数理解析研究所講究録
第 1923 巻 2014 年 99-104

Kannan mapping theorems in partially ordered sets

Masashi Toyoda* and Toshikazu Watanabe**

*Faculty of Engineering, Tamagawa University,
6–1–1 Tamagawa-gakuen, Machida-shi, Tokyo 194–8610, Japan
E-mail: mss-toyoda@eng.tamagawa.ac.jp

**College of Science and Technology, Nihon University,
1–8–14 Kanda-Surugadai, Chiyoda-ku, Tokyo 101–8308, Japan
E-mail: twatana@edu.tuis.ac.jp

1 Introduction

Let X be a metric space and let T be a mapping from X into itself. T is contractive if there exists $k\in[0,1)$ such that for any $x, y\in X$,

$$d(Tx, Ty)\leq kd(x, y).$$

Moreover T is Kannan if there exists $\alpha\in[0, \frac{1}{2})$ such that for any $x, y\in X$,

$$d(Tx, Ty)\leq\alpha d(x, Tx)+\alpha d(y, Ty).$$

For these mappings, we can consider the existence and uniqueness of fixed points; see, for example, [3].

On the other hand, in [2], Nieto and López consider fixed point theorems for contractive mappings in partially ordered sets. They introduce a mapping T such that there exists $k\in[0,1)$ such that for any $x, y\in X$,

$$x\geq y \text{ implies } d(Tx, Ty)\leq kd(x, y).$$

For this mapping, they show the existence and uniqueness of fixed points.

In this paper, motivated by [2], we consider fixed point theorems for Kannan mappings [1] in partially ordered sets.

2 Fixed point theorem for contractive mappings

The following theorem is proved in [2]. For the sake of completeness, we show the proof.

Let X be a partially ordered set with a metric d and let T be a mapping from X into itself. We say that T is monotone nondecreasing if for any $x, y \in X$, $x \leq y$ implies $Tx \leq Ty$.

Theorem 1 ([2]). *Let X be a partially ordered set with a metric d such that (X, d) is a complete metric space. If a nondecreasing sequence $\{x_n\}$ converges to x, then we have $x_n \leq x$ for any n. Let T be a monotone nonincreasing mapping from X into itself such that there exists $k \in [0, 1)$ such that for any $x, y \in X$,*

$$x \geq y \text{ implies } d(Tx, Ty) \leq kd(x, y)$$

Assume that there exists x_0 in X with $x_0 \leq Tx_0$. Then there exists a fixed point of T. Moreover, if for any $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.

Proof. Since $x_0 \leq Tx_0$ and T is monotone nondecreasing, we obtain that

$$x_0 \leq Tx_0 \leq T^2x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \cdots .$$

Since $x_0 \leq Tx_0$, we have

$$d(T^2x_0, Tx_0) \leq kd(Tx_0, x_0).$$

Moreover, since $Tx_0 \leq T^2x_0$, we have

$$d(T^3x_0, T^2x_0) \leq kd(T^2x_0, Tx_0) \leq k^2d(Tx_0, x_0).$$

Hence we have

$$d(T^{n+1}x_0, T^nx_0) \leq k^nd(Tx_0, x_0)$$

for any n. For $n < m$, we have

$$\begin{aligned}
&d(T^mx_0, T^nx_0) \\
&\leq d(T^mx_0, T^{m-1}x_0) + d(T^{m-1}x_0, T^{m-2}x_0)) + \cdots + d(T^{n+1}x_0, T^nx_0) \\
&\leq (k^{m-1} + k^{m-2} + \cdots + k^n)d(Tx_0, x_0) \\
&< (k^n + k^{n+1} + \cdots)d(Tx_0, x_0) \\
&= \frac{k^n}{1-k}d(Tx_0, x_0).
\end{aligned}$$

Then $\{T^n x_0\}$ is a Cauchy sequence in X. Since X is complete, there exists $p \in X$ such that $\lim_{n\to\infty} T^n x_0 = p$. Since $x_0 \le Tx_0 \le T^2 x_0 \le \cdots \le T^n x_0 \le T^{n+1} x_0 \le \cdots$ and $T^n x_0 \to p$, we have $T^n x_0 \le p$ for all n. Then we have

$$\begin{aligned} d(Tp, p) &\le d(Tp, T^{n+1}x_0) + d(T^{n+1}x_0, p) \\ &\le kd(p, T^n x_0) + d(T^{n+1}x_0, p). \end{aligned}$$

As $n \to \infty$, we have $d(Tp, p) = 0$. Hence we have $Tp = p$.

