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Why is the Dehornoy ordering so interesting?

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The braid group B_n is a group defined by the presentation

$$B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \left| \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \end{array} \right. \right\rangle.$$

The generators $\sigma_1, \ldots, \sigma_{n-1}$ are often called the *Artin generators.*

A left ordering of a group G is a total ordering $<_G$ of G that satisfies $\gamma\alpha <_G \gamma\beta$ for any $\alpha, \beta, \gamma \in G$ satisfying $\alpha <_G \beta$. A group having at least one left-ordering is called *left-orderable.*

The Dehornoy ordering $<_D$ is a left ordering of the braid group B_n discovered by Dehornoy [Deh], motivated from mathematical logic and set-theory. It turns out that the Dehornoy ordering is a natural but quite stimulating structure of the braid group that reflects several prospects of the braid groups. Nowadays there are more than seven equivalent definitions of the Dehonroy ordering. Each definition reflects and explains how naturally the Dehornoy ordering appears in various points of view. The discovery of the Dehornoy ordering inspired new directions of research including topology, geometry, dynamics, algebra and combinatorics. Moreover, one can use Dehornoy ordering to solve problems in topology.

In this note we give a brief explanation why the Dehornoy ordering is so interesting by giving several seemingly unexpected connections and presenting a new application of the Dehornoy ordering to knot theory.

1 Definitions of the Dehornoy orderings

Among many equivalent definitions of the Dehornoy ordering, we just give two definitions – it is surprising that they yield the same left-ordering of B_n. A standard reference for the Dehornoy ordering is [DDRW], where one can find and learn more diverse aspects.

Definition 1.1 (Dehornoy ordering – algebraic and combinatorial definition). A word w on the Atrin generators $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$ is *σ-positive* if there exists $i \in \{1, \ldots, n-1\}$ that satisfies the following two conditions:

1. w *contains* at least one letter σ_i.
2. w *does not contain* any letters in $\{\sigma_1^{\pm 1}, \ldots, \sigma_{i-1}^{\pm 1}, \sigma_i^{-1}\}$.

For example, $\sigma_2\sigma_3^{-1}$ is σ-positive ($i = 2$, in this case).

We define Dehornoy ordering $<_D$ as follows: For two braids α and β, we define $\alpha <_D \beta$ if $\alpha^{-1}\beta$ is represented by a σ-positive word.

The assertion that $<_D$ is a left-ordering is highly non-trivial. This particularly says that for any non-trivial braid β, either β or β^{-1} is represented by a σ-positive word. To understand the non-triviality of the Dehornoy ordering, the reader is encouraged to compare two braids, for example, $\alpha = \sigma_1^{-4}\sigma_2^3\sigma_1\sigma_2^{-5}\sigma_1^2$ and $\beta = \sigma_1^2\sigma_2^5\sigma_1^2\sigma_2^{-5}$ according to the definition. This will convince the readers that the fact the Dehornoy ordering reflects subtle combinatorics of words.

Definition 1.2 (Dehornoy ordering – topological and geometrical definition). Next we give a geometric definition of the Dehornoy ordering, following [SW]. Here we identify the braid group B_n with the mapping class group of n-punctured disc D_n.

Let $\pi : \widetilde{D_n} \to D_n$ be the universal covering. By equipping an hyperbolic metric on D_n, $\widetilde{D_n}$ can be isometrically embedded into the hyperbolic plane $\mathbb{H}^2$. By attaching points at infinity, we compactify $\widetilde{D_n}$ as a topological disk $\overline{D_n}$. Take a basepoint $* \in \partial D_n$ and its lift $\widetilde{*} \in \pi^{-1}(*)$. Then we may identify $\partial\overline{D_n} - \widetilde{*}$ with the real line $\mathbb{R}$.

