On the identification of noncausal functions from the SFCs *

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1 Introduction.

Let \( \{W_t, t \in [0, 1]\} \) be a Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\), starting at the origin. Let \( f(t, \omega) \) be a random function on \([0, 1] \times \Omega \) and \( \{\varphi_n(t)\} \) be a CONS in \( L^2([0,1]; \mathbb{C}) \). The system \( \{\tilde{f}_n(\omega) = \int_0^1 f(t, \omega) \overline{\varphi_n(t)} dW_t\} \) is called the stochastic Fourier coefficients (SFCs in abbr.) of \( f(t, \omega) \). Here \( \overline{z} (z \in \mathbb{C}) \) denotes the complex conjugate of \( z \). It is of course understood that the stochastic integral \( \int dW \) in the definition of SFCs should be chosen adequately. We are concerned with the problem whether \( f(t, \omega) \) is identified from the SFCs of \( f(t, \omega) \) or not.

We here roughly present some preceding studies of this problem without rigorous assumptions on \( f(t, \omega) \). Suppose \( f(t, \omega) \) be a square integrable Wiener functional. Then \( f(t, \omega) \) admits the Wiener-Itô expansion

\[
f(t, \omega) = \sum_{n=0}^{\infty} I_n(k_n^f(t; \cdot)),
\]

where \( I_n(k_n^f(t; \cdot)) \) denotes the multiple Wiener-Itô integral of \( n \) th degree of the kernel function \( k_n^f(t; \cdot) \). In S.Ogawa[4] and S.Ogawa and H.Uemura[6] the author(s) investigated this problem in this framework of the theory of the Homogeneous Chaos, and obtained some affirmative answers by determining the kernel functions \( k_n^f(t; \cdot) \), \( n \in \mathbb{N} \cup \{0\} \). In [4], \( f(t, \omega) \) was assumed to be causal and SFCs were defined by any uniformly bounded basis through the Itô integral, whereas in [6], \( f(t, \omega) \) was assumed to be noncausal and SFCs were defined by the system of trigonometric functions through the Skorokhod integral.

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In these procedures some simple multiple Wiener integrals constructed by \{\varphi_n(t)\} were used as test functions. Moreover, we needed the multiple Wiener integrals of these functions to reconstruct $f(t, \omega)$ from \{$k_n(t, \cdot)$\}. In the same framework of [6] we applied the Bohr convolution method to identify the Fourier coefficients of $f(t, \omega)$ in S.Ogawa and H.Uemura[7]. In this procedure we utilized $\int_0^1 \varphi_n(t)dW_t, \ n \in \mathbb{Z}$. In any studies mentioned above we needed the Brownian motion as 'a catalyst'.

Recently, however, S. Ogawa [5] obtained an affirmative answer \textit{without} the aid of a Brownian motion from the stochastic Fourier transform (SFT in abbr.) if $f(t, \omega)$ is a non-negative or nonpositive causal function. In [5] Itô type stochastic process $\int_0^t f(s, \omega)dW_s$ was gotten from the stochastic Fourier transform and the quadratic variation of this process gave $f(t, \omega)$ since $f(t, \omega)$ being of definite sign.

In this note we develop this SFT method to the case where $f(t, \omega)$ is noncausal. In Section 2, we state our setting. We next introduce the SFT in Section 3, and Section 4 is devoted to our main theorem and some remarks.

2 Our Setting.

Our settings are as follows:

[H1] $f(t, \omega)$ is a nonnegative function which is differentiable for almost all $\omega$ and satisfies $\int_0^1 f(t, \omega) dt \in L^2(\Omega, dP)$ and $f'(t, \omega) \in L^2([0,1] \times \Omega, dt dP)$, where $f'(t, \omega) = \partial f(t, \omega)/\partial t$. We do not assume $f(t, \omega)$ is causal.

[H2] We employ the system of trigonometric functions \{$e_n(t) = e^{2\pi int}, n \in \mathbb{Z}$\} to define the SFCs of $f(t, \omega)$.

[H3] We choose the Ogawa integral for the stochastic integral to define the SFCs of $f(t, \omega)$.

Here, $f(t, \omega)$ is called Ogawa integrable if

$$\sum_n \int_0^1 f(t, \omega)\varphi_n(t)dt \int_0^1 \varphi_n(t)dW_t$$

converges in probability for any CONS \{\varphi_n(t)\} and moreover if this limit is identical. We call this limit by the Ogawa integral of $f(t, \omega)$ and denote by $\int_0^1 f(t, \omega)d_*W_t$. We note that the equation

$$\int_0^1 f(t, \omega)d_*W_t = f(1, \omega)W_1 - \int_0^1 W_t f'(t, \omega)dt \quad (1)$$

holds under our hypothesis [H1]. Refer S.Ogawa [1, 2] for details. We employ the notation
\[ \tilde{f}_n(\omega) = \int_0^1 f(t, \omega) e_{n}(t) d_* W_t \]
for SFC.

