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Complete asymptotic analysis of second-order differential equations of Thomas-Fermi type in the framework of regular variation

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Abstract

We present a survey of results that have been obtained over the past years on asymptotic analysis of positive solutions of second order differential equations of Thomas-Fermi type in the framework of regular variation and prove some new results confirming that all solutions of sublinear equation are regularly varying providing coefficient is regularly varying.

1 Introduction.

The objective of this paper is to make a detailed survey of the recent progress in the study of the existence and the asymptotic behavior of positive solutions of the Thomas-Fermi differential equation

$$x'' = q(t)x^{\gamma}, \tag{A}$$

where $q(t)$ is a continuous regularly varying function on $[a,\infty)$, $a > 0$ and $\gamma > 0$. Equation (A) is called *sublinear* or *superlinear* according as $\gamma < 1$ or $\gamma > 1$. Our aim is to provide comprehensive overview of our present knowledge of the asymptotic analysis of positive solutions of Eq. (A) in both sublinear and superlinear case placing emphasis on some new results giving a complete answer to the three important questions: Are all solutions of (A) regularly varying? What are necessary and sufficient conditions for the existence of such solutions? Is it possible to determine the precise asymptotic formulas for such solutions?

Investigation of the equation of type (A) was inspired by the classical Thomas-Fermi atomic model described by the following nonlinear singular boundary value problem

$$x'' = \frac{1}{\sqrt{t}}x^{3/2}, \qquad x(0) = 1,\ x(\infty) = 0,$$

(see Thomas [28] and Fermi [5]).

The study of equation (A) (in fact of differential equations in general) in the framework of regular variation is initiated by V.G. Avakumović in [1]. For some physical reasons only solutions decreasing to zero of superlinear equation (A) were of interest in [1]. Later on, results on the decreasing solutions of (A) for the superlinear case were further developed in [19, 20, 21], while increasing solutions were studied recently in [16]. Sublinear case of (A) has been considered in [9, 10, 13, 14, 15, 18, 23, 26]. This paper is designed to present a survey of the main results developed in the papers listed above.

A comprehensive survey of results on the asymptotic analysis of ordinary differential equations in the framework of regular variation up to 2000 can be found in the monograph [22].

2 Regular varation

The set of regularly varying functions of index ρ is introduced by J. Karamata in 1930. by the following:

Definition 2.1. *A measurable function $f : [a, \infty) \to (0, \infty)$, $a > 0$, is said to be regularly varying at infinity of index $\rho \in \mathbb{R}$ if*

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

A measurable function $f : (0, a) \to (0, \infty)$ is said to be regularly varying at zero of index $\rho \in \mathbb{R}$ if $f\left(\frac{1}{t}\right)$ is regularly varying at ∞ i.e. if

$$\lim_{t\to 0+} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0. \tag{2.1}$$

With $\mathrm{RV}(\rho)$ we denote, the set of regularly varying functions of index ρ at infinity. If in particular, $\rho = 0$, the function f is called *slowly varying* at infinity. With SV we denote, the set of these. Saying only regularly or slowly varying function, we mean regularity at infinity.

The most complete presentation of Karamata theory and its generalizations as well as the majority of the applications are contained in Bingham et al. [2]. Comprehensive treatises on regular variation is given also in Seneta [27].

We present here a fundamental result which will be used throughout the paper.

The symbol $\sim$ denotes the asymptotic equivalence

$$f(t) \sim g(t), \quad t \to \infty \quad \iff \quad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

Proposition 2.1.

(i) *$f \in \mathrm{RV}(\rho)$ if and only if $f(t) = t^\rho \ell(t)$ with $\ell \in \mathrm{SV}$*

(ii) (Representation theorem) *$f \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t) \exp\left(\int_{t_0}^{t} \frac{\delta(s)}{s} ds \right), \quad t \geqq t_0,$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

(iii) If $f_1 \in \mathrm{RV}(\sigma_1)$*,* $f_2 \in \mathrm{RV}(\sigma_1)$*, then* $f_1 f_2 \in \mathrm{RV}(\sigma_1+\sigma_2)$*,* $f_1^\alpha \in \mathrm{RV}(\alpha\sigma_1)$ *for any* $\alpha \in \mathbb{R}$*. Moreover,* $f_1 \circ f_2 \in \mathrm{RV}(\sigma_1\sigma_2)$ *if* $f_2(t) \to \infty$*, as* $t \to \infty$*.*

(iv) If $f(t) \sim t^\alpha \ell(t)$ *as* $t \to \infty$ *with* $\ell(t) \in SV$*, then* $f(t)$ *is a regularly varying function of index* α*, i.e.* $f(t) = t^\alpha \ell^*(t)$*,* $\ell^*(t) \in SV$*, where in general* $\ell^*(t) \neq \ell(t)$*, but* $\ell^*(t) \sim \ell(t)$ *as* $t \to \infty$*.*

(v) Let $f(t)$ *be a positive, continuously differentiable for* $t > 0$ *and such that*

$$\lim_{t\to\infty} \frac{tf'(t)}{f(t)} = 0.$$

Then, $f(t)$ *is slowly varying.*

(vi) Regularly varying function of index $\sigma \neq 0$ *is almost monotone.*

Proposition 2.2. (Karamata's integration theorem) *Let* $L(t) \in \mathrm{SV}$*. Then,*

(i) *if* $\alpha > -1$*,*

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if* $\alpha < -1$*,*

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if* $\alpha = -1$*,*

$$m_1(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV}, \quad m_2(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV}$$

$$\text{and} \quad \lim_{t\to\infty} \frac{L(t)}{m_i(t)} = 0, \quad i = 1, 2.$$

If in particular for $f(t) = t^\rho \ell(t)$ the slowly varying part $\ell(t)$ tends to some positive constant as $t \to \infty$, it is called a *trivial slowly varying* one denoted by $\ell \in tr$-SV, while the function $f \in \mathrm{RV}(\rho)$ is called a *trivial regularly varying of index* ρ, denoted by $f \in tr\text{-RV}(\rho)$. Otherwise $\ell(t)$ is called a *nontrivial slowly varying* function denoted by $\ell \in ntr$-SV and $f(t)$ is called a *nontrivial* RV(ρ) *function*, denoted by $f \in ntr\text{-RV}(\rho)$.

3 The existence of all solutions of (A)

If a solution $x(t)$ of (A) exists on an interval of the form $[t_x, \infty)$, $t_x \geq a$, and is eventually nontrivial, then it is called *proper.* A nontrivial solution which is not proper is called *singular.* Further, a singular solution is classified into two types.

