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THE EXISTENCE OF A NON SPECIAL ARONSZAJN TREE AND TODORČEVIĆ ORDERINGS

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ABSTRACT. It is proved that it is consistent that every forcing notions with $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$, every Todorčević ordering for any second countable Hausdorff space also has precaliber $\aleph_1$, and there exists a non-special Aronszajn tree. This slightly extends the previous work [16, 18].

1. INTRODUCTION

Martin's Axiom was introduced by Martin and Solovay to solve Suslin's problem in [5]. In 1980's, Todorčević investigated Martin's Axiom from the view point of Ramsey theory, and introduced the countable chain condition for partitions on the set $[\omega_1]^{<\aleph_0}$. In [13], Todorčević and Veličiković proved that $\mathsf{MA}_{\aleph_1}$, which is Martin's Axiom for $\aleph_1$ many dense sets, is equivalent to the statement $\mathcal{K}'_{<\omega}$ that every ccc partition $K_0 \cup K_1$ on $[\omega_1]^{<\aleph_0}$ has an uncountable K_0-homogeneous set. Todorčević also introduced many fragments of $\mathsf{MA}_{\aleph_1}$ in his many papers e.g. [9, 13]. Some of them are as follows[(1)]: $\mathcal{K}_{<\omega}$ is the statement that every ccc forcing notion has precaliber $\aleph_1$. For each $n \in \omega$, $\mathcal{K}_n$ is the statement that every uncountable subset of a ccc forcing notion has an uncountable n-linked subset, and $\mathcal{K}'_n$ is the statement that every ccc partition $K_0 \cup K_1 = [\omega_1]^n$ has an uncountable K_0-homogeneous set. $\mathcal{C}^2$ is the statement that every product of ccc forcing notions has the countable chain condition. We note that they have many applications. For example, $\mathcal{C}^2$ implies Suslin's Hypothesis, every (ω_1, ω_1)-gap is indestructible, and the bounding number $\mathfrak{b}$ is greater than $\aleph_1$, and $\mathcal{K}'_2$ implies that every Aronszajn tree is special. (For other applications, see e.g. [3].) We also note the following diagram of implications

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(1) They are defined by Todorčević in several papers. In [3, Definition 4.9] and [13, §2], $\mathcal{K}_n$'s are defined as statements for ccc forcing notions, however in [4, §4] and [9, §7], $\mathcal{K}_n$'s are defined as statements for ccc partitions. To separate them, we use notation as above. In [13], $\mathcal{K}'_{<\omega}$ above is denoted by $\mathcal{H}$.

A forcing notion $\mathbb{P}$ has precaliber $\aleph_1$ if every uncountable subset I of $\mathbb{P}$ has an uncountable subset I' of I such that every finite subset of I' has a common extension in $\mathbb{P}$. A subset I of a forcing notion $\mathbb{P}$ is called n-linked if every member of the set $[I]^n$ has a common extension in $\mathbb{P}$. A forcing notion $\mathbb{P}$ has property K if every uncountable subset of $\mathbb{P}$ has an uncountable 2-linked subset.

between them:

$$\begin{tikzcd}
& \mathcal{K}_{<\omega} \arrow[r] \arrow[dd, leftrightarrow] \arrow[dl, leftrightarrow] & \cdots \arrow[r] & \mathcal{K}_{n+1} \arrow[r] \arrow[dd] & \mathcal{K}_n \arrow[r] \arrow[dd] & \cdots \arrow[r] & \mathcal{K}_2 \arrow[r] \arrow[dd] & \mathcal{C}^2 \\
\mathsf{MA}_{\aleph_1} \arrow[dr, leftrightarrow] & & & & & & & \\
& \mathcal{K}'_{<\omega} \arrow[r] & \cdots \arrow[r] & \mathcal{K}'_{n+1} \arrow[r] & \mathcal{K}'_n \arrow[r] & \cdots \arrow[r] & \mathcal{K}'_2 &
\end{tikzcd}$$

The equivalence of $\mathsf{MA}_{\aleph_1}$, $\mathcal{K}_{<\omega}$ and $\mathcal{K}'_{<\omega}$ are the theorem due to Todorčević and Veličković [13]. Other implications follows from definitions or trivial argumrnts. It is unknown whether any other implications hold in ZFC.

