数理解析研究所講究録
第 1929 巻 2014 年 100-111

TWO TYPICAL EXAMPLES ON INTEGRABLE GEOMETRY FLOWS

ZHIWEI WU

1. Introduction

Let $\mathcal{M}$ be a manifold of dimension n, and a group G acts on $\mathcal{M}$. An evolution equation for curves in a manifold $\mathcal{M}$ is called a *geometric curve flow*, if the equation can be written as

$$\gamma_t = \xi_0\gamma + \sum_{i=1}^{n}\xi_i e_i.$$

Here $(e_1, e_2, \cdots, e_n) \in G$ is a moving frame along γ under the group action G on $\mathcal{M}$. And all ξ_i's are smooth functions of the invariants (or curvatures) under the action of G on γ. A specific case is the Frenet frame, it is corresponding to the transitive action of $SO(3)$ on $\mathbb{R}^3$. Under the $SO(3)$ action, a generic curve can be determined up to a constant element in $SO(3)$ by two functions k and τ, which are called the curvature and torsion of γ respectively. Let (e_1, e_2, e_3) denote the Frenet frame of γ, then a geometric curve flow in this case can be written as

$$\gamma_t = \xi_0(k,\tau)\gamma + \sum_{i=1}^{3}\xi_i(k,\tau)e_i.$$

Different group actions usually leads to different type of moving frames. Let $\gamma : \mathbb{R} \to \mathbb{R}^2\backslash\{0\}$ be a parameterized curve such that $\Delta = \det(\gamma, \gamma_x)$ never vanishes, set $e_2 = \frac{\gamma_x}{\Delta}$, then $g = (\gamma, e_2) \in SL(2,\mathbb{R})$ is a moving frame for γ under the affine action. Moreover, the structure equation of g is

$$g_x = g\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}.$$

It can be proved that q, r form a set of differential invariants for $\gamma \in \mathbb{R}^2\backslash\{0\}$ under $SL(2,\mathbb{R})$ action. Here there exist two invariants because we do not specify the parameter. Consider a third ordered differential equation for γ:

$$\gamma_t = -\frac{1}{4}(q_x r - q r_x)\gamma - \frac{1}{4}(2qr^2 - r_{xx})\gamma_x.$$

From the previous setting, this is a geometric curve flow. Its solvability condition is equivalent to a couple PDE system in affine curvatures:

$$\begin{cases} q_t = \frac{1}{4}(q_{xxx} - 6qrq_x), \\ r_t = \frac{1}{4}(r_{xxx} - 6qrr_x). \end{cases}$$

This is the third flow in a soliton hierarchy–the AKNS hierarchy [1].

Fixing the parameter such that $\det(\gamma, \gamma_x) \equiv 1$, then there is a natural moving frame (γ, γ_x) with only one invariant q such that $\gamma_{xx} = q\gamma$. Pinkall consider the following geometric curve flow [15]:

$$\gamma_t = \frac{1}{4} q_x \gamma - \frac{1}{2} q \gamma_x. \tag{1.1}$$

It is solvable if and only if q is a solution of the KdV equation:

$$q_t = \frac{1}{4}(q_{xxx} - 6qq_x), \tag{1.2}$$

This article is based on the talk I gave in RIMS workshop on "Development of group actions and submanifold theory" (06/25/2014– 06/27/2014). In this article, we will focus on two examples connecting integrable systems and geometric curve flows. One of the equation is the nonlinear Schrödinger equation (NLS), and the other one is the KdV equation. They are two classical equations in the soliton theory literature.

2. Nonlinear Schrödinger Equation and Vortex Filament Equation

Let $\gamma(x, t) : \mathbb{R}^2 \to \mathbb{R}^3$ be a family of curves, and $\{e_1(\cdot, t), e_2(\cdot, t), e_3(\cdot, t)\}$ the Frenet frame of $\gamma(\cdot, t)$. Let $k(\cdot, t)$ and $\tau(\cdot, t)$ be the corresponding curvature and torsion. We say an evolution equation for $\gamma(x, t)$ is a geometric flow if it can be written as

$$\gamma_t = F_0(k, \tau)\gamma + \sum_{i=1}^{3} F_i(k, \tau) e_i, \tag{2.1}$$

where $\{F_i \mid 0 \le i \le 3\}$ are differential polynomials in k and τ with respect to arc-length parameter.

Remark 2.1. Given any curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ such that $||\gamma_x|| > 0$ for $\forall x \in \mathbb{R}$. The Frenet frame and k, τ are rational functions of $\gamma, \partial_x\gamma, \partial_{xx}\gamma$. $\{k, \tau\}$ is also called a differential invariants for the curve. Therefore, (2.1) is a partial differential equation for curves in $\mathbb{R}^3$.