Definition 3.1 *(i) A solution* $x(t)$ *of* (A) *defined on* $[t_0, \infty)$ *is said to be extinct at a finite time* t_1 *(type* (S_1)*) if there exists* $t_1 > t_0$ *such that*

$$x(t) \neq 0 \ \text{ on } \ [t_0, t_1) \quad \text{and} \quad x(t) \equiv 0 \ \text{ on } \ [t_1, \infty).$$

(ii) A solution $x(t)$ of (A) *defined on* $[t_0, \infty)$ *is said to blow up at a finite time t_1 (type (S_2)) if there exists $t_1 > t_0$ such that*

$$x(t) \neq 0 \;\; on \;\; [t_0, t_1) \quad and \quad \lim_{t \to t_1 - 0} |x(t)| = \lim_{t \to t_1 - 0} |x'(t)| = \infty.$$

For the existence of singular solutions we have the following result:

Theorem 3.1 *(i) Superlinear equation (A) has solutions of type (S_2), but has no solutions of type (S_1).*

(ii) Sublinear equation (A) has solutions of type (S_1), but has no solutions of type (S_2).

PROOF. Claim (i) follows from [25, Theorems 2.1, 2.9]. Claim (ii) follows from [25, Theorems 3.1, 3.9]. □

It is known that all proper solutions of (A) are nonoscillatory and eventually strictly monotone (see [24]). If $x(t)$ satisfies (A), then so does $-x(t)$, and so considering the equation (A) in the framework of regular variation, we focus our attention on *positive* proper solutions of (A). Each positive proper solution satisfies one of four different features:

- all possible positive decreasing solutions fall into following two types

(I)
$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x'(t) = 0$$

(II)
$$\lim_{t\to\infty} x(t) = \text{const} > 0, \quad \lim_{t\to\infty} x'(t) = 0$$

- all possible positive increasing solutions fall into following two types

(III)
$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0,$$

(IV)
$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t} = \infty\,.$$

A solution of type (I), (II), (III) or (IV) is called respectively *strongly decreasing*, *asymptotically constant*, *asymptotically linear* and *strongly increasing* solution of (A).

The existence in the above four classes is described by the convergence or divergence of the two integrals

$$I = \int_a^\infty tq(t)dt, \qquad J = \int_a^\infty t^\gamma q(t)dt\,.$$

It is known that the existence of solutions of types (II) and (III) can be fully characterized in both superlinear and sublinear case.

Proposition 3.1 *(i) Equation* (A) *either superlinear or sublinear, possesses a positive solution $x(t)$ satisfying* (II) *if and only if $I < \infty$.*
(ii) Equation (A) *either superlinear or sublinear, possesses a positive solution $x(t)$ satisfying* (III) *if and only if $J < \infty$.*

PROOF. Claim (i) follows from [25, Theorems 2.3, 3.6] and claim (ii) follows from [25, Theorems 2.4, 3.7]. □

As regards the existence of strongly decreasing and strongly increasing solutions, the problem of establishing necessary and sufficient conditions turns out to be extremely difficult to solve in some cases. In fact, the existence of strongly decreasing solutions in superlinear case and strongly increasing solutions in sublinear case is completely characterized, while for the existence of strongly decreasing solutions in sublinear case and strongly increasing solutions in superlinear case, only necessary or sufficient conditions are known.

Proposition 3.2 *(i) Superlinear equation* (A) *possesses a positive solution of type* (I) *if and only if* $I = \infty$.
(ii) Superlinear equation (A) *possesses a positive solution of type* (IV) *if* $J < \infty$.
(iii) Superlinear equation (A) *does not possess positive solutions of type* (IV) *if*

$$\liminf_{t\to\infty} t^{\gamma+1} q(t) > 0\,.$$

(iv) Sublinear equation (A) *possesses a positive solution of type* (I) *if* $I < \infty$.
(v) Sublinear equation (A) *does not possess positive solutions of type* (I) *if*

$$\liminf_{t\to\infty} t^2 q(t) > 0.$$

(vi) Sublinear equation (A) *possesses a positive solution of type* (IV) *if and only if* $J = \infty$.

PROOF. Claim (i)–(vi) follow, respectively, from [25, Theorems 2.2, 2.5, 2.6, 3.2, 3.3, 3.8]. □

While the asymptotic behavior (as $t \to \infty$) of asymptotically constant and asymptotically linear solutions is reasonably clear, this is not the case of the other two types of solutions for which determination of precise asymptotic formula is not an easy problem. At the beginning of the research in this area, assuming that coefficients $q(t) \sim t^{\sigma}$, $t \to \infty$, Kamo and Naito in [11, 12] showed that, under some specific assumptions on σ, strongly increasing and strongly decreasing solutions have the form $x(t) \sim k\,t^{\rho}$ where ρ is constant depending on σ and γ.

Considering RV functions as a (nontrivial) extension of functions asymptotically equivalent to power ones, natural question arises: *How about an extension in the sense that the coefficient in the equation (A) is a regularly varying function*? Such study of asymptotic of solutions of differential equations via regular variation was initiated in the seminal paper of V.G. Avakumović [1] and about 30 years later extended and developed in [19]. Avakumović showed that assuming that coefficient $q(t)$ is regularly varying of certain index all decreasing solutions of (A) are regularly varying with precise asymptotic behavior as $t \to \infty$. Initiated by Avakumović paper asymptotic analysis of differential equations in the framework of regularly varying functions (or Karamata functions) means considering equation (A) with regularly varying coefficient $q(t)$ and also more generally nonlinear equation with regularly varying function $\phi(x)$ instead of x^{γ}.

4 Asymptotic behavior

4.1 Superlinear Thomas-Fermi equation (A).

The first paper connecting regular variation and differential equations was Avakumović [1] in 1947.

Theorem 4.1 (Avakumović [1]) *Let $q(t) : [a, \infty) \to R$ be regularly varying function of index $\sigma > -2$, then any positive solutions $x(t)$ of superlinear equation* (A) *tending to zero is regularly varying and satisfies*

$$x(t) \sim \left(\frac{(\gamma-1)^2}{(\sigma+\gamma+1)(\sigma+2)} t^2 q(t) \right)^{-\frac{1}{\gamma-1}}, \quad t \to \infty .$$

His method of proof is rather involved and make use, in addition to several artifices, of an elementary Tauberian theorem. In 1991, Geluk in [6] presented a simple and elegant proof of Theorem 4.1 using results on smoothly varying functions proved meanvile by Balkema, Haan and himself (for the proof of Theorem 4.1 see also [22, Theorem 3.2]).