The author studied about this problem in [14, 15, 16, 17, 18]. In [16, 18], The author introduced the following property on chain conditions [16, 18, Definition 2.6]: A forcing notion $\mathbb{P}$ has the property $\mathsf{R}_{1,\aleph_1}$ if conditions of $\mathbb{P}$ are finite sets of countable ordinals, the order $\leq_{\mathbb{P}}$ is equal to the superset relation $\supseteq$, and for any large enough regular cardinal θ, any countable elementary submodel N of $H(\theta)$, any uncountable subset I of $\mathbb{P}$ which forms a Δ-system with root ν and any $\sigma \in \mathbb{P}$ with $\sigma \cap N = \nu$, there exists an uncountable subset I' of I such that every condition in I' is compatible with σ in $\mathbb{P}$. It is proved that $\mathcal{K}_2(\mathsf{R}_{1,\aleph_1})$(2) also implies that Suslin's Hypothesis holds, every (ω_1, ω_1)-gap is indestructible and $\mathfrak{b} > \aleph_1$. It is also proved that it is consistent that every forcing notion with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$ and there exists a non-special Aronszajn tree. This says that $\mathcal{K}_{<\omega}(\mathsf{R}_{1,\aleph_1})$ doesn't imply $\mathsf{MA}_{\aleph_1}(\mathsf{R}_{1,\aleph_1})$.

In this paper, we slightly develop this result by dealing with not only forcing notions with $\mathsf{R}_{1,\aleph_1}$ but also forcing notions defined due to Todorčević and Balcar-Pazák-Thümmel [10, 1], so called *Todorčević orderings*. Namely, it is shown that it is consistent that every forcing notion with the property $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$, Todorčević orderings for second countable Hausdorff spaces also have precaliber $\aleph_1$, and there exists a non-special Aronszajn tree.

2. Preliminaries

2.1. **Todorčević orderings.** As said in [1], when a topological space is applied to Todorčević ordering, it is natural to require it to be sequential and have the unique limit property. A topological space X is called sequential if for any $Z \subseteq X$, Z is closed in X iff for any $A \subseteq Z$ and $x \in X$ to which A converges, x belongs to Z. A topological space X has the unique limit property if any converging subset of X converges to the unique point. For example, Hausdorff spaces have the unique limit property. For a subset F of a topological space, let F^d denote the first Cantor-Bendixson derivative of F, that is, the set of all accumulation points of F.

Definition 2.1 (Todorčević [10], see also [1, 8]). *For a topological space X, $\mathbb{T}(X)$ is the set of all subsets of X which are unions of finitely many converging sequences*

(2)$\mathcal{K}_2(\mathsf{R}_{1,\aleph_1})$ is the statement that every forcing notion with the property $\mathsf{R}_{1,\aleph_1}$ has property K.

including their limit points, and for each p and q in $\mathbb{T}(X)$, $q \leq_{\mathbb{T}(X)} p$ iff $q \supseteq p$ and $q^d \cap p = p^d$.(3)

For $p, q \in \mathbb{T}(X)$, the statement $q \leq_{\mathbb{T}(X)} p$ means that q is an extension of p (as the subset relation) and the isolated points in p are still isolated in q. $\mathbb{T}(X)$ is called *Todorčević ordering for the space* X in [1, 8] (and [19]).

Todorčević orderings were firstly introduced by Todorčević in [10]. The motivation is to demonstrate a Borel definable ccc forcing which consistently does not have property K. He defined it on a separable metric space. By generalizing it and applying it to other topological spaces, Thümmel discovered a forcing notion which has the σ-finite chain condition but does not have the σ-bounded chain condition, and so he solved the problem of Horn and Tarski [8]. (For Horn-Tarski's problem, see [2, 11].) Right after Thümmel's result, Todorčević introduced a Borel definable solution of the problem of Horn and Tarski [12].

In [12], Todorčević introduced the Borel definable version of Todorčević orderings, which consists of all countable compact subsets whose first Cantor-Bendixson derivative is finite. In [1], Balcar-Pazák-Thümmel introduced a separative version of Todorčević orderings, which consists of all functions f from members p of $\mathbb{T}(X)$ into $\{0, 1\}$ such that $f^{-1}(1)$ is a finite set including p^d as a subset, ordered by the function-extension. In this paper, as in [19], we adopt the definition of Todorčević orderings in Definition 2.1.