For a braid $\alpha \in B_n$, let ϕ be an homeomorphism of D_n that represents α. Take a lift $\widetilde{\phi}$ of ϕ so that $\widetilde{\phi}(\widetilde{*}) = \widetilde{*}$. $\widetilde{\phi}$ extends to the homeomorphism of $\overline{D_n}$. By considering the restriction of $\widetilde{\phi}$ on $\partial\overline{D_n} - \widetilde{*} \cong \mathbb{R}$, we get an orientation-preserving homeomophism $\Theta(\phi) : \mathbb{R} \to \mathbb{R}$. Basic hyperbolic geometry shows that $\Theta(\phi)$ does not depend on a choice of representative homomorphism ϕ, so we get an homeomoprhism

$$\Theta : B_n \to \mathrm{Homeo}^+(\mathbb{R})$$

which is often called the *Nielsen-Thurston action.*

Using the Nielsen-Thurston action, one obtains a left-ordering of B_n in a following manner. Take a point $x \in \mathbb{R} = \partial\overline{D_n} - \widetilde{*}$. For braids α and β, we define $\alpha <_x \beta$ if $[\Theta(\alpha)](x) < [\Theta(\beta)](x)$. By taking x appropriately, the resulting ordering $<_x$ coincides with the Dehornoy ordering $<_D$ defined above.

2 Research inspired by the Dehornoy ordering

As another reason why the Dehornoy ordering is so fascinating, we point out that the discovery and studies of the Dehornoy ordering inspired several new research directions. Here we only give two examples which recently gather much attentions.

- Space of left orderings and isolated orderings

For a group G, let $LO(G)$ be the set of all left-ordering (possibly empty) on G. For $g \in G$, let U_g be the set of left-orderings of G such that g is greater than 1. We equip a topology on $LO(G)$ so that $\{U_g\}_{g \in G}$ forms a sub-basis of a topology, and call $LO(G)$ the *space of left orderings of* G. This space $LO(G)$ plays a fundamental role in a theory of left-orderable groups.

For a countable group G, $LO(G)$ is a totally disconnected, compact, and metrizable. Thus $LO(G)$ is close to Cantor set. One crucial difference is that $LO(G)$ may have isolated points, so it is an important problem to find an example of isolated orderings, isolated points in $LO(G)$. However, constructing an isolated ordering is often very hard.

By modifying Dehornoy ordering, one gets another interesting ordering called the *Dubrovina-Dubovin ordering* [DD]. This ordering is remarkable since it is an isolated ordering. It is interesting to note that the Dehornoy ordering itself is *not* isolated. This explains why the Dehornoy ordering is interesting – it provides a surprising phenomenon that non-isolated points yield isolated points !

- Orderablity of 3-manifold groups

The braid group is one of the most important group in low-dimensional topology. Inspired by the left-orderability of the braid groups, a natural question emerged: Which 3-manifolds have the left-orderable fundamental groups ? Surprisingly, there is a fantastic conjecture on the left-orderability of 3-manifold groups, which predicts unexpected relationships between Gauge theory.

Conjecture. [BGW] The fundamental group of an irreducible 3-manifold M is non-left-orderable if and only if M is an L-space.

Here L-space is a rational homology sphere whose Heegaard Floer homology group is the simplest. The L-space plays an important role in a theory of Heegaard Floer homology, and have various applications in topology and geometry of 3-manifolds.

3 A new application to knot theory

As the author showed in [Ito1, Ito2], the Dehornoy ordering also can be applied to study knots in S^3 – this gives still another explanation why the Dehornoy ordering is so interesting. Here we report a new application of the Dehornoy ordering to knot theory, which seems to be hard to reach by standard techniques in the knot theory. Details can be found in [Ito3].

Theorem 3.1. *[Big, Conjecture 3.2] Let N be a non-trivial normal subgroup of B_n. Then there exists $\beta \in N$ such that the closure of $\beta(\sigma_1 \cdots \sigma_{n-1})$ is a non-trivial knot. (Actually, we have more: for any $M > 0$, there exists $\beta \in N$ such that the closure of $\beta(\sigma_1 \cdots \sigma_{n-1})$ is an hyperbolic knot whose genus is greater than M.*

This theorem, although it sounds quite reasonable, would be hard to prove by standard techniques in knot theory. Usually showing a knot to be non-trivial is done by calculating certain knot invariants. However, calculating knot invariants for general case (for example, assume that β requires 100000 crossings!) is often hard. It is also hard to know, a given knot is indeed hyperbolic though in certain sense, "generic" knots are hyperbolic knots.