However, Avakumović's paper did not attract much attention and RV functions were totally distant from the theory of DE at that time, until the investigation of Tomić and Marić [19] in 1976. Neither Avakumović nor Geluk consider the border case $\sigma = -2$ when the solutions tending to zero may still exists. Thefore, Marić and Tomić in [19, 20, 21] considering in fact the more general equation

$$(B) \qquad\qquad x'' = q(t)\phi(x),$$

with ϕ be a regularly varying function at zero of index $\gamma > 1$ proved the following (for the proof see also [22, Theorem 3.4, 3.5]):

Theorem 4.2 *Let $q(t)$ be regularly varying function of index $\sigma \geq -2$. For every positive solution $x(t)$ tending to zero as $t \to \infty$ of superlinear equation* (A) *there holds:*

1. *if $\sigma > -2$ solution $x(t)$ is regularly varying of index $\rho = \dfrac{\sigma+2}{1-\gamma}$. All such decreasing solutions of* (A) *have one and the same asymptotic behavior*

$$(4.1) \qquad x(t) \sim \left[\frac{t^2 q(t)}{\rho(\rho-1)} \right]^{\frac{1}{1-\gamma}} \quad as \quad t \to \infty .$$

2. *if $\sigma = -2$ solution $x(t)$ is slowly varying. All such decreasing solutions of* (A) *have one and the same asymptotic behavior*

$$(4.2) \qquad x(t) \sim \left((\gamma-1) \int_a^t sq(s)ds \right)^{-\frac{1}{\gamma-1}}, \quad t \to \infty .$$

Now, we will give an answer to the question which naturally arises: *Is the requirement $\sigma \geq -2$ necessary for the superlinear equation (A) to have a regularly varying solution of negative index or a slowly varying solution*? The answer is the affirmative as the following lemma shows.

Lemma 4.1 *Let $q(t) \in \mathrm{RV}(\sigma)$, $q(t) = t^{\sigma} l(t)$, $l(t) \in \mathrm{SV}$.*

(i) If equation (A) has a regularly varying solution of index $\rho < 0$, then $\sigma > -2$.

(ii) If equation (A) has a nontrivial slowly varying solution, then

$$\sigma = -2, \qquad \text{and} \qquad \int_a^t s q(s) ds = \infty. \tag{4.3}$$

PROOF. **(i)** Let $q(t) = t^{\sigma} l(t)$, $l(t) \in \mathrm{SV}$ and let $x(t) \in \mathrm{RV}(\rho)$ with $\rho < 0$, be a solution of (A) on $[T, \infty)$. We express $x(t)$ as $x(t) = t^{\rho} \xi(t)$, $\xi(t) \in \mathrm{SV}$. Since $x'(t) \to 0$, $t \to \infty$, integrating (A) from t to ∞, we have

$$-x'(t) = \int_t^{\infty} q(s) x(s)^{\gamma} ds = \int_t^{\infty} s^{\sigma + \rho\gamma} l(s) \xi(s)^{\gamma} ds, \quad t \geq T. \tag{4.4}$$

The convergence of the last integral implies $\sigma + \rho\gamma \leq -1$. However, the possibility $\sigma + \rho\gamma = -1$ is excluded. If fact, if this is the case, then (4.4) reduces to

$$-x'(t) = \int_t^{\infty} s^{-1} l(s) \xi(s)^{\gamma} ds \in \mathrm{SV},$$

which is impossible, because taking that $\lim_{t\to\infty} x(t) = c \in [0, \infty)$ the left side is integrable on $[T, \infty)$, while the right side is SV-function and thus it is not integrable on any neighborhood of infinity. Thus, we have $\sigma + \rho\gamma < -1$. Then, by Proposition 2.2, from (4.4) we get

$$-x'(t) \sim \frac{t^{\sigma+\rho\gamma+1} l(t) \xi(t)^{\gamma}}{-(\sigma + \rho\gamma + 1)}, \quad t \to \infty. \tag{4.5}$$

From the integrability of the left side of (4.5) on $[T, \infty)$ we have $\sigma + \rho\gamma + 1 \leq -1$. If $\sigma + \rho\gamma = -2$, then (4.5) reduces to

$$-x'(t) \sim t^{-1} l(t) \xi(t)^{\gamma}, \qquad t \to \infty.$$

Integration of the last relation on $[t, \infty)$ yields

$$x(t) \sim \int_t^{\infty} \frac{l(s)\xi(s)^{\gamma}}{s} ds, \qquad t \to \infty,$$

which implies that $x(t) \in \mathrm{SV}$, i.e. $\rho = 0$, an impossibility. Therefore, we must have $\sigma + \rho\gamma < -2$, in which case, integrating (4.5) from t to ∞ shows that

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2} l(t) \xi(t)^{\gamma}}{(\sigma + \rho\gamma + 1)(\sigma + \rho\gamma + 2)} = \frac{t^{\sigma+2} l(t) x(t)^{\gamma}}{(\sigma + \rho\gamma + 1)(\sigma + \rho\gamma + 2)}, \quad t \to \infty,$$

or

$$x(t) \sim \Big((\sigma + \rho\gamma + 1)(\sigma + \rho\gamma + 2) \Big)^{\frac{1}{\gamma-1}} t^{-\frac{\sigma+2}{\gamma-1}} l(t)^{-\frac{1}{\gamma-1}}, \quad t \to \infty.$$

This shows that $x(t)$ is regularly varying of index $\rho = -(\sigma+2)/(\gamma-1) < 0$. Using this value of ρ, we see from $\sigma+\rho\gamma < -2$ that $\sigma > -2$. Moreover, since $(\sigma+\rho\gamma+1)(\sigma+\rho\gamma+2) = (\rho-1)\rho$, the asymptotic behavior of $x(t)$ is given by (4.1).