Some of Todorčević orderings may not be ccc [1, Theorem 2.3], but many of them are ccc. From the proof of [10], we note that for a space X, if each of finite powers of X is hereditarily separable, then Todorčević ordering for X has the ccc. In [1, Definition 2.1], Balcar-Pazák-Thümmel introduced the property of topological spaces which is a sufficient condition to introduce Todorčević orderings to have the ccc (see also [19]). In this paper, we use the following property of Todorčević orderings.

Lemma 2.2. *For a second countable Hausdorff space X, $\mathbb{T}(X)$ is powerfully ccc, that is, a finite support product of any number of copies of $\mathbb{T}(X)$ has the countable chain condition.*

Proof. It suffices to show that for any $n \in \omega$, the finite support product ${}^n\mathbb{T}(X)$ is ccc. Let I be an uncountable subset of ${}^n\mathbb{T}(X)$. By shrinking I if necessary, we may assume that for each $i < n$, the set $\{{p_i}^d; \langle p_j; j < n\rangle \in I\}$ forms a Δ-system with root d_i. Take a countable elementary submodel N of $H(\theta)$ (for some large enough regular cardinal θ) such that $\{X, I\} \in N$.

Take $\langle p_i; i < n\rangle$ and $\langle q_i; i < n\rangle$ in I(4) such that for each $i < n$,

- $({p_i}^d \setminus \mathrm{d}_i) \cap N = \emptyset$, and

(3)This definition is slightly different from the original one, in [10], which consists of all finite sets σ of convergent sequences in X including their limit points such that for any $A, B \in \sigma$,

$$\lim(A) \notin (B \setminus \{\lim(B)\}),$$

ordered by the reverse inclusion. But essentially, both are same. In fact, both are forcing-equivalent.

(4)Since the set $\{{p_i}^d; \langle p_j; j < n\rangle \in I\}$ forms an uncountable Δ-system for each $i < n$ and N is countable, we can find such a $\langle p_i; i < n\rangle \in I$. Similarly, since the set $N \cup \bigcup_{i<n} p_i$ is countable, we can find such a $\langle q_i; i < n\rangle \in I$.

- $(q_i{}^d \setminus \mathrm{d}_i) \cap (N \cup p_i) = \emptyset$.

Since X is second countable Hausdorff and N is an elementary submodel, there exists a sequence $\langle U_i, V_i; i < n\rangle \in N$ of open subsets of X such that for each $i < n$,

- $U_i \cap V_i = \emptyset$,
- $p_i{}^d \setminus \mathrm{d}_i \subseteq U_i$,
- $q_i{}^d \setminus \mathrm{d}_i \subseteq V_i$, and
- $V_i \cap (p_i \setminus U_i) = \emptyset$.

This can be done because the sets $p_i{}^d \setminus \mathrm{d}_i$, $q_i{}^d \setminus \mathrm{d}_i$ and $p_i \setminus U_i$ are finite and $(q_i{}^d \setminus \mathrm{d}_i) \cap p_i = \emptyset$. By the elementarity of N, there exists $\langle q_i'; i < n\rangle \in I \cap N$ such that for each $i < n$, $(q_i')^d \setminus \mathrm{d}_i \subseteq V_i$. Then for each $i < n$,

$$q_i' \cup p_i \leq_{\mathbb{T}(X)} p_i.$$

Since $q_i' \subseteq N^{(5)}$ and $(p_i{}^d \setminus \mathrm{d}_i) \cap N = \emptyset$ for each $i < n$, we notice that

$$q_i' \cup p_i \leq_{\mathbb{T}(X)} q_i'.$$

Thus the condition $\langle q_i' \cup p_i; i < n\rangle$ is a common extension of conditions $\langle p_i; i < n\rangle$ and $\langle q_i'; i < n\rangle$ in ${}^n\mathbb{T}(X)$. □

2.2. **The chapter IX of [6]: Souslin Hypothesis Does Not Imply "Every Aronszajn Tree Is Special.** In this section, we summarize Shelah's approach to show the consistency that Suslin's Hypothesis holds and there exists a non-special Aronszajn tree. All of definitions and proofsin this section are in [6, IX. Souslin Hypothesis Does Not Imply "Every Aronszajn Tree Is Special].

