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On the identification of noncausal functions from the SFCs *

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1 Introduction.

Let $\{W_t, t \in [0,1]\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$, starting at the origin. Let $f(t,\omega)$ be a random function on $[0,1] \times \Omega$ and $\{\varphi_n(t)\}$ be a CONS in $L^2([0,1];\mathbb{C})$. The system $\{\tilde{f}_n(\omega) = \int_0^1 f(t,\omega)\overline{\varphi_n(t)}dW_t\}$ is called the stochastic Fourier coefficients (SFCs in abbr.) of $f(t,\omega)$. Here $\overline{z}\,(z \in \mathbb{C})$ denotes the complex conjugate of z. It is of course understood that the stochastic integral $\int dW$ in the definition of SFCs should be chosen adequately. We are concerned with the problem whether $f(t,\omega)$ is identified from the SFCs of $f(t,\omega)$ or not.

We here roughly present some preceding studies of this problem without rigorous assumptions on $f(t,\omega)$. Suppose $f(t,\omega)$ be a square integrable Wiener functional. Then $f(t,\omega)$ admits the Wiener-Itô expansion

$$f(t,\omega) = \sum_{n=0}^{\infty} I_n(k_n^f(t;\cdot)),$$

where $I_n(k_n^f(t;\cdot))$ denotes the multiple Wiener-Itô integral of nth degree of the kernel function $k_n^f(t;\cdot)$. In S.Ogawa[4] and S.Ogawa and H.Uemura[6] the author(s) investigated this problem in this framework of the theory of the Homogeneous Chaos, and obtained some affirmative answers by determining the kernel functions $k_n^f(t;\cdot)$, $n \in \mathbb{N} \cup \{0\}$. In [4], $f(t,\omega)$ was assumed to be causal and SFCs were defined by any uniformly bounded basis through the Itô integral, whereas in [6], $f(t,\omega)$ was assumed to be noncausal and SFCs were defined by the system of trigonometric functions through the Skorokhod integral.

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In these procedures some simple multiple Wiener integrals constructed by $\{\varphi_n(t)\}$ were used as test functions. Moreover, we needed the multiple Wiener integrals of these functions to reconstruct $f(t,\omega)$ from $\{k_n^f(t;\cdot)\}$. In the same framework of [6] we applied the Bohr convolution method to identify the Fourier coefficients of $f(t,\omega)$ in S.Ogawa and H.Uemura[7]. In this procedure we utilized $\int_0^1 \varphi_n(t)dW_t, n \in \mathbb{Z}$. In any studies mentioned above we needed the Brownian motion as 'a catalyst'.

Recently, however, S. Ogawa [5] obtained an affirmative answer <u>without</u> the aid of a Brownian motion from the stochastic Fourier transform (SFT in abbr.) if $f(t,\omega)$ is a nonnegative or nonpositive causal function. In [5] Itô type stochastic process $\int_0^t f(s,\omega)dW_s$ was gotten from the stochastic Fourier transform and the quadratic variation of this process gave $f(t,\omega)$ since $f(t,\omega)$ being of definite sign.

In this note we develop this SFT method to the case where $f(t,\omega)$ is noncausal. In Section 2, we state our setting. We next introduce the SFT in Section 3, and Section 4 is devoted to our main theorem and some remarks.

2 Our Setting.

Our settings are as follows:

[H1] $f(t,\omega)$ is a nonnegative function which is differentiable for almost all ω and satisfies $\int_0^1 f(t,\omega)dt \in L^2(\Omega, dP)$ and $f'(t,\omega) \in L^2([0,1]\times\Omega, dtdP)$, where $f'(t,\omega) = \partial f(t,\omega)/\partial t$. We do not assume $f(t,\omega)$ is causal.

[H2] We employ the system of trigonometric functions $\{e_n(t) = e^{2\pi int}, n \in \mathbb{Z}\}$ to define the SFCs of $f(t,\omega)$.

[H3] We choose the Ogawa integral for the stochastic integral to define the SFCs of $f(t,\omega)$.

Here, $f(t,\omega)$ is called Ogawa integrable if

$$\sum_n \int_0^1 f(t,\omega)\overline{\varphi_n(t)}dt \int_0^1 \varphi_n(t)dW_t$$

converges in probability for any CONS $\{\varphi_n(t)\}$ and moreover if this limit is identical. We call this limit by the Ogawa integral of $f(t,\omega)$ and denote by $\int_0^1 f(t,\omega)d_*W_t$. We note that the equation

$$\int_0^1 f(t,\omega)d_*W_t = f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega)dt \tag{1}$$

holds under our hypothesis [H1]. Refer S.Ogawa [1, 2] for details. We employ the notation $\tilde{f}_n(\omega) = \int_0^1 f(t,\omega)\overline{e_n(t)}d_*W_t$ for SFC.