(ii) Let $x(t) \in ntr - \mathrm{SV}$ be a solution of (A) on $[t_0, \infty)$. Then $x(t) \to 0$ and $x'(t) \to 0$ as $t \to \infty$. From (A) we have

$$-x'(t) = \int_t^\infty q(s)x(s)^\gamma ds = \int_t^\infty s^\sigma l(s)x(s)^\gamma ds, \quad t \geq t_0\,, \tag{4.6}$$

implying that $\sigma \leq -1$. If $\sigma = -1$, the right side of (4.6) is an SV function, which is not integrable on $[t_0, \infty)$. This contradicts the integrability of the left side of (4.6) and accordingly, it must be $\sigma < -1$. In this case from (4.6) by application of Proposition 2.2 it follows that

$$-x'(t) \sim \frac{t^{\sigma+1}l(t)x(t)^\gamma}{-(\sigma+1)}, \quad t \to \infty, \tag{4.7}$$

which by integration on $[t, \infty)$ yields

$$x(t) \sim \int_t^\infty \frac{s^{\sigma+1}l(s)}{-(\sigma+1)}\, ds, \quad t \to \infty.$$

This means that $\sigma + 1 \leq -1$. We claim that the possibility that $\sigma < -2$ is not allowed. In fact, we rewrite (4.7) as

$$-x(t)^{-\gamma}x'(t) \sim \frac{t^{\sigma+1}l(t)}{-(\sigma+1)}, \quad t \to \infty. \tag{4.8}$$

Thus, if $\sigma < -2$ the right-hand side of (4.8) is integrable on $[t_0, \infty)$, implying that $x(t)^{1-\gamma}$ tends to a finite limit as $t \to \infty$, which is contradiction. Therefore, it must be $\sigma = -2$ and (4.8) becomes

$$-x(t)^{-\gamma}x'(t) \sim t\,q(t) = tq(t), \quad t \to \infty. \tag{4.9}$$

Since $x^{1-\gamma}(\infty) \to \infty$, $t \to \infty$ the right-hand side of is not integrable on $[t_0, \infty)$ implying $\int_a^\infty tq(t)dt = \infty$. If we integrate on $[t_0, t]$ we get

$$x(t)^{1-\gamma} \sim (\gamma - 1)\int_{t_0}^t s\,q(s)\,ds, \quad t \to \infty,$$

which yields (4.2). □

Combining Theorem 4.2 with Lemma 4.1 we have the following result.

Theorem 4.3 *Let $q(t) \in \mathrm{RV}(\sigma)$, $\sigma \in \mathbb{R}$.*

1. *All strongly decreasing solutions $x(t)$ of superlinear equation (A) are regularly varying of index $\rho < 0$ with $\rho = \dfrac{\sigma+2}{1-\gamma}$, if and only if $\sigma > -2$. All such solutions have the exact asymptotic behavior given by (4.1).*

2. *All strongly decreasing solutions $x(t)$ of superlinear equation (A) are nontrivial slowly varying if and only if (4.19) is satisfied. For all such solutions (4.2) holds.*

Results of the same type for all increasing solutions of superlinear equation (A) have been obtained by Kusano, Manojlović and Marić in [16].

Theorem 4.4 *Let $q(t) \in \mathrm{RV}(\sigma), \sigma \in \mathbb{R}$. Then, all increasing solutions $x(t)$ of superlinear equation* (A) *such that $x(t)/t \to \infty$ as $t \to \infty$ are:*

1. *Regularly varying of index $\rho > 1$ with $\rho = \dfrac{\sigma+2}{1-\gamma}$, if and only if $\sigma < -\gamma - 1$, and all such solutions have the exact asymptotic behavior given by* (4.1).
2. *Nontrivial regularly varying of index* 1 *if and only if*

$$\sigma = -\gamma - 1, \quad \text{and} \quad \int_a^\infty s^\gamma q(s)ds < \infty, \tag{4.10}$$

in which case any such solution has one and the same asymptotic behavior

$$x(t) \sim t\left((\gamma-1)\int_t^\infty s^\gamma q(s)ds\right)^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{4.11}$$

Proof. See [16, Theorem 2.2]. □

4.2 Sublinear Thomas-Fermi equation (A).

Sublinear Thomas-Fermi equation (A) has been considered first by Kusano, Marić and Tanigawa [14, 15] and later on in [9, 10, 18, 13, 23, 26] by other authors. Considering equation (A) with regularly varying coefficient necessary and sufficient conditions for the existence of two types of strongly increasing RV solutions and two types of strongly decreasing RV solutions have been obtained and precise asymptotic formulas have been derived for such solutions.

Theorem 4.5 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$.*

(i) Sublinear equation (A) *possesses decreasing regularly varying solutions of index $\rho < 0$ if and only if $\sigma < -2$, in which case ρ is given by*

$$\rho = \frac{\sigma+2}{1-\gamma}. \tag{4.12}$$

All such solutions have one and the same asymptotic behavior

$$x(t) \sim \left[\frac{t^2 q(t)}{\rho(\rho-1)}\right]^{\frac{1}{1-\gamma}} \quad \text{as} \quad t \to \infty. \tag{4.13}$$

(ii) Sublinear equation (A) *possesses a nontrivial* SV*-solution if and only if*

$$\sigma = -2 \quad \text{and} \quad I = \int_a^\infty tq(t) < \infty, \tag{4.14}$$

in which case any such solution has one and the same asymptotic behavior

$$x(t) \sim \left[(1-\gamma)\int_t^\infty sq(s)\,ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{4.15}$$

(iii) Equation (A) *possesses increasing regularly varying solutions of index* $\rho > 1$ *if and only if* $\sigma > -\gamma - 1$ *in which case* ρ *is given by* (4.12) *and any such solution* $x(t)$ *has one and the same asymptotic behavior given by* (4.13).

(iv) Sublinear equation (A) *possesses a nontrivial* RV(1)*-solution if and only if*

$$\sigma = -\gamma - 1 \qquad \textit{and} \qquad \int_a^\infty t^\gamma q(t)dt = \infty. \tag{4.16}$$

in which case any such solution has one and the same asymptotic behavior

$$x(t) \sim t\Big((1-\gamma)\int_a^t s^\gamma q(s)ds\Big)^{\frac{1}{1-\gamma}}, \quad t \to \infty. \tag{4.17}$$

PROOF. Claim (i) follows from [18, Theorems 2.1] and [10, Theorem 5.1].
Claim (ii) follows from [18, Theorems 2.3] and [15, Theorem 2.4].
Claim (iii) follows from [18, Theorems 2.1] and [10, Theorem 5.2].
Claim (iv) follows from [13, Theorems 3.3] and [15, Theorem 3.4]. □

In comparison with superlinear case, the answer to the question of whether all solutions are regularly varying assuming that $q(t)$ is regularly varying has not been given in these papers. However, Matucci, Rehák [23] and Rehák [26] partially solve this problem recently. They proved more general results for positive decreasing solutions of a system of two coupled nonlinear second-order equations of Thomas-Fermi type in [23] and for positive increasing solutions of a cyclic system of n nonlinear differential equations of Thomas-Fermi type in [25]. The above-mentioned systems includes, as special cases, nonlinear scalar differential equation of type (A) and so applications of results from [23, 26] gives improvement of Theorem 4.5 by giving a positive answer to the above question in the case $\sigma < -2$ and $\sigma > -\gamma - 1$. To complete the story, we will adapt proofs in [23, 26] and presented them in Theorem 4.6 and Theorem 4.8. However, we note that in neither one of these two papers border cases $\sigma = -2$ and $\sigma = -\gamma - 1$ have not been treated, so the answer to the above question in these cases is still an open problem, which we will work out here in Theorem 4.7 and Theorem 4.9.