Definition 2.3 (Shelah, [6, IX 3.3 Definition]). *For an Aronszajn tree T and a subset S of ω_1, T is called S-st-special if there exists a function f from the set $\{t \in T; \mathsf{rk}_T(t) \in S\}$ into ω such that for each $n \in \omega$, the set $f^{-1}[\{n\}]$ forms an antichain in T.*

We note that if S is uncountable and an Aronszajn tree T is S-st-special, then T is still Aronszajn in the forcing extension where S is still uncountable. And then T has an uncountable antichain, hence then T is not a Suslin tree. For a costationary subset S of ω_1, if T is a special Aronszajn tree, then there exists an antichain A through T such that the set $\mathsf{rk}_T[A] \setminus S^{(6)}$ is stationary. Therefore if S is an uncountable costationary subset of ω_1 and T^* satisfies the property

> $(*)$ for every antichain A through T^*, the set $\mathsf{rk}_{T^*}[A] \setminus S$ is nonstationary,

then T^* is a non-special Aronszajn tree.

In [6, IX 4.8 Conclusion], Shelah introduced the iterated proper forcing which forces that Suslin's Hypothesis holds and there are a stationary and costationary subset S of ω_1 and an S-st-special Aronszajn tree T^* which satisfies the property $(*)$. The S-st-speciality of T^* guarantees that T^* is still Aronszajn in any proper forcing extension. To guarantee the property $(*)$ of T^*, we shoot a club on ω_1 for the complement of $\mathsf{rk}_{T^*}[\dot{A}]$ which is disjoint from S in some intermediate stage of the iteration [6, IX 4.7, 4.8]. However, the iteration is required to be a proper forcing. To do this, Shelah introduced the following preservation property.

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(5) q_i' is a countable subset of X.

(6) $\mathsf{rk}_T[A] := \{\mathsf{rk}_T(t); t \in A\}$.

Definition 2.4 (Shelah [6, IX 4.5 Definition]). *Let T be an Aronszajn tree and S a subset of ω_1.*

A forcing notion $\mathbb{P}$ is (T,S)-preserving if for a large enough regular cardinal θ, a countable elementary submodel N of $H(\theta)$ which has the set $\{\mathbb{P},T,S\}$ and $p \in \mathbb{P}\cap N$, there exists $q \leq_{\mathbb{P}} p$ which is $(N,\mathbb{P})$-generic such that if $\omega_1 \cap N \notin S$, then

for any $x \in T$ of height $\omega_1 \cap N$,
if $\forall A \in \mathcal{P}(T) \cap N\,(x \in A \to \exists y \in A\,(y <_T x))$,
then for every $\mathbb{P}$-name $\dot{A}$, which is in N, for a subset of T,

$$q \Vdash_{\mathbb{P}} \text{“} x \in \dot{A} \to \exists y \in \dot{A}\,(y <_T x) \text{”}.$$

If T^* is a Suslin tree, then for every countable elementary submodel N of $H(\theta)$ (for some large enough regular cardinal θ) and $x \in T^*$ of height $\omega_1 \cap N$ and $A \in \mathcal{P}(T^*) \cap N$, if $x \in A$, then there exists $y \in A$ such that $y <_{T^*} x$(7). It follows that T^* satisfies $(*)$. So we start from a Suslin tree T^* and a stationary and costationary subset S of ω_1 and make each Aronszajn tree to be S-st-special and T^* to be S-st-special which satisfies the property $(*)$ by the iterated proper forcing extension such that each iterand is (T^*,S)-preserving and the whole iteration is also (T^*,S)-preserving. For Aronszajn trees T and T^* and a stationary subset S of ω_1, Shelah introduced the forcing notion $Q(T,S)$ which forces T to be S-st-special and is (T^*,S)-preserving [6, IX 4.2, 4.3, 4.6]. Moreover, Shelah introduced the new forcing iteration, so called a free limit iteration, which preserves the (T^*,S)-preserving property [6, IX §1, §2 and 4.7].