3 Stochastic Fourier Transform.

In this section, we first introduce the stochastic Fourier transform of $f(t,\omega)$. Let $\{\varphi_n(t)\}$ be a CONS. Let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n \neq 0$ for all n and that

$$\mathcal{T}_{\varphi,\varepsilon}(f) = \sum_n \varepsilon_n \tilde{f}_n(\omega)\varphi_n(t)$$

converges. Then $\mathcal{T}_{\varphi,\varepsilon}(f)$ is called the $\{\varepsilon_n\}$-stochastic Fourier transform ($\{\varepsilon_n\}$-SFT in abbr.) of $f(t,\omega)$. Refer S.Ogawa[3] for details.

We construct a SFT of $f(t,\omega)$ for some suitable sequence. To this end we first note the following proposition:

Proposition 1. *Under the hypotheses [H.1], [H.2] and [H.3],* $\{\tilde{f}_n(\omega), n \in \mathbb{Z}\}$ *is uniformly bounded in* $L^1(dP)$.

Proof. From (1) we have

$$\begin{aligned}\tilde{f}_n(\omega) = &\int_0^1 f(t,\omega)dt \int_0^1 e_{-n}(t)dW_t \\ &+ (f(1,\omega) - f(0,\omega)) \sum_{k\neq -n} \frac{1}{-2\pi i(n+k)} \int_0^1 e_k(t)dW_t \\ &- \sum_{k\neq -n} \int_0^1 f'(t,\omega)\overline{e_{n+k}(t)}dt \frac{1}{-2\pi i(n+k)} \int_0^1 e_k(t)dW_t. \end{aligned} \tag{2}$$

Applying the Schwarz inequality

$$\begin{aligned} E|\tilde{f}_n(\omega)| \leqq &\sqrt{E\left|\int_0^1 f(t,\omega)dt\right|^2} \\ &+ \sqrt{E\left|\int_0^1 f'(t,\omega)dt\right|^2}\sqrt{\sum_{k\neq -n}\frac{1}{4\pi^2(n+k)^2}} \\ &+ \sqrt{E\left[\int_0^1 |f'(t,\omega)|^2 dt\right]}\sqrt{\sum_{k\neq -n}\frac{1}{4\pi^2(n+k)^2}} < \infty, \end{aligned}$$

which complete the proof since the right hand side is independent of n. □

Thus it is enough to employ an ℓ^1 sequence to construct a SFT, and we here choose the following $\{\tau_n\}$:

$$\tau_n = \begin{cases} \dfrac{1}{-4\pi^2 n^2} & \text{if } n \neq 0 \\ 1 & \text{if } n = 0. \end{cases}$$

Then

$$\left\{ \sum_{|n| \leqq N} \tau_n \tilde{f}_n(\omega) e_n(t) \right\}_{N=1,2,\ldots}$$

forms a Cauchy sequence in $(L^1(\Omega \to C([0,1];\mathbb{C})), dP), \|\cdot\|)$, where

$$\|X(\cdot)\| = E\left[\sup_{0 \leqq t \leqq 1} |X(t)| \right].$$

This is because

$$\begin{aligned} & E\left[\sup_{0 \leqq t \leqq 1} \left| \sum_{n \neq 0, |n| \leqq N} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t) - \sum_{n \neq 0, |n| \leqq M} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t) \right| \right] \\ & \leqq \sum_{N < |n| \leqq M} \frac{1}{4\pi^2 n^2} E|\tilde{f}_n(\omega)|, \end{aligned}$$

which goes to 0 as $N, M \to \infty$. Thus we obtain the $\{\tau_n\}$-SFT of $f(t,\omega)$, say $S(t,\omega)$:

Proposition 2. *Under the hypotheses [H.1], [H.2] and [H.3], there exists a random function* $S(t,\omega) \in C([0,1])$ *a.s. such that*

$$\lim_{N \to \infty} E\left[\sup_{0 \leqq t \leqq 1} \left| \sum_{|n| \leqq N} \tau_n \tilde{f}_n(\omega) e_n(t) - S(t,\omega) \right| \right] = 0.$$