Throughout proofs all minimizing constants will be denoted by the same letter m and all majorizing ones by M.

Theorem 4.6 *Suppose that* $q(t) \in \mathrm{RV}(\sigma)$, $\sigma < -2$. *All possible strongly decreasing solutions* $x(t)$ *of sublinear equation* (A) *are regularly varying of index* ρ *given by* (4.12).

PROOF. Let $q(t) = t^\sigma \ell(t) \in \mathrm{RV}(\sigma)$, $\sigma < -2$. First, we show that for each strongly decreasing solution $x(t)$ there exist positive constants m, M such that

$$m\, t^\rho \ell(t)^{\frac{1}{1-\gamma}} \le x(t) \le M\, t^\rho \ell(t)^{\frac{1}{1-\gamma}}. \tag{4.18}$$

Since $x'(\infty) = x(\infty) = 0$, integrating (A) twice first from t to ∞ we have

$$-x'(t) = \int_t^\infty q(s)x(s)^\gamma\, ds, \qquad x(t) = \int_t^\infty \int_s^\infty q(r)x(r)^\gamma dr\, ds, \quad t \ge T,$$

which using that $x(t)$ is decreasing implies

$$(4.19)\qquad -x'(t) \le x(t)^\gamma \int_t^\infty q(s)\,ds, \qquad x(t) \le x(t)^\gamma \int_t^\infty \int_s^\infty q(r)dr\,ds, \quad t \ge T.$$

Because $\sigma < -2$, application of Proposition 2.2 to the both integrals in (4.19) yields that there exists $M > 0$ such that

$$(4.20)\qquad -x'(t) \le M\,x(t)^\gamma t^{\sigma+1}\ell(t), \qquad x(t) \le M\,x(t)^\gamma t^{\sigma+2}\ell(t).$$

Second inequality in (4.20) implies directly the right-hand side inequality in (4.18).

Next we prove the left-hand side inequality in (4.18). Setting $w(t) = x(t)|x'(t)|$ and

$$(4.21)\qquad \nu = \frac{\gamma+1}{\gamma+3}, \quad \mu = \frac{2}{\gamma+3}, \quad \kappa = \frac{1-\gamma}{\gamma+3}$$

an application of Young's inequality gives

$$\begin{aligned} -w'(t) &= w(t)\left(\frac{q(t)x(t)^\gamma}{|x'(t)|} + \frac{|x'(t)|}{x(t)}\right) \ge \frac{w(t)}{\mu^\mu\nu^\nu}\left(\frac{q(t)x(t)^\gamma}{|x'(t)|}\right)^\mu \left(\frac{|x'(t)|}{x(t)}\right)^\nu \\ &= \frac{w(t)}{\mu^\mu\nu^\nu}\, x(t)^{\gamma\mu-\nu}|x'(t)|^{\nu-\mu}q(t)^\mu . \end{aligned}$$

Since, $\gamma\mu - \nu = \nu - \mu = -\kappa$, we get

$$(4.22)\qquad -w'(t) \ge m\,w(t)^{1-\kappa}q(t)^\mu .$$

After dividing (4.22) with $w(t)^{1-\kappa}$, using $\kappa > 0$ and $w(\infty) = 0$, by integration on $[t,\infty]$ we obtain

$$(4.23)\qquad w(t)^\kappa \ge m \int_t^\infty q(s)^\mu\,ds = m\int_t^\infty s^{\sigma\mu}\ell(s)^\mu\,ds \quad \text{for } m > 0 .$$

Since $-\frac{\gamma+3}{2} > -2$, assumption $\sigma < -2$ implies $\sigma\mu + 1 < 0$. Thus, application of Proposition 2.2 on the right hand side of the previous inequality together with the first inequality in (4.20) gives

$$(4.24)\qquad x(t)^\kappa \ge m\,(-x'(t))^{-\kappa}\,t^{\sigma\mu+1}\ell(t)^\mu \ge m\,x(t)^{-\kappa\gamma}t^{\sigma\mu+1-(\sigma+1)\kappa}l(t)^{\mu-\kappa} .$$

Using (4.21) we have

$$(4.25)\qquad \sigma\mu + 1 - (\sigma+1)\kappa = \rho\kappa(\gamma+1), \qquad \frac{\mu-\kappa}{\kappa(\gamma+1)} = \frac{1}{1-\gamma}$$

so that from (4.24) we get the left-hand side inequality in (4.18).

It remains to prove that solutions satisfying (4.18) are regularly varying of index $\rho = \frac{\sigma+2}{1-\gamma}$. We define the function

$$(4.26)\qquad X(t) = t^\rho \ell(t)^{\frac{1}{1-\gamma}}\big(\rho(\rho-1)\big)^{-\frac{1}{1-\gamma}}, \quad \ell(t) \in \mathrm{SV}$$

It is a matter of straightforward computation with application of Proposition 2.2 to verify that $X(t)$ satisfies integral asymptotic relation

$$\int_t^\infty \int_s^\infty q(r)X(r)^\gamma dr ds \sim X(t), \quad t \to \infty \tag{4.27}$$

Put

$$k = \liminf_{t\to\infty} \frac{x(t)}{X(t)}, \qquad K = \limsup_{t\to\infty} \frac{x(t)}{X(t)}. \tag{4.28}$$

and

$$J(t) = \int_t^\infty \int_s^\infty q(r)X(r)^\gamma dr ds, \quad t \geq T,$$

In view of (4.18) it is clear that $0 < k \leq K < \infty$. Application of generalized L'Hospital's rule (see [7]) two times gives

$$\begin{aligned} k = \liminf_{t\to\infty} \frac{x(t)}{X(t)} = \liminf_{t\to\infty} \frac{x(t)}{J(t)} &\geq \liminf_{t\to\infty} \frac{x'(t)}{J'(t)} = \liminf_{t\to\infty} \frac{\int_t^\infty q(s)x(s)^\gamma ds}{\int_t^\infty q(s)X(s)^\gamma ds} \\ &\geq \liminf_{t\to\infty} \frac{x(t)^\gamma}{X(t)^\gamma} = \left(\liminf_{t\to\infty} \frac{x(t)}{X(t)}\right)^\gamma = k^\gamma. \end{aligned}$$

It follows that $k \geq k^\gamma$, implying that $k \geq 1$ because $\gamma < 1$. Similarly, we are led to the inequality $K \leq K^\gamma$, which implies that $K \leq 1$. Thus we conclude that $k = K = 1$, i.e. $x(t) \sim X(t)$, $t \to \infty$, which yields that $x(t)$ is a regularly varying function of index ρ. □