## 3 Stochastic Fourier Transform.

In this section, we first introduce the stochastic Fourier transform of \( f(t, \omega) \). Let \( \{ \varphi_n(t) \} \) be a CONS. Let \( \{ \epsilon_n \} \) be a sequence such that \( \epsilon_n \neq 0 \) for all \( n \) and that

\[ T_{\varphi, \epsilon}(f) = \sum_n \epsilon_n \tilde{f}_n(\omega) \varphi_n(t) \]

converges. Then \( T_{\varphi, \epsilon}(f) \) is called the \( \{ \epsilon_n \} \)-stochastic Fourier transform (\( \{ \epsilon_n \} \)-SFT in abbr.) of \( f(t, \omega) \). Refer S.Ogawa[3] for details.

We construct a SFT of \( f(t, \omega) \) for some suitable sequence. To this end we first note the following proposition:

**Proposition 1.** Under the hypotheses \([H.1], [H.2] \) and \([H.3], \{ \tilde{f}_n(\omega), n \in \mathbb{Z} \} \) is uniformly bounded in \( L^1(dP) \).

**Proof.** From (1) we have

\[
\tilde{f}_n(\omega) = \int_0^1 f(t, \omega) dt \int_0^1 e_{-n}(t) dW_t \\
+ (f(1, \omega) - f(0, \omega)) \sum_{k \neq -n} \frac{1}{-2\pi i(n+k)} \int_0^1 e_k(t) dW_t \\
- \sum_{k \neq -n} \int_0^1 f'(t, \omega) \overline{e_{n+k}(t)} dt \frac{1}{-2\pi i(n+k)} \int_0^1 e_k(t) dW_t.
\]

(2)

Applying the Schwarz inequality

\[
E|\tilde{f}_n(\omega)| \leq \sqrt{E \left| \int_0^1 f(t, \omega) dt \right|^2} \\
+ \sqrt{E \left| \int_0^1 f'(t, \omega) dt \right|^2} \sqrt{\sum_{k \neq -n} \frac{1}{4\pi^2(n+k)^2}} \\
+ \sqrt{E \left[ \int_0^1 |f'(t, \omega)|^2 dt \right]} \sqrt{\sum_{k \neq -n} \frac{1}{4\pi^2(n+k)^2}} < \infty,
\]

which complete the proof since the right hand side is independent of \( n \). \( \square \)
Thus it is enough to employ an \( \ell^1 \) sequence to construct a SFT, and we here choose the following \( \{\tau_n\} \):

\[
\tau_n = \begin{cases} 
\frac{1}{-4\pi^2n^2} & \text{if } n \neq 0 \\
1 & \text{if } n = 0.
\end{cases}
\]

Then

\[
\left\{ \sum_{|n| \leq N} \tau_n \tilde{f}_n(\omega)e_n(t) \right\}_{N=1,2,...}
\]

forms a Cauchy sequence in \( (L^1(\Omega \to C([0,1];\mathbb{C}),dP), \| \cdot \|) \), where

\[
\| X(\cdot) \| = E \left[ \sup_{0 \leq t \leq 1} |X(t)| \right].
\]

This is because

\[
E \left[ \sup_{0 \leq t \leq 1} \left| \sum_{|n| \neq 0, |n| \leq N} \frac{1}{-4\pi^2n^2} \tilde{f}_n(\omega)e_n(t) - \sum_{|n| \neq 0, |n| \leq M} \frac{1}{-4\pi^2n^2} \tilde{f}_n(\omega)e_n(t) \right| \right]
\]

\[
\leq \sum_{N < |n| \leq M} \frac{1}{4\pi^2n^2} E|\tilde{f}_n(\omega)|,
\]

which goes to 0 as \( N,M \to \infty \). Thus we obtain the \( \{\tau_n\} \)-SFT of \( f(t,\omega) \), say \( S(t,\omega) \):

**Proposition 2.** Under the hypotheses \([H.1],[H.2]\) and \([H.3]\), there exists a random function \( S(t,\omega) \in C([0,1]) \) a.s. such that

\[
\lim_{N \to \infty} E \left[ \sup_{0 \leq t \leq 1} \left| \sum_{|n| \leq N} \tau_n \tilde{f}_n(\omega)e_n(t) - S(t,\omega) \right| \right] = 0.
\]