Da Rios modeled the movement of a thin tube in viscous fluid as an evolution equation for curve in $\mathbb{R}^3$:

$$\gamma_t = \gamma_x \times \gamma_{xx}. \tag{2.2}$$

This equation is called Vortex Filament equation (VFE).

One of the important properties of VFE is that it is arc-length preserving. That is, if $\gamma(x, t)$ is a solution of VFE and $\gamma(\cdot, 0)$ is parameterized by arc-length, so is $\gamma(\cdot, t)$ for all t. Therefore, (2.2) can be rewritten in terms of the Frenet frame $\{e_1(\cdot, t), e_2(\cdot, t), e_3(\cdot, t)\}$:

$$\gamma_t = \gamma_x \times \gamma_{xx} = e_1 \times k e_2 = k e_3.$$

In other words, the curve evolves along the bi-normal direction e_3 with curvature k as the speed. If we consider the initial value problem for (2.2) such that $\gamma(x,0)$ is a circle, then under the flow it will evolve as a smoke ring. That is why sometime this equation is also called the smoke ring equation.

In 1972, Hasimoto pointed out the connection between the VFE and the NLS:

Theorem 2.2. *([11]) If $\gamma(x,t)$ is a solution to VFE (2.2) with curvature $k(x,t)$ and torsion $\tau(x,t)$, then there exists a function $c(t)$ of t such that*

$$q(x,t) = k(x,t)e^{i(\int_0^x \tau(s,t)ds + c(t))}$$

is a solution of the nonlinear Schrödinger equation:

$$q_t = i(q_{xx} + \frac{1}{2}|q|^2 q).$$

Remark 2.3. The formula $q(x,t) = k(x,t)e^{i(\int_0^x \tau(s,t)ds+c(t))}$ is called the Hasimoto transformation.

Hasimoto transformation provides a explicit formula from solutions of VFE to NLS. But on the other hand, given a solution of the NLS, it is not clear from the formula that how to construct a curve flow solution for the VFE. Therefore, we need to explore more on the NLS.

From integrable system point of view, NLS is equivalent to the following $su(2)$-value flat connection 1-from (Lax pair):

$$\theta_\lambda = (a\lambda + u)dx + Q(u,\lambda)dt, \tag{2.3}$$

where $a = \mathrm{diag}(i,-i)$, $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$, $q \in C^\infty(\mathbb{R},\mathbb{C})$, and

$$Q(u,\lambda) = \begin{pmatrix} i\lambda^2 - \frac{i}{2}|q|^2 & q\lambda + \frac{i}{2}q_x \\ -i\bar{q}\lambda + \frac{i}{2}\bar{q}_x & -i\lambda^2 + \frac{i}{2}|q|^2 \end{pmatrix}.$$

Note that if we carry out the computation, the NLS is of the form:

$$q_t = i(q_{xx} + |q|^2 q). \tag{2.4}$$

We will see the reason using this equation in the following part of this section.

We call $E(x,t,\lambda) \in SU(2)$ an *extended* frame of q if $E(x,t,\lambda)$ is a parallel frame of θ_λ. That is, $E(x,t,\lambda)$ satisfies the following equations:

$$\begin{cases} E(x,t,\lambda)_x = E(x,t,\lambda)(a\lambda + u), \\ E(x,t,\lambda)_t = E(x,t,\lambda)Q(u,\lambda). \end{cases}$$

Associate $su(2)$ with the following inner product: $\langle X,Y\rangle = -\frac{1}{2}Tr(XY)$, and consider the isomorphism from $su(2)$ to $\mathbb{R}^3$: let

$$a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad b = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

Then under the inner product, (a, b, c) forms an ordered orthonormal basis for $su(2)$. Moreover, we note that

$$[a, b] = 2c, \quad [b, c] = 2a, \quad [c, a] = 2b.$$

Hence we can make the following identification between $\mathbb{R}^3$ and $su(2)$:

$$e_1 = (1, 0, 0)^t \mapsto a, \quad , e_2 = (0, 1, 0)^t \mapsto b, \quad e_3 = (0, 0, 1)^t \mapsto c,$$

and

$$x \times y \to \frac{1}{2}[X, Y], \quad x, y \in \mathbb{R}^3, \quad X, Y \in su(2).$$

Pohlmeyer-Sym construction (cf. [14], [17]) presents a converse version of the Hasimoto transformation:

Theorem 2.4. [Sym's formula]
Let q be a solution of the NLS equation (2.4) and $E(x, t, \lambda)$ is an extended frame for q. Then $\gamma(x, t) = \frac{\partial E}{\partial \lambda} E^{-1}(x, t, \lambda) \mid_{\lambda=0}$ is a solution of VFE of the form

$$\gamma_t = \frac{1}{2}\gamma_x \times \gamma_{xx}. \tag{2.5}$$

Moreover, if F is another frame for q, then $\tilde{\gamma} = \frac{\partial F}{\partial \lambda} F^{-1}(x, t, \lambda) \mid_{\lambda=0}$ is also a solution of (2.5). And there exists $A_0 \in O(3)$, $b_0 \in \mathbb{R}^3$ constant, such that $\tilde{\gamma} = A_0\gamma + b_0$.

Proof. This proof is by direct computation. Let $g(x, t) = E(x, t, 0)$, then

$$\gamma_t = (\frac{\partial E}{\partial \lambda} E^{-1}(x, t, \lambda) \mid_{\lambda=0})_t = (\frac{\partial}{\partial_t}(\frac{\partial E}{\partial \lambda})E^{-1}) \mid_{\lambda=0} - \frac{\partial E}{\partial \lambda} E^{-1} E_t E^{-1} \mid_{\lambda=0}$$
$$= gug^{-1}$$

Similarly,

$$\gamma_x = gag^{-1}, \quad \gamma_{xx} = g[u, a]g^{-1}.$$

Therefore, from the isometric between $su(2)$ and $\mathbb{R}^3$, we have

$$\gamma_x \times \gamma_{xx} = \frac{1}{2} g[a, [u, a]]g^{-1} = 2gug^{-1} = 2\gamma_t.$$

This proves the theorem. □

Note that if $\gamma = \frac{\partial E}{\partial \lambda} E^{-1}(x, t, \lambda) \mid_{\lambda=0}$, then $\gamma_x = gag^{-1}$, where $g = E(x, t, 0)$ is the value of extended frame at $\lambda = 0$. It is a unit vector in $\mathbb{R}^3$. Therefore, x is the arc-length parameter. From direct computation,

$$(\gamma_x)_x = g[g^{-1}g_x, a]g^{-1} = 2\mathrm{Re}(q)b + 2\mathrm{Im}(q)c$$

In other words, $(gag^{-1}, gbg^{-1}, gcg^{-1})$ is a *parallel frame* along γ with principle curvatures $k_1 = 2Re(q)$ and $k_2 = 2\mathrm{Im}(q)$. Therefore, if γ is a solution of $\gamma_t = \frac{1}{2}\gamma_x \times \gamma_{xx}$, and k_1, k_2 its principle curvatures, then $q := \frac{1}{2}(k_1 + k_2 i)$ is a solution of the NLS equation:

$$q_t = i(q_{xx} + |q|^2 q). \tag{2.6}$$

3. Central affine plane curves and the KdV equation

In this section, we discuss the geometric explanation of the KdV equation:

$$q_t = \frac{1}{4}(q_{xxx} - 6qq_x). \tag{3.1}$$

In this part, we will give a brief scheme of constructing commutative geometric curve flows from the KdV hierarchy. More details can be found in [15], [20].

As we mentioned before, the moving frame depends on the group action on the space. On the plane, instead of the rigid motion, we consider the affine action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2\backslash\{0\}$ such that $A \cdot y = Ay$ for $A \in SL(2,\mathbb{R})$ and $y \in \mathbb{R}^2\backslash\{0\}$. Then the moving frame g of a curve $\gamma \in \mathbb{R}^2\backslash\{0\}$ belongs to $SL(2,\mathbb{R})$, and the invariant set $g^{-1}g_x \in sl(2,\mathbb{R})$.

Given a smooth curve $\gamma(s)$ in $\mathbb{R}^2\backslash\{0\}$, if $\det(\gamma,\gamma_s)$ never vanishes, we can change the parameter s to x ($\frac{ds}{dx} = \det(\gamma,\gamma_s)^{-1}$) such that $\det(\gamma,\gamma_x) = 1$. Such parameter is called the *central affine arc-length parameter* for γ. Take the derivative of $\det(\gamma,\gamma_x) = 1$ with respect to x, then $\det(\gamma,\gamma_{xx}) = 0$. Hence there exists a unique smooth function q such that

$$\gamma_{xx} = q\gamma.$$

This q is called the *central affine curvature* of γ. From the uniqueness of ordinary differential equations, $\{q\}$ is a complete set of local invariants of curves in $\mathbb{R}^2\backslash\{0\}$ under the affine action.