Theorem 4.7 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$ satisfies (4.14). All possible strongly decreasing solutions $x(t)$ of sublinear equation (A) are slowly varying.*

Proof. Let $q(t) = t^{-2}\ell(t) \in \mathrm{RV}(-2)$. First, we show that for each strongly decreasing solution $x(t)$ there exist positive constants m, M such that

$$m\left(\int_t^\infty s^{-1}\ell(s)ds\right)^{\frac{1}{1-\gamma}} \leq x(t) \leq M\left(\int_t^\infty s^{-1}\ell(s)ds\right)^{\frac{1}{1-\gamma}}. \tag{4.29}$$

Integrating (A) twice first from t to ∞, applying Proposition 2.2 and using that $x(t)$ is decreasing gives

$$-x'(t) \leq x(t)^\gamma t^{-1}\ell(t), \tag{4.30}$$

and

$$x(t) \leq x(t)^\gamma \int_t^\infty s^{-1}\ell(s)\, ds, \quad t \geq T,$$

implying the right-hand side inequality in (4.29).

To prove the left-hand side inequality in (4.29), first, note that in view of Proposition 2.1-(vi), there exist numbers p, r, $(r < -\sigma < p)$ such that

$$t^p q(t) \text{ almost increases and } t^r q(t) \text{ almost decreases}. \tag{4.31}$$

Bearing in mind $x(t)$ decreases, by integrating on both sides of (A) over (t, kt) with an arbitrary fixed $k > 1$, in view of (4.31), one obtains for $t \geq T$,

$$-x'(t) \geq mt^p q(t) x(kt)^\gamma \int_t^{kt} s^{-p} ds,$$

which leads to

$$-x'(t) \geq mtq(t)x(kt)^\gamma, \quad t \geq T. \tag{4.32}$$

On the other hand, by multiplying on both sides of (A) by $-x'(t)$, integrating over (t, kt) and using (4.31), one obtains for any fixed $k > 1$ and $t \geq T$

$$x'(t)^2 \geq mt^p q(t) \int_t^{kt} s^{-p} x(s)^\gamma (-x'(s)) ds,$$

implying that

$$-x'(t) \geq m \left(q(t)x(t)^{\gamma+1}\right)^{1/2} \left[1 - \left(\frac{x(kt)}{x(t)}\right)^{\gamma+1}\right]^{1/2}. \tag{4.33}$$

From (4.32) and (4.33) we shall derive the following inequality, holding for all $t \geq T$

$$-x'(t) \geq mtq(t)x(t)^\gamma. \tag{4.34}$$

Obviously, the behavior of the quotient $0 < x(kt)/x(t) < 1$ is essential in that. For, if e.g. $\limsup_{t\to\infty} x(kt)/x(t) = 1$, inequality (4.33) is useless. Therefore consider the following alternative:

Take a fixed $k > 1$, and an arbitrary fixed α such that $0 < \alpha < 1$. There holds:

Either

$$\frac{x(kt)}{x(t)} \geq \alpha \tag{4.35}$$

for all t belonging to some intervals $\bar{I}_n$, $n \geq 1$ which might be all ultimately neighbouring when $\bigcup_{n=1}^{\infty} \bar{I}_n = [T, \infty)$ for some $T \geq a$, or

$$\frac{x(kt)}{x(t)} < \alpha \tag{4.36}$$

for all to belonging to some intervals $\underline{I}_n$, $n \geq 1$, which again might be all ultimately neighbouring when $\bigcup_{n=1}^{\infty} \underline{I}_n = [T, \infty)$ for some $T \geq a$.

In general, due to the continuity of $x(t)$, one has

$$\bigcup_{n\geq 1} (\underline{I}_n \cup \bar{I}_n) = [T, \infty). \tag{4.37}$$

Now, if (4.35) holds, inequality (4.32) gives (4.34) for all $t \in \overline{I}_n$.

However, if all $\overline{I}_n$ are ultimately neighbouring then $\underline{I}_n$ do not exist and so (4.34) holds for all $t \geq T$.

If, on the other hand, (4.36) holds, choose a sequence $\{t_n\}$, $n \geq 1$ of arbitrary points $t_n \in \underline{I}_n$ so that (4.36) holds for $t = t_n$. But then, because of Lemma [22, Lemma 3.1], there exist numbers $0 < \mu < 1$ and $0 < \alpha' < 1$ such that $x(kt)/x(t) < \alpha'$ for all $t \in [\mu t_n, t_n]$. Hence, from (4.33) and the preceding inequality, one obtains

$$-x'(t) \geq m\big(q(t)x(t)^{\gamma+1}\big)^{1/2}, \quad t \in [\mu t_n, t_n], \tag{4.38}$$

so after dividing by $x(t)^{\frac{\gamma+1}{2}}$ and integrating over $[\mu t_n, t_n]$, since $\gamma < 1$, we get

$$x(\mu t_n)^{\frac{1-\gamma}{2}} \geq m \int_{\mu t_n}^{t_n} \big(t^p q(t)\big)^{1/2} t^{-p/2} dt \geq m \left(t_n^p q(\mu t_n)\right)^{1/2} \int_{\mu t_n}^{t_n} t^{-p/2} dt \geq m t_n q(\mu t_n)^{1/2}, \tag{4.39}$$

which multiplying by $q(\mu t_n)^{1/2} x(\mu t_n)^{\gamma}$ gives

$$\big(q(\mu t_n)x(\mu t_n)^{\gamma+1}\big)^{1/2} \geq m t_n q(\mu t_n) x(\mu t_n)^{\gamma}. \tag{4.40}$$

Since t_n is arbitrary in $\underline{I}_n$, inequalities (4.38) and (4.40) together give (4.34) for all $t \in \underline{I}_n$.

Again, if all $\underline{I}_n$ are ultimately neighbouring, then $\overline{I}_n$ do not exist, t_n is arbitrary in $[T, \infty)$ and (4.34) holds for all $t \geq T$. Finally, if both sequences of considered intervals exist, then again (4.34) holds for all $t \geq T$ due to (4.37).

To conclude the proof divide (4.34) by $x(kt)^{\gamma}$, integrate over $(t/k, \infty)$ to obtain for $t \geq T$

$$x(t)^{1-\gamma} \geq m \int_t^{\infty} s q(s) ds = m \int_t^{\infty} s^{-1} \ell(s) ds,$$

which because $1 - \gamma > 0$ is the same as the left-hand side of inequality (4.29).