The following is Shelah's iterated forcing in [6, Chapter IX, 4.8 Conclusion](8). We start in the ground model where $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, and there exists a Suslin tree T^*. Let S be a stationary and costationary subset of the set ω_1. We define an $\aleph_1$-free iteration $\langle P_\xi, Q_\eta; \xi \leq \omega_2 \;\&\; \eta < \omega_2\rangle$ such that

- $Q_0 = Q(T^*,S)$,
- each Q_η satisfies one of the following:
  1. Q_η is proper and (T^*,S)-preserving of size $\aleph_1$,
  2. for some P_ξ-name of an antichain $\dot{A}$ of T^*, $\mathsf{rk}_{T^*}[\dot{A}] \cap S = \emptyset$ and $Q_\eta = Q_{\text{club}}(\omega_1 \setminus \mathsf{rk}_{T^*}[\dot{A}])$, which shoots a club through the set $\omega_1 \setminus \mathsf{rk}_{T^*}[\dot{A}]$ by countable approximations.

In this extension (with some bookkeeping argument), S is still stationary and co-stationary, every Aronszajn tree is S-st-special (hence not Suslin), and T^* is an S-st-special Aronszajn tree which satisfies $(*)$.

Combining Shelah's iteration above, some bookkeeping device, theorems in [16, 18] and the next section, we can conclude the following.

Theorem 2.5. *It is consistent that every forcing notions with $\mathsf{R}_{1,\aleph_1}$ has precaliber $\aleph_1$, every Todorčević ordering for any second countable Hausdorff space also has precaliber $\aleph_1$ and there exists a non-special Aronszajn tree.*

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(7)Let $D := \{t \in T^*; t \in A$ or for every $s \in T^*$ with $t <_{T^*} s$, $s \notin A\}$. Since D is a dense subset of T^* and T^* is Suslin, there exists $y \in D \cap N$ which is compatible with x in T^*. Then it have to be true that $y <_{T^*} x$. Since $x \in A$, it have to be true that $y \in A$.

This statement is equivalent that there are no uncountable antichain through T^*.

(8)Shelah's proof uses an $\aleph_1$-free iteration. This is different from a countable support iteration. But Schlindwein proved in [7] that the same proof works for a countable support iterations. So our theorem can be shown by a countable support iteration.

3. PROOF

Suppose that S is a stationary subset of ω_1, X is a second countable Hausdorff space and I is an uncountable subset of $\mathbb{T}(X)$. By shrinking I if necessary, we may assume that

- the size of I is $\aleph_1$,
- the set $\{p^d; p \in I\}$ forms a Δ-system with root d,
- for some $q \in \mathbb{T}(X)$,

$$q \Vdash_{\mathbb{T}(X)} \text{“}I \cap \dot{G} \text{ is uncountable”}.$$

Let $\vec{M} = \langle M_\alpha; \alpha \in \omega_1\rangle$ be a sequence of countable elementary submodels of $H(\aleph_2)$ such that $\{S, X, I\} \in M_0$, and for every $\alpha \in \omega_1$, $\langle M_\beta; \beta \in \alpha\rangle \in M_\alpha$. By shrinking I if necessary again, we may assume that

- for each $p \in I$ and $\alpha \in \omega_1$, if $p^d \cap (M_{\alpha+1} \setminus M_\alpha) \neq \emptyset$, then $p^d \subseteq M_{\alpha+1} \setminus M_\alpha$.

We have to notice then that it may happen that I does *not* belong to M_0. From now on, we do *not* assume that $I \in M_\alpha$ for any $\alpha \in \omega_1$.

We define the forcing notion $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ which consists of pairs $\langle h, f\rangle$ such that

- h is a finite partial function from ω_1 into ω_1,
- for any $\alpha, \beta \in \mathrm{dom}(h)$, $\alpha \leq h(\alpha)$, and if $\alpha < \beta$, then $h(\alpha) < \beta$,
- for any $\alpha \in \mathrm{dom}(h) \cap S$, $h(\alpha) = \alpha$,
- f is a finite partial function from I into ω,
- for any $\alpha \in \mathrm{dom}(h)$ and $p \in \mathrm{dom}(f)$,

$$p^d \cap \left(M_{h(\alpha)} \setminus M_\alpha\right) = \emptyset,$$

- for any $p \in \mathrm{dom}(f)$, the set $\bigcup f^{-1}[\{f(p)\}]$ is a common extension of members of the set $f^{-1}[\{f(p)\}]$ in $\mathbb{T}(X)$,

ordered by extension, that is, for any $\langle h, f\rangle$ and $\langle h', f'\rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$,

$$\langle h, f\rangle \leq_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} \langle h', f'\rangle :\iff h \supseteq h \;\&\; f \supseteq f'.$$

By a density argument, if $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is proper, then $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ adds an uncountable subset of I which satisfies the finite compatibility property. Therefore, under the approach due to Shelah in §2, it suffices to show that $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is proper and (T^*, S)-preserving.

