its parametric problem

$$\begin{aligned} f(\lambda) &= \inf_{x \in \mathbb{R}^3} \left\{ x_1^2 + 1 - \lambda(x_2^2 + 1) : g(x) = x_1^2 + 2x_3 - 1 \leq 0 \right\} \\ &= \begin{cases} 1 - \lambda, & \text{if } \lambda \leq 0 \\ -\infty, & \text{if } \lambda > 0 \end{cases}, \end{aligned}$$

we observe that $f(\lambda^*) = f(0) = 1 > 0$.

Lemma 2 (The attainment problem) *Suppose that problem (P) is well defined. Then, $\lambda^* = v(P)$ is attained at $x^* \in X$ if and only if λ^* is a root of $f(\lambda) = 0$ and x^* is an optimal solution to $(P)_{\lambda^*}$.*

Proof. Suppose first that $\lambda^* \in \mathbb{R}$ such that $f(\lambda^*) = 0$ and $x^* \in$ argmin $\{f_1(x) - \lambda^* f_2(x) : x \in X\}$. Then,

$$f(\lambda^*) = \inf_{x \in X} \left\{ f_1(x) - \lambda^* f_2(x) \right\} = f_1(x^*) - \lambda^* f_2(x^*) = 0.$$

Since $f_2(x^*) > 0$, this is equivalent to

$$\frac{f_1(x^*)}{f_2(x^*)} = \lambda^*.$$

Moreover, for all $x \in X$,

$$f_1(x) - \lambda^* f_2(x) \geq f_1(x^*) - \lambda^* f_2(x^*) = 0.$$

It implies that

$$\frac{f_1(x)}{f_2(x)} \geq \lambda^* = \frac{f_1(x^*)}{f_2(x^*)}, \forall x \in X.$$

In other words, $\lambda^* = v(P)$ and x^* attains λ^*.

Conversely, suppose $\lambda^* = v(P)$ is attained at $x^* \in X$ such that $\lambda^* = \dfrac{f_1(x^*)}{f_2(x^*)}$. Then we have

$$\frac{f_1(x)}{f_2(x)} \geq \frac{f_1(x^*)}{f_2(x^*)} = \lambda^*, \forall x \in X.$$

This implies that

$$f_1(x) - \lambda^* f_2(x) \geq 0, \forall x \in X \text{ and } f_1(x^*) - \lambda^* f_2(x^*) = 0.$$

Consequently, x^* is a minimizer of $(P)_{\lambda^*}$ with $f(\lambda^*) = 0$. □

Remark 1 *In our proofs above we do not use the assumption that (P) is a quadratic fractional programming problem, so that Lemmas 1 and 2 hold for any well-defined fractional programming problem.*

3 Main results

Assume in this section that problem (P) satisfies the Slater condition, i.e., there exists $\bar{x} \in \mathbb{R}^n$ such that $g(\bar{x}) < 0$. Otherwise, the problem (P) is either infeasible or reduced to an unconstrained fractional programming problem.

Lemma 3 *If Problem (P) has no Slater point, it is either infeasible or equivalent to an unconstrained quadratic fractional programming problem.*

Proof. The Slater condition is violated only when $g(x) \geq 0, \forall x \in \mathbb{R}^n$. This implies that $B \succeq 0$, i.e, $g(x)$ is convex, and $d \in \mathcal{R}(B)$, where $\mathcal{R}(B)$ is the range space of B. That is, the affine space

$$\{x \in \mathbb{R}^n : Bx + d = 0\} \neq \emptyset.$$

Then $Bx + d = 0 \Leftrightarrow x = -B^+ d + Wz$, where B^+ is the Moore-Penrose generalized inverse of B and W is a matrix whose columns form a basis for the null space of B if B is singular; and $W = 0$ if B is nonsingular. Since $g(x)$ is convex, $x = -B^+ d + Wz$ is the global minimizer of $g(x)$ with the minimum value $-d^T B^+ d + \alpha$. If $-d^T B^+ d + \alpha > 0, g(x) \geq$

$-d^TB^+d+\alpha>0$ implies that (P) is infeasible. If $-d^TB^+d+\alpha=0$, then $g(x)\geq 0$. In this case, the feasible domain $X=\{x|g(x)\leq 0\}$ is reduced to $X=\{x|g(x)=0\}$. That is

$$\{x\in\mathbb{R}^n: g(x)\leq 0\}=\begin{cases}\{-B^+d+Wz,\ z\in\mathbb{R}^m\}, & \text{if } d^TB^+d=\alpha\\ \emptyset, & \text{if } d^TB^+d<\alpha\end{cases}$$

where m is the dimension of the null space of B. In the case that (QF1QC) is feasible, it can be expressed in term of $z\in\mathbb{R}^m$ and becomes the following unconstrained fractional programming problem:

$$\lambda^*=\inf_{x\in\mathbb{R}^n}\left\{\frac{f_1(x)}{f_2(x)}: g(x)\leq 0\right\}=\inf_{z\in\mathbb{R}^m}\frac{\bar f_1(z)}{\bar f_2(z)}, \tag{3}$$

where $\bar f_i(z)=f_i(-B^+d+Wz)=z^TQ_iz-2q_i^Tz+\gamma_i$, $Q_i=W^TA_iW$, $q_i^T=(d^TB^+A_i-b_i^T)W$, $\gamma_i=d^TB^+A_iB^+d-2b_i^TB^+d+c_i, i=1,2$. □