Let $I = S^1$ or $\mathbb{R}$, denote

$$\mathcal{M}_2(I) = \{\gamma : I \to \mathbb{R}^2 \mid \det(\gamma,\gamma_x) = 1\}.$$

Then the tangent space of $\mathcal{M}_2(I)$ at γ is of the form:

$$T_\gamma\mathcal{M}_2(I) = \{\tilde{\xi} = -\frac{\xi_x}{2}\gamma + \xi\gamma_x \mid \xi \in C^\infty(I,\mathbb{R})\}.$$

An equation on $\mathcal{M}_2(I)$ is called a *central affine curve flow* if

$$\gamma_x = -\frac{\xi_x}{2}\gamma + \xi\gamma_x,$$

where ξ is a differential polynomial of q. In [15], Pinkall show that if $\gamma \in \mathcal{M}_2(\mathbb{R})$ is a solution of the following equation:

$$\gamma_t = \frac{1}{4}q_x\gamma - \frac{1}{2}q\gamma_x, \tag{3.2}$$

then its central affine curvature q is a solution of the KdV equation:

$$q_t = \frac{1}{4}(q_{xxx} - 6qq_x). \tag{3.3}$$

There is a natural S^1 action on $\mathcal{M}_2(S^1)$ such that $e^{i\theta}\cdot\gamma(x) = \gamma(x+\theta)$. Then this equation is the Hamiltonian for $H(q) = \frac{1}{2}\oint q\,dx$ with respect to the following symplectic form on $\mathcal{M}_2(S^1)/S^1$:

$$\omega_\gamma(\tilde{\xi},\tilde{\eta}) = \oint \det(\tilde{\xi},\tilde{\eta})dx = -\oint \xi_x \eta dx.$$

Periodic solutions in x-part of (3.2) with finite-gap central affine curvature is studied in [3]. Higher-order central affine curve flows, conservation law and bi-Hamiltonian structure are given in [4, 5, 8, 9].

As in the previous section, it is natural to ask whether we can construct central affine curve flows from the knowledge of KdV equation. In the soliton theory literature, Lie algebra splitting theory is a powerful method to generate soliton equations (cf. [1], [7], [12], [16], [19] and etc.)

I. There are several ways to derive the KdV equation. First we take a look of the construction from certain constrain on the $SL(2)$-hierarchy (or the 2×2 ANKS hierarchy [1]).

Let $L(SL(2))$ be the group of smooth loops on $SL(2)$ and $\mathcal{L}(sl(2))$ its Lie algebra. Consider the following splitting of $\mathcal{L}(sl(2))$:

$$\begin{cases} \mathcal{L}(sl(2))_+ = \{\sum_{i\geq 0} A_i\lambda^i \mid A_i \in sl(2)\}, \\ \mathcal{L}(sl(2))_- = \{\sum_{i<0} A_i\lambda^i \mid A_i \in sl(2)\}. \end{cases} \tag{3.4}$$

Let $a = \mathrm{diag}(1,-1)$, given $u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$, set $Q(u,z) = z\lambda + Q_0 + Q_{-1}z^{-1} + \cdots$, then we can solve $Q(u,z)$ uniquely from the following condition:

$$\begin{cases} [\partial_x + az + u, Q(u,z)] = 0, \\ Q^2 = z^2. \end{cases} \tag{3.5}$$

The j-*th flow* in the $SL(2)$-hierarchy is a coupled system for q and r:

$$u_t = [\partial_x + az + u, (Q(u,z)z^{j-1})_+] = [\partial_x + u, Q_{1-j}]. \tag{3.6}$$

For example, the first several terms of $Q(u,z)$ is

$$Q_0 = u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, Q_{-1} = \frac{1}{2}\begin{pmatrix} -qr & -q_x \\ r_x & qr \end{pmatrix},$$
$$Q_{-2} = \frac{1}{4}\begin{pmatrix} q_x r - q r_x & q_{xx} - 2q^2 r \\ r_{xx} - 2qr^2 & q r_x - q_x r \end{pmatrix}.$$

We write down the first three flows:

$$q_{t_1} = q_x, \quad r_{t_1} = r_x,$$
$$q_{t_2} = \frac{1}{2}(q_{xx} - 2q^2 r), \quad r_{t_2} = -\frac{1}{2}(r_{xx} - 2q^2 r),$$
$$q_{t_3} = \frac{1}{4}(q_{xxx} - 6qq_x r), \quad r_{t_3} = \frac{1}{4}(r_{xxx} - 6qrr_x).$$