Lemma 3.1. *$\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is proper.*

Proof. Let θ be a large enough regular cardinal, a countable elementary submodel N of $H(\theta)$ which has the set $\left\{X, I, \vec{M}, S\right\}$, $\langle h, f\rangle \in \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$, and δ a countable ordinal not smaller than the ordinal $\omega_1 \cap N$ (if $\omega_1 \cap N \in S$, then we define $\delta := \omega_1 \cap N$). We show that $\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$ is $(N, \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S))$-generic.

Let $\langle h', f'\rangle \leq_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} \langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$ and D a dense open subset of $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$. We will find a condition in $D \cap N$ which is compatible with $\langle h', f'\rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$.

By extending the condition $\langle h', f'\rangle$ if necessary, we may assume that $\langle h', f'\rangle \in D$. We note that $\langle h' \restriction N, f' \restriction N\rangle$ is in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S) \cap N$ because $\omega_1 \cap N \in \mathrm{dom}(h')$.

Let

$$D' := \left\{ \langle k, g\rangle \in D; \langle k, g\rangle \leq_{\mathcal{Q}(\mathbb{T}(X),I,\vec{M},S)} \langle h'\restriction N, f'\restriction N\rangle \ \& \ \mathrm{ran}(g) = \mathrm{ran}(f') \right\}.$$

We note that D' is in N[9], $\langle h', f'\rangle \in D'$ and D' is dense in $\mathcal{Q}(\mathbb{Q}, I, \vec{M})$ below $\langle h'\restriction N, f'\restriction N\rangle$. Since the product forcing ${}^{\mathrm{ran}(f')}\mathbb{T}(X)$ of $\mathbb{T}(X)$ is ccc in the model N, by the elementarity of N, there exists a countable subset J of ${}^{\mathrm{ran}(f')}\mathbb{T}(X)$ in N such that

- J is a subset of the set
$$\left\{ \left\langle \bigcup g^{-1}[\{n\}]; n \in \mathrm{ran}(f') \right\rangle ; \langle k, g\rangle \in D' \right\},$$
- for every $\langle k, g\rangle \in D'$, there exists $\langle \mu_n; n \in \mathrm{ran}(f')\rangle \in J$ such that for each $n \in \mathrm{ran}(f')$, μ_n and $\bigcup g^{-1}[\{n\}]$ are compatible in $\mathbb{T}(X)$.

Since $\langle h', f'\rangle \in D'$, there exists $\langle \mu_n; n \in \mathrm{ran}(f')\rangle \in J$ such that for each $n \in \mathrm{ran}(f')$, μ_n and $\bigcup (f')^{-1}[\{n\}]$ are compatible in $\mathbb{T}(X)$. Since $\langle \mu_n; n \in \mathrm{ran}(f')\rangle \in J$ holds in N, there exists $\langle k, g\rangle \in D' \cap N$ such that

$$\left\langle \bigcup g^{-1}[\{n\}]; n \in \mathrm{ran}(f') \right\rangle = \langle \mu_n; n \in \mathrm{ran}(f')\rangle .$$

Then $\langle h' \cup k, f' \cup g\rangle$ is a common extension of $\langle h', f'\rangle$ and $\langle k, g\rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$. □

Lemma 3.2. *For any Aronszajn tree T, $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ is (T, S)-preserving.*

Proof. Let T, θ, N be as in the statement of the definition of the (T, S)-preservation, (moreover we suppose $\vec{M} \in N$) and $\langle h, f\rangle \in \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S) \cap N$. Suppose that $\omega_1 \cap N \notin S$, because if $\omega_1 \cap N \in S$, then the condition $\langle h \cup \{\langle \omega_1 \cap N, \omega_1 \cap N\rangle\}, f\rangle$ is as desired.

Let

$$\delta := \sup \{F(\omega_1 \cap N) + 1; F \in ({}^{\omega_1}\omega_1) \cap N\} .$$

Since N is countable, δ is a countable ordinal. We will show that the condition $\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$ of $\mathcal{Q}(\mathbb{Q}, I, \vec{M}, S)$ is our desired one.