Finally, we show the uniqueness of fixed points of T. Let q is another fixed point of T. If q is comparable to p, then $T^n q = q$ is comparable to $T^n p = p$ for any n. Then we have

$$d(p, q) = d(T^n p, T^n q)) \le k^n d(p, q),$$

which implies $d(p, q) = 0$.

If q is not comparable to p, then there exists $z \in X$ comparable to p and q. Then $T^n z$ is comparable to $T^n p = p$ and $T^n q = q$ for all n. Then we have

$$\begin{aligned} d(p, q) &\le d(T^n p, T^n z) + d(T^n z, T^n q)) \\ &\le k^n d(p, z) + k^n d(z, q). \end{aligned}$$

As $n \to \infty$, we have $d(p, q) = 0$. □

3 Fixed point theorem for Kannan mappings

In this section, we consider Kannan mappings in partially ordered sets.

Let X be a partially ordered set with a metric d and let T be a mapping from X into itself. To prove the uniqueness of fixed point of Kannan mappings in partially ordered sets, we assume that X satisfies the following;

for any x, y there exists z with $z \le Tz$, which is comparable to x, y. (1)

Theorem 2. *Let X be a partially ordered set with a metric d such that (X, d) is a complete metric space. If a nondecreasing sequence $\{x_n\}$ converges to x, then we have $x_n \le x$ for any n. Let T be a monotone nonincreasing mapping from X into itself such that there exists $\alpha \in [0, \frac{1}{2})$ such that for any $x, y \in X$,*

$$x \ge y \text{ implies } d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty).$$

Assume that there exists x_0 in X with $x_0 \le Tx_0$. Then there exists a fixed point of T. Moreover, if X satisfies (1), the fixed point of T is unique.

Proof. Since $x_0 \leq Tx_0$ and T is monotone nondecreasing, we obtain that

$$x_0 \leq Tx_0 \leq T^2x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \cdots.$$

Then we have

$$d(T^nx_0, T^{n-1}x_0) \leq \alpha d(T^{n-1}x_0, T^nx_0) + \alpha d(T^{n-2}x_0, T^{n-1}x_0).$$

Thus $(1-\alpha)d(T^{n-1}x_0, T^nx_0) \leq \alpha d(T^{n-2}x_0, T^{n-1}x_0)$ holds. Therefore we have

$$d(T^{n-1}x_0, T^nx_0) \leq \frac{\alpha}{1-\alpha} d(T^{n-2}x_0, T^{n-1}x_0)$$

for any n. Then we have

$$\begin{aligned} d(T^nx_0, T^{n+1}x_0) &\leq \frac{\alpha}{1-\alpha} d(T^{n-1}x, T^nx) \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^2 d(T^{n-2}x_0, T^{n-1}x_0) \\ &\leq \cdots \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, Tx_0). \end{aligned}$$

Therefore we obtain that for any n,

$$d(T^nx_0, T^{n+1}x_0) \leq \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, Tx_0).$$

Then $\{T^nx_0\}$ is a Cauchy sequence in X. In fact, for $n \leq m$, we have

$$\begin{aligned} & d(T^mx_0, T^nx_0) \\ & \leq d(T^mx_0, T^{m-1}x_0) + \cdots + d(T^{n+1}x_0, T^nx_0) \\ & \leq \left(\frac{\alpha}{1-\alpha}\right)^{m-1} d(x_0, Tx_0) + \cdots + \left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, Tx_0) \\ & < \frac{1-\alpha}{1-2\alpha}\left(\frac{\alpha}{1-\alpha}\right)^n d(x_0, Tx_0). \end{aligned}$$

Therefore we have $d(T^mx_0, T^nx_0) \to 0$. Since X is complete, there exists $p \in X$ such that $\lim_{n\to\infty} T^nx_0 = p$. Since $T^nx_0 \to p$ and $\{T^nx_0\}$ is nondecreasing, we obtain that $T^nx_0 \leq p$ for any n. Then we have

$$d(Tp, T^{n+1}x_0) \leq \alpha d(p, Tp) + \alpha d(T^nx_0, T^{n+1}x_0).$$

As $n \to \infty$, we have

$$d(Tp, p) \leq \alpha d(p, Tp).$$

So we have $(1-\alpha)d(p, Tp) \leq 0$. Thus $d(p, Tp) \leq 0$ holds. Hence we have $Tp = p$.

