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RAAGs in Braids

SANG-HYUN KIM* AND THOMAS KOBERDA

1. Right-angled Artin groups

In this article, we survey some of the known results regarding right-angled Artin subgroups of right-angled Artin groups and also of mapping class groups. While doing so, we introduce techniques that can improve given embeddings between these groups to simpler ones, which in turn will help us understanding rigidity of embeddings into such groups.

Definition 1. Let (G, d_G) and (H, d_H) be two groups with metrics. A group homomoprhism $f\colon G \to H$ is called a *quasi-isometric group embedding from G to H* if f is injective and there exists $C \geq 1$ such that for every x and y in G we have

$$d_G(x,y)/C - C \leq d_H(f(x), f(y)) \leq C d_G(x,y) + C.$$

Remark. A finitely generated group will be equipped with a word metric.

For a finite graph Γ, let us define the *RAAG (Right-Angled Artin Group)* on Γ by the group presentation

$$G(\Gamma) = \langle V(\Gamma) \mid [a,b] = 1 \text{ if } \{a,b\} \notin E(\Gamma)\rangle.$$

For example, $G(\bullet) \cong \mathbb{Z}$, $G(\Delta) \cong F_3$ and $G(\text{two edges}) \cong F_2 \times F_2$.

Question 1. *Which groups arise as subgroups of RAAGs?*

Fact.
1. ([9, 7]) *Let S be a closed surface with $\xi(S) < -1$. Then $\pi_1(S)$ admits a quasi-isometric group embedding into some $G(\Gamma)$.*
2. ([1]) *The π_1 of every closed hyperbolic 3–manifold virtually admits a quasi-isometric group embedding into some $G(\Gamma)$.*
3. ([3]) *For $d \geq 2$, there exists a closed hyperbolic d–manifold M_d such that $\pi_1(M_d)$ admits a quasi-isometric group embedding into some RAAG.*

Question 2.
1. *Which groups arise as subgroups of a given $G(\Gamma)$?*
2. *Which RAAGs arise as subgroups of a given $G(\Gamma)$?*

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Universal property of RAAGs

Suppose $\phi_1, \ldots, \phi_n \in \mathrm{Diff}(M)$ for a manifold M, and Γ be the intersection graph of $\{\mathrm{supp}(\phi_1), \ldots, \mathrm{supp}(\phi_n)\}$. Then there exists a "natural" group homomorphism $G(T)$ to $\mathrm{Diff}(M)$.

Question 3. *Which RAAGs arise as subgroups of* Diff(M) *or* Mod(M)*?*

2. RAAGs in RAAGs and in Mods

Notation. For two graphs X and Y, we write $X \leq Y$ if

$$X \subseteq Y \text{ and } EX = EY \cap \binom{VX}{2}.$$

Theorem 2 ([13]). *Let Γ be a finite graph such that $\mathbb{Z}^3 \not\hookrightarrow G(\Gamma)$. Then there exists a combinatorially defined, locally infinite graph Γ^e such that for a finite graph Λ, we have*

$$G(\Lambda) \hookrightarrow G(\Gamma) \Leftrightarrow \Lambda \leq \Gamma^e.$$

Theorem 3 ([12] ($\Rightarrow$), [14] ($\Leftarrow$)). *Let S be a surface possibly with punctures such that $\mathbb{Z}^3 \not\hookrightarrow Mod(S)$. For a finite graph Λ, we have*

$$G(\Lambda) \hookrightarrow Mod(S) \Leftrightarrow \Lambda^{\mathrm{opp}} \leq C(S).$$

Remark.
(1) Not true as-is for rank>2. ([6] for RAAGs, [12] for Mods) But, there is a version for rank>2 using "multi-curves".
(2) Abelian ranks are not the only obstructions for $G(\Gamma) \hookrightarrow Mod(S)$.
(3) Γ^e is a quasi-tree and $G(\Gamma)$ acts on the opposite graph of Γ^e acylindrically. So we have a "canonical" classification of elements in RAAGs (cf. Bowditch).

3. RAAGs on trees

Theorem A ([11]). *For each finite graph Γ, there exists a finite tree T such that $G(\Gamma)$ admits a quasi-isometric group embedding into $G(T)$.*

Here is the recipe. Without loss of generality, we may assume Γ is connected. Consider its universal cover $p\colon \tilde{\Gamma} \to \Gamma$. For a finite subtree T of $\tilde{\Gamma}$, we have a group homomorphism $\phi(\Gamma, T)\colon G(\Gamma) \to G(T)$ defined by

$$\phi(v) = \prod_{u \in p^{-1}(v) \cap T} u.$$

The proof would be complete by showing that for a sufficiently large T, the map $\phi(\Gamma, T)$ is a quasi-isometric group embedding.