As seen in the proof of the previous lemma, the condition $\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$ is $(N, \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S))$-generic. Suppose that $x \in T$ of height $\omega_1 \cap N$ such that for any subset $A \in N$ of T, if $x \in A$, then there is $y \in A$ such that $y <_T x$. Let $\dot{A} \in N$ be a $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$-name for a subset of T. We will show that

$$\langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X),I,\vec{M},S)} \text{“} x \notin \dot{A} \text{ or } \exists y \in \dot{A}\, (y <_T x) \text{”}.$$

Let $\langle h', f'\rangle \leq_{\mathcal{Q}(\mathbb{Q},I,\vec{M},S)} \langle h \cup \{\langle \omega_1 \cap N, \delta\rangle\}, f\rangle$, and assume that

$$\langle h', f'\rangle \nVdash_{\mathcal{Q}(\mathbb{T}(X),I,\vec{M},S)} \text{“} x \notin \dot{A} \text{”}.$$

By strengthening $\langle h', f'\rangle$ if necessary, we may assume that

$$\langle h', f'\rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X),I,\vec{M},S)} \text{“} x \in \dot{A} \text{”}.$$

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(9) $\mathrm{ran}(f')$ is a finite subset of ω.

We note that $\langle h'\restriction N, f'\restriction N\rangle$ is in N (because $\omega_1\cap N \in \mathrm{dom}(h')$), and by the definition of $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$, for every $p \in \mathrm{dom}(f')$, if $\mathrm{ran}(p) \notin N$, then

$$(p^d \setminus \mathsf{d}) \cap M_\delta = \emptyset.$$

Let $\gamma \in \omega_1 \cap N$ be such that for every $p \in \mathrm{dom}(f')$, if the set $p^d \setminus \mathsf{d}$ intersects N, then $p^d \subseteq M_\gamma$[10]. Since X is second countable Hausdorff and N is an elementary submodel, there exists a finite set $\mathcal{U}$ of pairwise disjoint open subsets of X in N such that for each $n \in \mathrm{ran}(f')$, the finite set $\left(\bigcup (f')^{-1}[\{n\}]\right)^d$ is separated by $\mathcal{U}$. We define a function F with the domain

$$\{t \in T; \mathsf{ht}_T(t) > \max(\mathrm{dom}(h'\restriction N))\}$$

such that for each $t \in T$ of height larger than $\max(\mathrm{dom}(h'\restriction N))$,

$F(t) := \sup\Big\{\beta \in \omega_1;$ there exists $\langle k, g\rangle \in \mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ such that

- $\min(\mathrm{dom}(k)) = \mathsf{rk}_T(t)$,
- $k(\mathsf{rk}_T(t)) = \beta$,
- $\langle (h'\restriction N) \cup k, (f'\restriction N) \cup g\rangle$ is a condition of $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$,
- for each $p \in \mathrm{dom}(g)$, $(p^d \setminus \mathsf{d}) \cap M_\gamma = \emptyset$,
- $\mathrm{ran}(g) = \mathrm{ran}(f' \setminus N)$,
- for each $n \in \mathrm{ran}(f' \setminus N)$, the set $\left(\bigcup g^{-1}[\{n\}]\right)^d \setminus \mathsf{d}$ is separated by $\mathcal{U}$, and
- $\langle (h'\restriction N) \cup k, (f'\restriction N) \cup g\rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)}$ "$t \in \dot{A}$" $\Big\}$.

Then F belongs to N. Let

$$B := \{t \in T; \mathsf{rk}_T(t) > \max(\mathrm{dom}(h'\restriction N))\ \&\ F(t) = \omega_1\},$$

which is also in N. We define a function F' with the domain

$$[\max(\mathrm{dom}(h'\restriction N)) + 1, \omega_1)$$

such that for a countable ordinal β larger than $\max(\mathrm{dom}(h'\restriction N))$,

$$F'(\beta) := \sup\left\{F(t) + 1; t \in T \setminus B\ \&\ \mathsf{rk}_T(t) \in \left(\max(\mathrm{dom}(h'\restriction N)), \beta\right]\right\}.$$

This F' is a function from ω_1 into ω_1 and also in N. Hence $F'(\omega_1 \cap N) < \delta$ by the definition of δ. Since $\langle h', f'\rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)}$ "$x \in \dot{A}$" and $h'(\mathsf{rk}_T(x)) = h'(\omega_1 \cap N) = \delta$, $F(x) \geq \delta$ holds. Therefore x have to belong to B. Thus by our assumption, there exists $y \in B$ such that $y <_T x$.