It remains to prove that solutions satisfying (4.29) are slowly varying. Therefore, in view of (4.30) and (4.29) we have

$$0 \leq t \frac{-x'(t)}{x(t)} \leq M x(t)^{\gamma-1} \ell(t) \leq M \ell(t) \left(\int_t^{\infty} s^{-1} \ell(s) ds \right)^{-1} \tag{4.41}$$

Since, by Proposition (2.2)-(iii)

$$\lim_{t\to\infty} \ell(t) \left(\int_t^{\infty} s^{-1} \ell(s) ds \right)^{-1} = 0,$$

we conclude that

$$\lim_{t\to\infty} t \frac{x'(t)}{x(t)} = 0.$$

Thus, $x(t) \in \mathrm{SV}$ by Proposition 2.2-(v). $\square$

Theorem 4.8 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$, $\sigma > -\gamma - 1$. All possible strongly increasing solutions $x(t)$ of sublinear equation* (A) *are regularly varying of index ρ given by* (4.12).

PROOF. Let $q(t) = t^{\sigma}\ell(t) \in \mathrm{RV}(\sigma)$, $\sigma > -\gamma - 1$. First, we show that for each strongly increasing solution $x(t)$ there exist positive constants m, M such that

$$m\, t^{\rho}\ell(t)^{\frac{1}{1-\gamma}} \le x(t) \le M\, t^{\rho}\ell(t)^{\frac{1}{1-\gamma}}\,. \tag{4.42}$$

for all large t. Using that $x(t) \to \infty$, $t \to \infty$ we have

$$x(t) \sim \int_T^t x'(s)ds, \quad t \ge T\,,$$

which since x' is increasing gives

$$x(t) \le t\,x'(t), \quad t \ge T\,. \tag{4.43}$$

Integration of (A) from T to t, since $x'(t) \to \infty$, $t \to \infty$, in view of (4.43), gives

$$x'(t) \sim \int_T^t q(s)x(s)^{\gamma}\, ds \le x'(t)^{\gamma} \int_T^t q(s)s^{\gamma}\, ds\,.$$

Using $\sigma + \gamma > -1$ application of Proposition 2.2 to the above integral yields that there exists $M > 0$ such that

$$x'(t)^{1-\gamma} \le M\, t^{\gamma+1}q(t) = M\, t^{\gamma+\sigma+1}\ell(t) \tag{4.44}$$

which together with (4.43) implies the right-hand side inequality in (4.42).

Next we prove the left-hand side inequality in (4.42). Setting $w(t) = x(t)x'(t)$ and ν, μ, κ as in (4.21), application of Young's inequality gives

$$\begin{aligned} w'(t) &= w(t)\left(\frac{q(t)x(t)^{\gamma}}{x'(t)} + \frac{x'(t)}{x(t)}\right) \ge \frac{w(t)}{\mu^{\mu}\nu^{\nu}}\left(\frac{q(t)x(t)^{\gamma}}{x'(t)}\right)^{\mu}\left(\frac{x'(t)}{x(t)}\right)^{\nu} \\ &= \frac{w(t)}{\mu^{\mu}\nu^{\nu}}\, x(t)^{\gamma\mu-\nu}x'(t)^{\nu-\mu}q(t)^{\mu} = \frac{1}{\mu^{\mu}\nu^{\nu}}\, w(t)^{1-\kappa}q(t)^{\mu}\,. \end{aligned}$$

and integration on $[T, t]$ implies

$$w(t)^{\kappa} \ge m \int_T^t q(s)^{\mu}\, ds\,. \tag{4.45}$$

Since $-\frac{\gamma+3}{2} < -\gamma - 1$, assumption $\sigma > -\gamma - 1$ implies $\sigma\mu + 1 > 0$. Thus, application of Proposition 2.2 on the right hand side of the previous inequality together with (4.44) gives

$$x(t)^{\kappa} \ge m\, x'(t)^{-\kappa}\, t^{\sigma\mu+1}\ell(t)^{\mu} \ge m\, t^{\sigma\mu+1-\kappa\frac{\gamma+\sigma+1}{1-\gamma}}\,\ell(t)^{\mu-\frac{\kappa}{1-\gamma}}, \tag{4.46}$$

for some $m > 0$. Using (4.21) we have

$$\sigma\mu + 1 - \kappa\frac{\gamma+\sigma+1}{1-\gamma} = \frac{\sigma+2}{\gamma+3} = \kappa\,\rho, \qquad \mu - \frac{\kappa}{1-\gamma} = \frac{1}{\gamma+3}$$

so that from (4.46) we get the left-hand side inequality in (4.42).

To prove that solutions satisfying (4.29) are regularly varying of index $\rho = \frac{\sigma+2}{1-\gamma}$, we define the function $X(t)$ with (4.26) and with application of Proposition 2.2 verify that $X(t)$ satisfies integral asymptotic relation

$$\int_T^t \int_T^s q(r)X(r)^\gamma dr ds \sim X(t), \quad t \to \infty \tag{4.47}$$

Put k, K as in (4.28) and in view of (4.29) it is clear that $0 < k \leq K < \infty$. Application of L'Hospital's rule gives $k \geq k^\gamma$ and $K \leq K^\gamma$, implying that $k \geq 1$ and $K \leq 1$. Thus we conclude that $k = K = 1$, i.e. $x(t) \sim X(t)$, $t \to \infty$, which yields that $x(t)$ is a regularly varying function of index ρ.

Theorem 4.9 *Suppose that $q(t) \in \mathrm{RV}(\sigma)$ satisfies* (4.16). *All possible strongly increasing solutions $x(t)$ of sublinear equation* (A) *are regularly varying of index* 1.

PROOF. Let $q(t) = t^{-\gamma-1}\ell(t) \in \mathrm{RV}(-\gamma-1)$. First, we show that for each strongly increasing solution $x(t)$ there exist positive constants m, M such that

$$mt\left(\int_T^t s^{-1}\ell(s)ds\right)^{\frac{1}{1-\gamma}} \leq x(t) \leq Mt\left(\int_T^t s^{-1}\ell(s)ds\right)^{\frac{1}{1-\gamma}}. \tag{4.48}$$

Integration of (A) from T to t, since $x'(t) \to \infty$, $t \to \infty$, in view of (4.43), gives

$$x'(t)^{1-\gamma} \leq M \int_T^t s^{-1}\ell(s)\, ds \in \mathrm{SV}.$$

which together with (4.43) and application of Proposition 2.2, implies the right-hand side inequality in (4.48).