4. Application I : RAAGs in braids

We consider the *pure braid group* on n–strands:

$$\begin{aligned} PB_n &= \pi_1(\{(z_1,\dots,z_n) \parallel z_i \neq z_j\}) \\ &= PMod(D^2 \setminus \{p_1,\dots,p_n\}, \partial D^2). \end{aligned}$$

Theorem 4 ([8]). *For each finite planar graph Γ, we have a quasi-isometric group embedding from $G(\Gamma)$ into some pure braid group.*

Corollary B ([11]). *Every RAAG admits a quasi-isometric group embedding from into some pure braid group.*

Actually, one can give a self-contained proof.

proof of Corollary B. We have only to embed $G(T)$ for an arbitrary finite tree T. Consider a collection of disks $\{D_v \quad v \in V(T)\}$ in D^2 such that the intersection graph is T. Puncture D^2 sufficiently many times so that there exists a pseudo-Anosov ψ_v supported on $D_v \setminus \{\text{punctures}\}$. There exists a group homomorphism from $G(T)$ to $\text{PMod}(D^2 \setminus \{\text{punctures}\}) = PB_n$ and this map is a quasi-isometric group embedding, possibly after raising to sufficiently high powers (Clay–Leininger–Mangahas '12). □

Question 4 (Farb). *Is the isomorphism problem solvable for f.p. subgroups of* $\text{Mod}(S)$*?*

Theorem 5 ([4]).
(1) *The isomorphism problem is not solvable for f.p. subgroups of a certain $G(\Gamma_0)$.*
(2) *The isomorphism problem is not solvable for f.p. subgroups of* $\text{Mod}(S_g)$ *for $g >> 0$.*

Corollary 6 ([11]). *The isomorphism problem is not solvable for f.p. subgroups of PB_n for $n >> 0$.*

5. Application II : Symps

We let $\text{Symp}(S^2)$ be the group of area- and orientation-preserving diffeomorphisms (*symplectomorphisms*) of the 2–sphere. For a path $\{\phi_t\}_{t\in I}$ in $\text{Symp}(S^2)$, its L^p–length is defined by

$$\ell_p(\{\phi_t\}) = \int_I \left(\int_{S^2} \left| \frac{\partial \phi_t}{\partial t} \right|^p dx \right)^{1/p}$$

and the (right–invariant) L^p–metric is given by the corresponding length metric.

Theorem C ([11]). *Every RAAG admits a quasi-isometric group embedding into* $(\text{Symp}(S^2), d_p)$ *for $p > 2$.*

Remark.
(1) ([2, 8]) True for D^2 and $p \geq 1$.
(2) Kapovich ('12) showed that every RAAG admits a group embedding into $\text{Symp}(S^2)$ [10].

The proof (independent from Kapovich) goes as follows. From the proof of "RAAGs in braids", we have a quasi-isometric group embedding

$$\begin{tikzcd} G(T) \arrow[r] \arrow[dr, "\text{qi gp emb}"'] & \mathcal{P}_n \arrow[d, "q"] \\ & \mathrm{PMod}(S^2 \setminus \{p_1, p_2, \ldots, p_n\}). \end{tikzcd}$$

Here, $\mathcal{P}_n \leq \mathrm{Symp}(S^2)$ consists of diffeomorphisms that fix some neighborhoods of punctures pointwise. It suffices to show that q does not "contract too much". For this, we consider

$$\begin{tikzcd} \widetilde{\mathrm{Symp}(S^2)} \arrow[d, "p"', "\text{univ. cover, 2-to-1}"] & p^{-1}(\mathcal{P}_n) \arrow[l, "incl."'] \arrow[r] \arrow[d, "\text{2-to-1}"] & PB_n(S^2) \arrow[d, "\text{2-to-1}"] \\ \mathrm{Symp}(S^2) & \mathcal{P}_n \arrow[l, "incl."'] \arrow[r, "q"] & \mathrm{PMod}(S^2 \setminus \{p_1, \ldots, p_n\}) \end{tikzcd}$$

The proof would be complete if the upper-right arrow is shown not to contract much. Use Gauss linking integral. This idea is due to (Benaim–Gambaudo '01, Gambaudo–Ghys '04, Brandenbursky–Shelukin '14).

Theorem A is also used in the proof of the following.

Theorem 7 ([5]). *Every RAAG embeds into* $PL_{\mathrm{Area}}(I^2, \partial I^2)$.

Finally, we consider one less dimsionsion:

Theorem D (Baik–K.–Koberda). *Every RAAG embeds into* $\mathrm{Diff}^\infty(\mathbb{R})$.

Question 5. *Does every RAAG embed into* $\mathrm{Diff}^\infty(S^1)$*?*

Note that every mapping class group embeds into $\mathrm{Homeo}(S^1)$ but not in $\mathrm{Diff}^2(S^1)$. Each RAAG embeds into some mapping class group.

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Department of Mathematical Sciences, KAIST, Daejeon 305-701, Republic of Korea

E-mail address: `shkim@kaist.edu`

Department of Mathematics, Yale University, 20 Hillhouse Ave, New Haven, CT 06520, USA

E-mail address: `thomas.koberda@gmail.com`