4 Main Theorem.

From (2) we have

$$\begin{aligned} S(t,\omega) &= \tilde{f}_0(\omega) + \sum_{n \neq 0} \frac{1}{-4\pi^2 n^2} \tilde{f}_n(\omega) e_n(t) \\ &= \tilde{f}_0(\omega) - \frac{1}{2}\left(f(1,\omega) W_1 - \int_0^1 W_t f'(t,\omega) dt \right) \left(\frac{1}{6} - t + t^2 \right) \\ &\quad - \left(\int_0^1 \left(\int_0^t W_s f'(s,\omega) ds \right) dt - \int_0^1 W_t f(t,\omega) dt \right) \left(\frac{1}{2} - t \right) \end{aligned}$$

$$-\left(\int_0^t\int_0^s W_u f'(u,\omega)duds - \int_0^1\int_0^t\int_0^s W_u f'(u,\omega)dudsdt\right)$$
$$+\left(\int_0^t W_s f(s,\omega)ds - \int_0^1\int_0^t W_s f(s,\omega)dsdt\right) \tag{3}$$

for all $t \in (0,1)$ and almost all ω, noting that

$$\lim_{N\to\infty}\sum_{n\neq 0,|n|\leqq N}\frac{1}{2\pi i n}e_n(t) = \frac{1}{2} - t$$

and

$$\lim_{N\to\infty}\sum_{n\neq 0,|n|\leqq N}\frac{1}{-4\pi^2 n^2}e_n(t) = -\frac{1}{2}\left(\frac{1}{6} - t + t^2\right)$$

for all $t \in (0,1)$. Since the right hand side of (3) is differentiable with respect to $t \in (0,1)$, so is the left hand side, and

$$S'(t,\omega) = -\frac{1}{2}\left(f(1,\omega)W_1 - \int_0^1 W_t f'(t,\omega)dt\right)(-1+2t)$$
$$+\left(\int_0^1\left(\int_0^t W_s f'(s,\omega)ds\right)dt - \int_0^1 W_t f(t,\omega)dt\right)$$
$$-\int_0^t W_u f'(u,\omega)du + W_t f(t,\omega).$$

We note that

$$\left|\int_s^t W_u f'(u,\omega)du\right| \leqq \sqrt{|t-s|}\sup_{u\in[0,1]}|W_u|\sqrt{\int_0^1 |f'(u,\omega)|^2 du}.$$

Hence if we fix $s \in (0,1)$ arbitrary then we have

$$\limsup_{t\downarrow s}\frac{S'(t,\omega)-S'(s,\omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = \limsup_{t\downarrow s}\frac{W_t f(t,\omega)-W_s f(s,\omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega) \quad a.s.$$

from the law of iterated logarithm of the Brownian motion, recalling that we assume $f(t,\omega)$ is nonnegative. Set $\mathbb{S}$ be a countable dense subset of $(0,1)$. Then we have the following theorem:

Theorem 1. *Let $f(t,\omega)$ be a nonnegative function which is differentiable for almost all ω satisfying $\int_0^1 f(t,\omega)dt \in L^2(\Omega, dP)$ and $f'(t,\omega) \in L^2([0,1]\times\Omega, dtdP)$. Then we have*

$$P\left(\limsup_{t\downarrow s}\frac{S'(t,\omega)-S'(s,\omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega) \quad \text{for all } s\in\mathbb{S}\right) = 1.$$

Remark 1. *Since we assume $f(t,\omega)$ is continuous, the theorem above is sufficient to identify $f(t,\omega)$ for all $t \in [0,1]$.*

Remark 2. *Note that* 0 *th SFC $\tilde{f}_0(\omega)$ does not appear in the construction of $S'(t,\omega)$, that is, $\tilde{f}_0(\omega)$ is not necessary to identify $f(t,\omega)$.*

Moreover, since $\frac{1}{-4\pi^2 n^2}\tilde{f}_n(\omega)e_n(t)$ is a continuously differentiable function, we get the following corollary:

Corollary 1. *Let $f(t,\omega)$ be a nonnegative function which is differentiable for almost all ω satisfying $\int_0^1 f(t,\omega)dt \in L^2(\Omega, dP)$ and $f'(t,\omega) \in L^2([0,1]\times\Omega, dtdP)$. Let Λ be a finite subset of $\mathbb{Z}$ containing* 0. *Set*

$$S_\Lambda(t,\omega) = \sum_{n\in\Lambda^c} \frac{1}{-4\pi^2 n^2}\tilde{f}_n(\omega)e_n(t).$$

Then we have

$$P\left(\limsup_{t\downarrow s} \frac{S'_\Lambda(t,\omega) - S'_\Lambda(s,\omega)}{\sqrt{2(t-s)\log\log\frac{1}{t-s}}} = f(s,\omega) \quad \text{for all } s \in \mathbb{S}\right) = 1.$$

Remark 3. *Corollary1 does not mean that $\tilde{f}_n(\omega)$, $n \in \Lambda$, is reconstructed from $\{\tilde{f}_n(\omega),\ n \in \Lambda^c\}$ by deterministic procedures. Indeed, if $f(t,\omega) = 1$, then $\tilde{f}_0(\omega) = W_1$ is independent of $\{\tilde{f}_n(\omega) = \int_0^1 \overline{e_n(t)}dW_t,\ n \in \Lambda^c\}$.*

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