Take $\varepsilon \in \omega_1$ such that $f' \subseteq M_\varepsilon$. Let

$E := \Big\{ \langle k, g\rangle \in \mathcal{Q}(\mathbb{Q}, I, \vec{M}, S);$

- $\min(\mathrm{dom}(k)) = \mathsf{rk}_T(y)$,
- $\langle (h'\restriction N) \cup k, (f'\restriction N) \cup g\rangle$ is a condition of $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$,
- for each $p \in \mathrm{dom}(g)$, $(p^d \setminus \mathsf{d}) \cap M_\gamma = \emptyset$,
- $\mathrm{ran}(g) = \mathrm{ran}(f')$,
- for each $n \in \mathrm{ran}(f')$, the set $\left(\bigcup g^{-1}[\{n\}]\right)^d \setminus \mathsf{d}$ is separated by $\mathcal{U}$, and
- $\langle (h'\restriction N) \cup k, (f'\restriction N) \cup g\rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)}$ "$y \in \dot{A}$" $\Big\}$.

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[10] Then for every $p \in \mathrm{dom}(f')$, $(p^d \setminus \mathsf{d}) \cap M_\gamma = \emptyset$ iff $(p^d \setminus \mathsf{d}) \cap M_\delta = \emptyset$.

We note that E is in N, and the set

$$\{k(\mathsf{rk}_T(y)); \langle k, g\rangle \in E\}$$

is uncountable because $F(y) = \omega_1$. So there exists $\langle k, g\rangle \in E$ such that for each $p \in \mathrm{dom}(g)$, $(p^d \setminus \mathtt{d}) \cap M_\varepsilon = \emptyset$. Then for each $n \in \mathrm{ran}(f' \setminus N)$,

$$\left(\left(\bigcup g^{-1}[\{n\}]\right)^d \setminus \mathtt{d}\right) \cap \left(\bigcup (f' \setminus N)^{-1}[\{n\}]\right) = \emptyset.$$

Since X is second countable Hausdorff and N is an elementary submodel, there exists disjoint open subsets U and V of X in N(11) such that for each $n \in \mathrm{ran}(f' \setminus N)$,

$$\left(\bigcup (f' \setminus N)^{-1}[\{n\}]\right)^d \setminus \mathtt{d} \subseteq U,$$

$$\left(\bigcup g^{-1}[\{n\}]\right)^d \setminus \mathtt{d} \subseteq V$$

and

$$V \cap \left(\left(\bigcup (f' \setminus N)^{-1}[\{n\}]\right) \setminus U\right) = \emptyset.$$

By the elementarity of N, we can find $\langle k', g'\rangle \in E$ such that

$$\left(\bigcup (g')^{-1}[\{n\}]\right)^d \setminus \mathtt{d} \subseteq V.$$

Then for each $n \in \mathrm{ran}(f' \setminus N)$, the set

$$\bigcup (f')^{-1}[\{n\}] \cup \bigcup (g')^{-1}[\{n\}] \leq_{\mathbb{T}(X)} \bigcup (f')^{-1}[\{n\}].$$

Since $g' \subseteq N$ and $\left(\bigcup (f' \setminus N)^{-1}[\{n\}]\right)^d \cap N = \emptyset$, we note that for each $n \in \mathrm{ran}(f' \setminus N)$, the set

$$\bigcup (f')^{-1}[\{n\}] \cup \bigcup (g')^{-1}[\{n\}] \leq_{\mathbb{T}(X)} \bigcup (g')^{-1}[\{n\}].$$

Therefore $\langle k' \cup h', g' \cup f'\rangle$ is an extension of $\langle h', f'\rangle$ in $\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)$ and

$$\langle k' \cup h', g' \cup f'\rangle \Vdash_{\mathcal{Q}(\mathbb{T}(X), I, \vec{M}, S)} \text{“}y \in \dot{A}\text{”}.$$

$\square$

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(11)This can be done because the set $\left(\bigcup (f' \setminus N)^{-1}[\{n\}]\right) \setminus U$ is finite if U satisfies that $\left(\bigcup (f' \setminus N)^{-1}[\{n\}]\right)^d \setminus \mathtt{d} \subseteq U$.

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