Proposition 3.1. *The third flow admits the constraints $r = 1$, which gives the KdV equation.*

Lax pair of the KdV equation is the following $sl(2)$-value connection 1-from:

$$\theta_z = \begin{pmatrix} z & q \\ 1 & -z \end{pmatrix} dx + \begin{pmatrix} z^3 - \frac{1}{2}qz + \frac{1}{4}q_x z & qz^2 - \frac{1}{2}q_x z + \frac{1}{4}q_{xx} - \frac{1}{2}q^2 \\ z^2 - \frac{1}{2}q & -z^3 + \frac{1}{2}qz - \frac{1}{4}q_x z \end{pmatrix} dt.$$

II. To construct the Bäcklund transformation, it is more efficient to consider the following splitting which is gauge equivalent to the $SL(2)$-hierarchy. Families of both pure soliton and rational solutions for central affine curvatures and solution of (3.2) are constructed from this splitting [18], [20].

Define $\phi(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. Let

$$\mathcal{L} = \{A(z) \in \mathcal{L}(sl(2)) \mid \phi(z)^{-1}A(z)\phi(z) = \phi(-z)^{-1}A(-z)\phi(-z)\}.$$

Consider the following splitting of $\mathcal{L}$:

$$\mathcal{L}_+ = \{\sum_{i\geq 0} A_i z^i\}, \quad \mathcal{L}_- = \{\sum_{i<0} A_i z^i\}.$$

Then the Lax pair is gauge equivalent to

$$\begin{aligned} \hat{\theta}_z &= \phi^{-1}(z)\theta_z\phi(z) \\ &= \begin{pmatrix} 0 & z^2+q \\ 1 & 0 \end{pmatrix} dx + \begin{pmatrix} \frac{q_x}{4} & z^4 + \frac{q}{2}z^2 + \frac{q_{xx}}{4} - \frac{q^2}{2} \\ z^2 - \frac{q}{2} & -\frac{q_x}{4} \end{pmatrix} dt. \end{aligned}$$

III. In the end, we consider another algebraic structure of the KdV equation from the Drifeld-Sokolov construction [7]. From this construction, the curve flows will rise naturally.

Let $\mathcal{B} : sl(2,\mathbb{R}) \to \mathbb{R}e_{12}$ defined as

$$\mathcal{B}\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

Consider the splitting $(\tilde{\mathcal{L}}_+, \tilde{\mathcal{L}}_-)$ of $\mathcal{L}(sl(2))$ such that for $A(\lambda) = \sum_i A_i\lambda^i \in \mathcal{L}(sl(2))$,

$$A(\lambda)_+ = \sum_{i>0} A_i\lambda^i + A_0 - \mathcal{B}(A_{-1}). \tag{3.7}$$

Let $J = e_{12}\lambda + e_{21}$, given a smooth function $q : \mathbb{R} \to \mathbb{R}$, let $u = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$, there exists unique $Q(q,\lambda) = e_{12}\lambda + \sum_{i\geq 0} Q_i(q)\lambda^{-i}$ such that

$$\begin{cases} [\partial_x + J + u, Q(q,\lambda)] = 0, \\ Q(q,\lambda)^2 = \lambda I_2. \end{cases} \tag{3.8}$$

The $(2j+1)$*-th flow in the KdV hierarchy* is

$$u_{t_{2j+1}} = [\partial_x + J + u, (Q(q,\lambda))_+^{2j+1}]. \tag{3.9}$$

In fact, set $Q_j(q) = \begin{pmatrix} A_j & B_j \\ C_j & -A_j \end{pmatrix}$, then the previous recursive formula induces:

$$\begin{aligned} C_{j+1}(q) &= -((A_j(q))_x + qC_j(q) - B_j(q)), \\ A_{j+1}(q) &= \frac{1}{2}((B_j(q))_x - qA_j(q), \\ A_j(q) &= -\frac{1}{2}(C_j(q))_x. \end{aligned} \tag{3.10}$$

Then written in q, the $2j+1$-th flow (3.9) is

$$q_{t_{2j+1}} = (B_j(q) - C_{j+1}(q))_x - 2qA_j(q).$$

In particular, the first, third, and fifth flows are:

$$q_{t_1} = q_x, \tag{3.11}$$

$$q_{t_3} = \frac{1}{4}(q_{xxx} - 6qq_x), \tag{3.12}$$

$$q_{t_5} = \frac{1}{16}(\partial_x^5 q - 10\, q\partial_x^3 q - 20(\partial_x q)\partial_{xx} q + 30\, q^2 \partial_x q). \tag{3.13}$$

Note that (3.12) is the KdV equation.