Next we show the uniqueness of fixed points of T. We assume that X satisfies (1) and $q \in X$ is another fixed point of T.

If p is comparable to q, then $p \geq q$ implies $T^n p \geq T^n q$. Thus $T^n p = p$ is comparable to $T^n q = q$ for any n. Then we have

$$\begin{aligned}
d(p, q) &= d(T^n p, T^n q) \\
&\leq \alpha d(T^{n-1}p, T^n p) + \alpha d(T^{n-1}q, T^n q) \\
&\leq \alpha \left(\frac{\alpha}{1-\alpha}\right)^{n-1} d(p, Tp) + \alpha \left(\frac{r}{1-r}\right)^{n-1} d(q, Tq)
\end{aligned}$$

for any n. As $n \to \infty$, we have $p = q$.

If p is not comparable to q. By (1), for p and q, there exists $z \in X$ such that $z \leq Tz$ and z is comparable to p, q. Since $Tz \leq z$ and T is monotone nondecreasing, we obtain that

$$z \leq Tz \leq T^2 z \leq \cdots \leq T^n z \leq T^{n+1} z \leq \cdots.$$

Then we have

$$d(T^{n-1}z, T^n z) \leq \frac{\alpha}{1-\alpha} d(T^{n-2}z, T^{n-1}z)$$

for any n. Then we have

$$\begin{aligned}
d(p, q) &= d(T^n p, T^n q) \\
&\leq d(T^n p, T^n z) + d(T^n z, T^n q) \\
&\leq \alpha(d(T^{n-1}p, T^n p) + d(T^{n-1}z, T^n z)) \\
&\quad + \alpha(d(T^{n-1}z, T^n z) + d(T^{n-1}q, T^n q)) \\
&= 2\alpha d(T^{n-1}z, T^n z) \\
&\leq 2\alpha \cdot \frac{\alpha}{1-\alpha} d(T^{n-2}z, T^{n-1}z) \\
&\leq \cdots \\
&\leq 2\alpha \left(\frac{\alpha}{1-\alpha}\right)^{n-1} d(z, Tz).
\end{aligned}$$

As $n \to \infty$, we have $d(p, q) = 0$. Hence we have $p = q$. □

The following mappings satisfy conditions of Theorem 2.

Example 3. *Let* $X = \{0,1,2\}$ *and the distance function* p *is the ordinary Euclidean distance on the line. Let* T *be a mapping of* X *into itself defined by* $Tx = 1$ *for* $x \in X$. *Then* T *is a monotone nondecreasing mapping satisfying* (1). *Moreover if we take* $\alpha = \frac{1}{2}$, *then* $x \geq y$ *implies* $d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty)$.

Example 4. *Let* $X = [0,1]$ *and the distance function* p *is the ordinary Euclidean distance on the line. Let* T *be a mapping of* X *into itself defined by*

$$Tx = \begin{cases} \frac{1}{5}x & 0 \leq x < \frac{1}{2}, \\ \frac{1}{4}x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Here T *is a monotone nondecreasing mapping satisfying* (1). *Moreover if we take* $\alpha = \frac{1}{3}$, *then* $x \geq y$ *implies* $d(Tx,Ty) \leq \alpha d(x,Tx) + \alpha d(y,Ty)$.

Remark. In [4], we apply Theorem 1 to boundary value problems for fourth order differential equations. We want to apply Theorem 2 to some problems for differential equations. This is a further topic.

References

[1] R. Kannan, *Some results on fixed points II*, American Mathematical Monthly, **76**(1969), 405–408.

[2] J J. Nieto and R. R. López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, **22**(2005), 223–239.

[3] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, 2005.

[4] M. Toyoda and T. Watanabe, *Application of a fixed point theorem in partially ordered sets to boundary value problems for fourth order differential equations*, to appear in Proceedings of 8th international conference on Nonlinear Analysis and Convex Analysis, Yokohama Publishers.