Theorem 1 *For any well-defined problem (P), its optimal value λ^* can be determined by solving the following semi-definite programming problem*

$$\lambda^*=\sup_{\lambda\in\mathbb{R},\mu\geq 0}\left\{\lambda:\begin{pmatrix}A_1-\lambda A_2+\mu B & b_1-\lambda b_2+\mu d\\ b_1^T-\lambda b_2^T+\mu d^T & c_1-\lambda c_2+\mu\alpha\end{pmatrix}\succeq 0\right\}. \tag{4}$$

Proof. We have

$$\begin{aligned}\lambda^*&=\inf_{x\in\mathbb{R}^n}\left\{\frac{f_1(x)}{f_2(x)}: g(x)\leq 0\right\}\\ &=\sup\left\{\lambda:\{x\in\mathbb{R}^n|\lambda>\frac{f_1(x)}{f_2(x)}, g(x)\leq 0\}=\emptyset\right\}\\ &=\sup\{\lambda:\{x\in\mathbb{R}^n|f_1(x)-\lambda f_2(x)<0, g(x)\leq 0\}=\emptyset\} &(5a)\\ &=\sup\{\lambda: f_1(x)-\lambda f_2(x)+\mu g(x)\geq 0, \forall x\in\mathbb{R}^n, \mu\geq 0\} &(5b)\\ &=\sup\left\{\lambda:\begin{pmatrix}A_1 & b_1\\ b_1^T & c_1\end{pmatrix}-\lambda\begin{pmatrix}A_2 & b_2\\ b_2^T & c_2\end{pmatrix}+\mu\begin{pmatrix}B & d\\ d^T & \alpha\end{pmatrix}\succeq 0, \mu\geq 0\right\}\\ &=\sup_{\lambda\in\mathbb{R},\mu\geq 0}\left\{\lambda:\begin{pmatrix}A_1-\lambda A_2+\mu B & b_1-\lambda b_2+\mu d\\ b_1^T-\lambda b_2^T+\mu d^T & c_1-\lambda c_2+\mu\alpha\end{pmatrix}\succeq 0\right\},\end{aligned}$$

where the equivalence of (5a) and (5b) is due to a standard S-lemma [17]. □

To know whether (P) is attained and to find x^* that solves (P), we need to check whether λ^* found in (4) satisfies $f(\lambda^*)=0$ and to solve $(P)_{\lambda^*}$. We have

$$f(\lambda^*)=\sup\{\nu\in\mathbb{R}:\{x\in\mathbb{R}^n|f_1(x)-\lambda^*f_2(x)<\nu, g(x)\leq 0\}=\emptyset\}. \tag{6}$$

Since the Slater condition is assumed, we can apply S-lemma to (6) and obtain

$$f(\lambda^*) = \sup\left\{\nu \in \mathbb{R} : f_1(x) - \lambda^* f_2(x) - \nu + \eta g(x) \geq 0, \forall x \in \mathbb{R}^n, \eta \geq 0\right\},$$

which is equivalent to a convex SDP formulation:

$$f(\lambda^*) = \sup\left\{\nu \in \mathbb{R} : \begin{pmatrix} A_1 - \lambda^* A_2 + \eta B & b_1 - \lambda^* b_2 + \eta d \\ b_1^T - \lambda^* b_2^T + \eta d^T & c_1 - \lambda^* c_2 + \eta\alpha - \nu \end{pmatrix} \succeq 0, \eta \geq 0\right\}. \quad (7)$$

We notice that (7) is the Lagrange dual problem of $(P)_{\lambda^*}$ [20]. It means that, the strong duality holds for $(P)_{\lambda^*}$. Therefore, $(P)_{\lambda^*}$ has the following tight SDP relaxation:

$$\begin{array}{ll} \inf & M(f_1 - \lambda^* f_2) \bullet Z \\ \text{subject to} & M(g) \bullet Z \leq 0 \\ & Z \succeq 0, I_{nn} \bullet Z = 1. \end{array} \quad (8)$$

Then an optimal solution x^* of $(P)_{\lambda^*}$, if exists, can be obtained from an optimal solution of (8) followed by the matrix rank-one decomposition procedure. See [18, 17].

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