Now we can write down Commuting higher order central affine curve flows for (3.2). Let $E(x,t,\lambda) \in SL(2,\mathbb{R})$ be an extended frame for the $2j+1$-th flow in the KdV hierarchy, that is

$$\begin{cases} E_x(x,t,\lambda) = E(x,t,\lambda) \begin{pmatrix} 0 & \lambda + q \\ 1 & 0 \end{pmatrix}, \\ E_t(x,t,\lambda) = E(x,t,\lambda)(e_{12}\lambda^{j+1} + \cdots + Q_j(q)). \end{cases} \tag{3.14}$$

Let $g(x,t) = E(x,t,0)$, then from the first equation of (3.14), $g(x,t) = (\gamma, \gamma_x)$, and $\gamma \in \mathcal{M}_2(I)$.

Recall that $\tilde{\xi}(\gamma) = \xi_1\gamma + \xi_2\gamma_x$ is a tangent vector field on $\mathcal{M}_2(I)$ if and only if $\xi_1 = -\frac{1}{2}(\xi_2)_x$. So it follows from (3.10) that $A_j(q)\gamma + C_j(q)\gamma_x$ is tangent to $\mathcal{M}_2(I)$ at γ and

$$\gamma_{t_{2j+1}} = A_j(q)\gamma + C_j(q)\gamma_x = -\frac{1}{2}(C_j(q))_x\gamma + C_j(q)\gamma_x \tag{3.15}$$

is a central affine curve flow on $\mathcal{M}_2(I)$ of order $2j+1$, where $Q_j(q) = \begin{pmatrix} A_j(q) & B_j(q) \\ C_j(q) & -A_j(q) \end{pmatrix}$ is the coefficient of λ^{-j} of the solution $Q(q,\lambda)$ of (3.8). We call this the $(2j+1)$-*th central affine curve flow* on $\mathcal{M}_2(I)$.

For example, the first, third, and fifth central affine curve flow on $\mathcal{M}_2(I)$ is

$$\gamma_{t_1} = \gamma_x,$$
$$\gamma_{t_3} = \frac{1}{4}q_x\gamma - \frac{1}{2}q\gamma_x,$$
$$\gamma_{t_5} = \frac{1}{16}(q_{xxx} - 6qq_x)\gamma + \frac{1}{8}(3q^2 - q_{xx})\gamma_x.$$

Note that the third central affine curve flow is the curve flow (3.2).

The concept of central affine curve can be generalized to higher dimensional case naturally. If $\gamma : \mathbb{R} \to \mathbb{R}^n\backslash\{0\}$ is a smooth curve such that $\det(\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)}) > 0$, then there exists a parameter $x = x(s)$ unique up to translation such that

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) \equiv 1.$$

Take x-derivative of the above equation to see that

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-2)}, \gamma_x^{(n)}) \equiv 0.$$

Therefore, $\gamma_x^{(n)} = u_1\gamma + u_2\gamma_x + \cdots + u_{n-1}\gamma_x^{(n-2)}$, for some smooth functions $u_1, \cdots, u_{n-1}$. Such parameter x is called the *central affine arc-length parameter*, and $u_1, \ldots, u_{n-1}$ are the *central affine curvatures.* The frame $g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)})$ is called the *central affine moving frame along* γ (cf. [4], [21]). Moreover,

$$g^{-1}g_x = b + u, \quad b = \sum_{i=1}^{n-1} e_{i+1,i}, \quad u = \sum_{i=1}^{n-1} u_i e_{in}.$$

Let

$$\mathcal{M}_n(\mathbb{R}) = \{\gamma \in C^\infty(\mathbb{R}, \mathbb{R}^n\backslash\{0\}) \mid \det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) = 1\}.$$

Define $\Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n)$, $V_n = \oplus_{i=1}^{n-1}\mathbb{R}e_{in}$, to be the *central affine curvature map* by

$$\Psi(\gamma) = u = \sum_{i=1}^{n-1} u_i e_{in},$$

where $u_1, \ldots, u_{n-1}$ are the central affine curvatures of γ.

In [21], we construct a sequence of commuting higher order central affine curves on $\mathcal{M}_n(\mathbb{R})$ such that the second flow is

$$\gamma_t = -\frac{2}{n}u_{n-1}\gamma + \gamma_{xx}.$$

These equations are all integrable, in the sense that under the central affine curvature map Ψ, $u = \sum_{i=1}^{n-1} u_i e_{in}$ satisfies equations belonging to the Gelfand-Dickey (GD_n) hierarchy (or the $A_{n-1}^{(1)}$-KdV hierarchy) [6]. From the Cauchy problems for the GD_n-hierarchy with rapidly decaying initial data and periodic initial data(cf. [2], [10], [13]), we can solve the Cauchy problem for the central affine curve flows with periodic initial data and with rapidly decaying initial data (i.e., the central affine curvatures are rapidly decaying).