4 Main Theorem.

From (2) we have

\[
S(t,\omega) = \tilde{f}_0(\omega) + \sum_{n \neq 0} \frac{1}{-4\pi^2n^2} \tilde{f}_n(\omega)e_n(t)
\]

\[
= \tilde{f}_0(\omega) - \frac{1}{2} \left( f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega) dt \right) \left( \frac{1}{6} - t + t^2 \right)
\]

\[
- \left( \int_0^1 \left( \int_0^t W_s f'(s,\omega) ds \right) dt - \int_0^1 W_t f(t,\omega) dt \right) \left( \frac{1}{2} - t \right)
\]
\begin{equation}
- \left( \int_0^t \int_0^s W_u f'(u, \omega) du ds - \int_0^1 \int_0^t \int_0^s W_u f'(u, \omega) du ds dt \right) + \left( \int_0^s W_s f(s, \omega) ds - \int_0^1 \int_0^t W_s f(s, \omega) ds dt \right) \tag{3}
\end{equation}

for all $t \in (0, 1)$ and almost all $\omega$, noting that

$$\lim_{N \to \infty} \sum_{n \neq 0, |n| \leq N} \frac{1}{2\pi i n} e_n(t) = \frac{1}{2} - t$$

and

$$\lim_{N \to \infty} \sum_{n \neq 0, |n| \leq N} \frac{1}{-4\pi^2 n^2} e_n(t) = -\frac{1}{2} \left( \frac{1}{6} - t + \iota^2 \right)$$

for all $t \in (0, 1)$. Since the right hand side of (3) is differentiable with respect to $t \in (0, 1)$, so is the left hand side, and

$$S'(t, \omega) = -\frac{1}{2} \left( f(1, \omega) W_1 - \int_0^1 W_t f'(t, \omega) dt \right) (-1 + 2t) + \left( \int_0^1 \left( \int_0^t W_s f'(s, \omega) ds \right) dt - \int_0^1 W_t f(t, \omega) dt \right) - \int_0^t W_u f'(u, \omega) du + W_t f(t, \omega).$$

We note that

$$\left| \int_s^t W_u f'(u, \omega) du \right| \leq \sqrt{|t - s|} \sup_{u \in [0,1]} |W_u| \sqrt{\int_0^1 |f'(u, \omega)|^2 du}.$$

Hence if we fix $s \in (0, 1)$ arbitrary then we have

$$\limsup_{t \downarrow s} \frac{S'(t, \omega) - S'(s, \omega)}{\sqrt{2(t - s) \log \log \frac{1}{t - s}}} = \limsup_{t \downarrow s} \frac{W_t f(t, \omega) - W_s f(s, \omega)}{\sqrt{2(t - s) \log \log \frac{1}{t - s}}} = f(s, \omega) \quad a.s.$$

from the law of iterated logarithm of the Brownian motion, recalling that we assume $f(t, \omega)$ is nonnegative. Set $\mathbb{S}$ be a countable dense subset of $(0, 1)$. Then we have the following theorem:

**Theorem 1.** Let $f(t, \omega)$ be a nonnegative function which is differentiable for almost all $\omega$ satisfying $\int_0^1 f(t, \omega) dt \in L^2(\Omega, dP)$ and $f'(t, \omega) \in L^2([0,1] \times \Omega, dt dP)$. Then we have

$$P \left( \limsup_{t \downarrow s} \frac{S'(t, \omega) - S'(s, \omega)}{\sqrt{2(t - s) \log \log \frac{1}{t - s}}} = f(s, \omega) \quad \text{for all } s \in \mathbb{S} \right) = 1.$$
Remark 1. Since we assume $f(t, \omega)$ is continuous, the theorem above is sufficient to identify $f(t, \omega)$ for all $t \in [0, 1]$.

Remark 2. Note that $0$ th SFC $\tilde{f}_0(\omega)$ does not appear in the construction of $S'(t, \omega)$, that is, $\tilde{f}_0(\omega)$ is not necessary to identify $f(t, \omega)$.

Moreover, since $\frac{1}{-4\pi^2n^2}\tilde{f}_n(\omega)e_n(t)$ is a continuously differentiable function, we get the following corollary:

**Corollary 1.** Let $f(t, \omega)$ be a nonnegative function which is differentiable for almost all $\omega$ satisfying $\int_0^1 f(t, \omega)dt \in L^2(\Omega, dP)$ and $f'(t, \omega) \in L^2([0,1]\times\Omega, dtdP)$. Let $\Lambda$ be a finite subset of $\mathbb{Z}$ containing 0. Set

$$S_\Lambda(t, \omega) = \sum_{n \in \Lambda^c} \frac{1}{-4\pi^2n^2}\tilde{f}_n(\omega)e_n(t).$$

Then we have

$$P \left( \limsup_{t \downarrow s} \frac{S_\Lambda'(t, \omega) - S_\Lambda'(s, \omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s, \omega) \quad \text{for all } s \in \mathbb{S} \right) = 1.$$

Remark 3. Corollary 1 does not mean that $\tilde{f}_n(\omega)$, $n \in \Lambda$, is reconstructed from $\{\tilde{f}_n(\omega), n \in \Lambda^c\}$ by deterministic procedures. Indeed, if $f(t, \omega) = 1$, then $\tilde{f}_0(\omega) = W_1$ is independent of $\{\tilde{f}_n(\omega) = \int_0^1 e_n(t)dW_t, n \in \Lambda^c\}$.

**References**


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