Theorem 3.1 is a consequence of the following two theorems. Surprisingly, the first theorem is a purely algebraic statement that concerns the Dehornoy ordering.

Theorem 3.2. *A non-trivial normal subgroup N of B_n is unbounded with respect to the Dehornoy ordering $<_D$: that is, for any $\alpha \in B_n$, there exists $\beta \in K$ such that $\alpha <_D \beta$. Moreover, we can always choose β so that it is pseudo-Anosov.*

We remark that this property is specific for the Dehornoy ordering. There is a left-ordering $<$ on B_n which admits a bounded (convex) normal subgroup. For example, the group extension

$$1 \to [B_n, B_n] \to B_n \to \mathbb{Z} \to 1$$

can be used to construct a left-ordering $<$ of B_n such that for any $\beta \in [B_n, B_n]$, $\sigma_1^{-1} < \beta < \sigma_1$ holds.

The second theorem states relationships between the Dehornoy ordering and knots.

Theorem 3.3. *[Ito1, Ito2] Let K be an oriented knot represented as a closure of an n-braid β.*

1. *If $(\sigma_1\sigma_2\cdots\sigma_{n-1})^{nM} <_D \beta$ then $g(K) > M$.*
2. *If $(\sigma_1\sigma_2\cdots\sigma_{n-1})^{2n} <_D \beta$ and β is pseudo-Anosov, then K is an hyperbolic knot.*

As a consequence, we prove another naturally sounding, but unsolved problem in knot theory. For a complex semi-simple Lie algebra $\mathfrak{g}$, one has a quantum group (quantum enveloping algebra) $U_q(\mathfrak{g})$ and for each $U_q(\mathfrak{g})$-module V we have a linear representation

$$\rho_V : B_n \to \mathrm{GL}(V^{\otimes n})$$

called a *quantum representation*.

For a knot K represented as a closure of an n-braid β, By taking a variant of trace, called a *quantum trace*, of $\rho_V(\beta)$, we get an invariant $Q^V(K)$ of K, called a *quantum V-invariant.*

Jones polynomial is a typical and the most fundamental example of a quantum invariant: This corresponds to the standard 2-dimensional representation of $U_q(\mathfrak{sl}_2)$. Whether Jones polynomial (or, various other quantum invariants like HOMFLY or Kauffman polynomials) detects the unknot or not is one of the most important open problem in knot theory.

By a construction of quantum invariants, it is natural to expect that the quantum V-invariant Q_V fails to detect the unknot if the quantum representation ρ_V is not faithful. Namely, there is a non-trivial knot whose quantum V-invariant agrees with that of the unknot. Theorem 3.2 shows that this is indeed the case.

Theorem 3.4. *Let $\rho_V : B_n \to GL(V^{\otimes n})$ be a quantum representation. If ρ_n is not faithful, then there exists a non-trivial knot K such that the quantum V-invariant of K and the unknot are equal. That is, quantum V-invariant fails to detect the unknot. Moreover, one may choose such K so that it is a hyperbolic knot with arbitrary large genus.*

Proof. If ρ_V is non-faithful, Theorem 3.1 says that there exists $\beta \in \mathrm{Ker}\rho_V$ such that a closure of $\beta(\sigma_1 \cdots \sigma_{n-1})$ is a non-trivial knot (with arbitrary large genus and hyperbolic), say K. Then K and the unknot, the closure of a braid $(\sigma_1 \cdots \sigma_{n-1})$, has the same quantum V-invariants since they have the same image under ρ_V. $\square$

Thus, beyond the theory of braid groups, the Dehornoy ordering is also interesting and provides useful techniques to study knots and links.

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