Furthermore, these flows are all integrable, we obtain a bi-Hamiltonian structure and a sequence of Poisson structures $\{\,,\}_j^\wedge$ on $\mathcal{M}_n(S^1)$ for the central affine curve flow hierarchy. We prove that these Poisson structures arise naturally from the Poisson structures of certain co-adjoint orbits.

Let $\mathcal{D}$ be the algebra of pseudo-differential operators $\sum_i f_i\partial_x^i$, where f_i's are smooth functions on $\mathbb{R}$. Consider the following splitting of $\mathcal{D}$:

$$\mathcal{D}_+ = \{\sum_{i\geq 0} f_i\partial_x^i\}, \quad \mathcal{D}_- = \{\sum_{i<0} f_i\partial_x^i\}.$$

The GD_n-hierarchy is generated by n-th order differential operator:

$$L = \partial^n - \sum_{i=1}^{n-1} u_i\partial_x^{i-1},$$

and the j-th ($j \neq 0 \mod (n)$) flow is a PDE system for u_i's:

$$L_{t_j} = [L_+^{\frac{j}{n}}, L].$$

For example, the second flow in the GD_3-KdV hierarchy is

$$\begin{cases} (u_1)_t = (u_1)_{xx} - \frac{2}{3}(u_2)_{xxx} + \frac{2}{3}u_2(u_2)_x, \\ (u_2)_t = -(u_2)_{xx} + 2(u_1)_x. \end{cases} \tag{3.16}$$

The second flow in the GD_4-KdV hierarchy is the following system for $u = u_1e_{14} + u_2e_{24} + u_3e_{34}$:

$$\begin{cases} (u_1)_t = (u_1)_{xx} - \frac{1}{2}u_3^{(4)} + \frac{1}{2}u_3(u_3)_{xx} + \frac{1}{2}u_2(u_3)_x, \\ (u_2)_t = 2(u_1)_x + (u_2)_{xx} - 2(u_3)_{xxx} + u_3(u_3)_x, \\ (u_3)_t = u_1 + 2(u_2)_x - 2(u_3)_{xx}. \end{cases} \tag{3.17}$$

And the third flow is

$$(u_1)_t = \frac{3}{8}(u_3^{(5)} - 2u_2^{(4)} - u_3^{(3)}u_3 + 2u_2^{(2)}u_3 - u_3^{(2)}u_2 + 2u_2'u_2 - 2u_1'u_3) + u_1^{(3)},$$

$$(u_2)_t = \frac{3}{4}u_3^{(4)} - 2u_2^{(3)} + 2u_1^{(2)} + u_1u_3 + \frac{3}{4}(u_2u_3)_x,$$

$$(u_3)_t = \frac{1}{4}u_3^{(3)} - \frac{3}{2}u_2^{(2)} + u_1' + \frac{3}{4}u_3'u_3.$$

In [21], we give a systematic method to construct higher ordered commuting central affine curve flows on $\mathcal{M}_n(I)$. Here we give some examples.

Example 3.2. [Higher order central affine curve flows]
(1) For $n \neq 3$, the third central affine curve flow on $\mathcal{M}_n(\mathbb{R})$ is the flow:

$$\gamma_t = \left(-\frac{3}{n}u_{n-2} + \frac{3(n-3)}{2n}(u_{n-1})_x\right)\gamma - \frac{3}{n}u_{n-1}\gamma_x + \gamma_{xxx}. \tag{3.18}$$

When $n = 2$, this is the Pinkall's central affine curve flow on $\mathbb{R}^2\backslash\{0\}$,

$$\gamma_t = \frac{1}{4}q_x\gamma - \frac{1}{2}q\gamma_x.$$

So (3.18) is a natural analogue of Pinkall's flow in n-dimension ($n \neq 3$).

When $n = 4$, (3.18) is

$$\gamma_t = (\frac{3}{8}(u_3)_x - \frac{3}{4}u_2)\gamma - \frac{3}{4}u_3\gamma_x + \gamma_{xxx}.$$

(2) The fourth and the fifth central affine curve flows on $\mathcal{M}_3(\mathbb{R})$ are

$$\gamma_t = -\frac{1}{9}(2u_2'' - 3u_1' - 2u_2^2)\gamma + \frac{1}{3}(u_2' - u_1)\gamma_x - \frac{u_2}{3}\gamma_{xx},$$

$$\gamma_t = \frac{1}{9}(-u_1'' + u_1u_2)\gamma - \frac{1}{9}(u_2'' - 3u_1' + u_2^2)\gamma_x + \frac{1}{3}(u_2' - 2u_1)\gamma_{xx}.$$

Acknowledgment.
The author would like to thank Professor Takashi SAKAI from Tokyo Metropolitan University for the kind invitation of RIMS workshop and the supports during his visit last summer.

References

[1] Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H., *The inverse scattering transform - Fourier analysis for nonlinear problems*, Stud. Appl. Math **53** (1974), 249–315.

[2] Beals, R., Deift, P., Tomei, C., *Direct and inverse scattering on the line*, Mathematical Surveys and Monographs, **28** (1988) American Mathematical Society, Providence, RI, xiv+209 pp.

[3] Calini, A., Ivey, T., Marí Beffa,G., *Remarks on KdV-type flows on star-shaped curves*, Phys. D **238(8)** (2009), 788–797.

[4] Calini, A., Ivey, T., Marí Beffa,G., *Integrable flows for starlike curves in centroaffine space*, SIGMA Symmetry Integrability Geom. Methods Appl. **9** (2013) 1–21.

[5] Chou,K.S., Qu, C.Z., *The KdV equation and motion of plane curves*, J. Phys. Soc. Japan **70(7)** (2001), 1912–1916.

[6] Dickey, L.A., *Soliton equations and Hamiltonian systems*, second edition, Advanced Series in Mathematical Physics **26** (2003), World Scientific Publishing Co. Inc., River Edge, NJ.

[7] Drinfel'd, V.G., Sokolov, V.V., *Lie algebras and equations of Korteweg-de Vries type*, (Russian) Current problems in mathematics, **24** (1984), 81–180, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow.

[8] Fujioka, A., Kurose, T., *Hamiltonian formalism for the higher KdV flows on the space of closed complex equicentroaffine curves*, Int. J. Geom. Methods Mod. Phys. **7(1)** (2010), 165-175.

[9] Fujioka, A., Kurose, T., *Multi-Hamiltonian structures on space of closed equicentroaffine plane curves associated to higher KdV flows*, SIGMA Symmetry Integrability Geom. Methods Appl. **10** (2014), 11 pp.

[10] Gardner, C.S., Greene, J.M., Kruskal, M.D., Miura, R.M., *Korteweg-de Vries equation and generalization. VI. Methods for exact solution*, Comm. Pure Appl. Math. **27** (1974), 97–133.

[11] Hasimoto, H., *A soliton on a vertex filament*, J. Fluid Mech. **51(3)** (1972), 477–485.

[12] Jimbo, M., Miwa, T., *Solitons and infinite-dimensional Lie algebras*, Publ. Res. Inst. Math. Sci. **19(3** (1983), 943–1001.

[13] Lax, Peter D., *Periodic solutions of the KdV equation*, Comm. Pure Appl. Math. **28** (1975), 141-188.

[14] Pohlmeyer, K., *Integrable Hamiltonian systems and interactions through quadratic constraints*, Comm. Math. Phys. **46(3)** (1976), 207–221.

[15] Pinkall, U., *Hamiltonian flows on the space of star-shaped curves*, Results Math. **27(3-4)** (1995), 328–332.

[16] Segal, G., Wilson, G., *Loop groups and equations of KdV type*, Inst. Hautes Études Sci. Publ. Math. **61** (1985), 5–65.

[17] Sym, A., *Soliton surfaces and their applications, in: Geomet- rical aspects of the Einstein equations and integrable systems*, Lecture Notes in Physics **239** (1985), 154–231.

[18] Terng, C.L., Uhlenbeck, K., *Bäcklund transformations and loop group actions*, Comm. Pure Appl. Math. **53** (2000), 1–75.

[19] Terng, C.L., Uhlenbeck, K., *The $n \times n$ KdV hierarchy*, JFPTA **10** (2011), 37–61.

[20] Terng, C.L., Wu, Z., *Central affine curve flow on the plane*, JFPTA Festschrift Volume for Mme Choquet-Bruhat textbf14 (2013), 375–396.

[21] Terng, C.L., Wu, Z., *N-dimensional central affine curve flows*, preprint, arXiv:1411.2725.

Department of Mathematics, Ningbo University, Zhejiang, 315211, China., Email: wuzhiwei@nbu.edu.cn