To prove the left-hand side inequality in (4.48) we perform the substitution $x(t) = ty(t)$ in (A) and obtain the following differential equation for $y(t)$:

$$\left(t^2y'(t)\right)' = t^{\gamma+1}q(t)y(t)^\gamma, \tag{C}$$

Obviously $y(t)$ increases and $y(t) \to \infty$, as $t \to \infty$. Clearly, in order to prove the left-hand side inequality in (4.48) it suffices to prove that $y(t)$ satisfies inequalities

$$y(t) \geq m\left(\int_T^t s^{-1}\ell(s)ds\right)^{\frac{1}{1-\gamma}}, \qquad t \geq T. \tag{4.49}$$

Bearing in mind $y(t)$ increases, by integrating on both sides of (C) over (t, kt) with an arbitrary fixed $k > 1$, in view of (4.31), one obtains for $t \geq T$,

$$y'(kt) \geq mt^r q(kt)y(t)^\gamma \int_t^{kt} s^{\gamma+1-r} ds,$$

which leads to

$$y'(kt) \geq mt^\gamma q(kt)y(t)^\gamma, \quad t \geq T. \tag{4.50}$$

On the other hand, by multiplying on both sides of (C) by $t^2y'(t)$, integrating over (t, kt) and using that the function $s^{r+\gamma+3}q(s)$ is almost decreasing for some r, one obtains for any fixed $k > 1$ and $t \geq T$

$$y'(kt)^2 \geq mt^{\gamma-1-r}q(kt)t^r \int_t^{kt} s^{-r}y(s)^\gamma y'(s)ds,$$

implying that

$$y'(kt) \geq m\left(t^{\gamma-1}q(kt)y(kt)^{\gamma+1}\right)^{1/2}\left\{1-\left(\frac{y(t)}{y(kt)}\right)^{\gamma+1}\right\}^{1/2}. \tag{4.51}$$

From (4.50) and (4.51) we shall derive the following inequality, holding for all $t \geq T$

$$y'(kt) \geq mt^\gamma q(kt)y(kt)^\gamma. \tag{4.52}$$

Obviously, the behavior of the quotient $0 < y(t)/y(kt) < 1$ is essential in that. For, if e.g. $\limsup_{t\to\infty} y(t)/y(kt) = 1$, inequality (4.51) is useless. Therefore consider the following alternative:

Take a fixed $k > 1$ and an arbitrary fixed α such that $0 < \alpha < 1$. There holds:

Either

$$\frac{y(t)}{y(kt)} \geq \alpha \tag{4.53}$$

for all t belonging to some intervals $\bar{I}_n$, $n \geq 1$ which might be all ultimately neighbouring when $\bigcup\limits_{n=1}^{\infty} \bar{I}_n = [T, \infty)$ for some $T \geq a$, or

$$\frac{y(t)}{y(kt)} < \alpha \tag{4.54}$$

for all to belonging to some intervals $\underline{I}_n$, $n \geq 1$, which again might be all ultimately neighbouring when $\bigcup\limits_{n=1}^{\infty} \underline{I}_n = [T, \infty)$ for some $T \geq a$.

In general, due to the continuity of $y(t)$, one has

$$\bigcup_{n\geq1}(\underline{I}_n \cup \bar{I}_n) = [T, \infty). \tag{4.55}$$

Now, if (4.53) holds, inequality (4.50) gives (4.52) for all $t \in \bar{I}_n$.

However, if all $\bar{I}_n$ are ultimately neighbouring then $\underline{I}_n$ do not exist and so (4.52) holds for all $t \geq T$.

If, on the other hand, (4.54) holds, choose a sequence $\{t_n\}$, $n \geq 1$ of arbitrary points $t_n \in \underline{I}_n$ so that (4.54) holds for $t = t_n$. But then, because of Lemma [16, Lemma1.1, Remark1.1] , there exists $0 < \alpha' < 1$ such that $y(t)/y(kt) < \alpha'$ for all $t \in [t_n, kt_n]$. Hence, from (4.51) and the preceding inequality, one obtains

$$y'(kt) \geq m\left(t^{\gamma-1}q(kt)y(kt)^{\gamma+1}\right)^{1/2}, \quad t \in [t_n, kt_n], \tag{4.56}$$

so after dividing by $y(kt)^{\frac{\gamma+1}{2}}$ and integrating over $[t_n, kt_n]$, since $\gamma > 1$, we get

$$y(kt_n)^{\frac{1-\gamma}{2}} \geq m \int_{t_n}^{kt_n} \left(t^{\gamma-1} q(kt)\right)^{1/2} dt. \tag{4.57}$$

Using (4.31) for the integral on the right-hand side of (4.39) we have

$$\int_{t_n}^{kt_n} \left(t^{\gamma-1} q(kt)\right)^{1/2} dt \geq m \left(t_n^r q(kt_n)\right)^{1/2} \int_{t_n}^{kt_n} t^{\frac{\gamma-1-r}{2}} dt \geq m\left(t_n^{\gamma+1} q(kt_n)\right)^{1/2},$$

which together with (4.57) gives

$$\left(t_n^{\gamma-1} q(kt_n) y(kt_n)^{\gamma+1}\right)^{1/2} \geq t_n^\gamma q(kt_n) y(kt_n)^\gamma. \tag{4.58}$$

Since t_n is arbitrary in $\underline{I}_n$, inequalities (4.56) and (4.58) together give (4.52) for all $t \in \underline{I}_n$.

Again, if all $\underline{I}_n$ are ultimately neighbouring, then $\bar{I}_n$ do not exist, t_n is arbitrary in $[T, \infty)$ and (4.52) holds for all $t \geq T$. Finally, if both sequences of considered intervals exist, then again (4.52) holds for all $t \geq T$ due to (4.55).

At this point we observe that one could not use such a procedure with the intervals $\underline{I}_n$ instead of $[t_i, kt_i]$, since the former may tend to 0 when $n \to \infty$.

To conclude the proof divide (4.52) by $y(kt)^\gamma$, integrate over $[T/k, t/k]$ to obtain for $t \geq T$

$$y(t)^{1-\gamma} \geq m \int_T^t s^\gamma q(s) ds = m \int_T^t s^{-1} \ell(s) ds,$$

which because $1 - \gamma > 0$ is the same as the right-hand side of inequality (4.49) implying the left hand side inequality in (4.48) for $x(t)$.

It remains to prove that solutions satisfying (4.48) are RV(1). Therefore, in view of (4.43) and (4.29) we have

$$0 \leq t\frac{x''(t)}{x'(t)} \leq t^2 q(t) x(t)^{\gamma-1} = t^{1-\gamma} \ell(t) x(t)^{\gamma-1} \leq M\ell(t) \left(\int_T^t s^{-1} \ell(s) ds\right)^{-1} \tag{4.59}$$

which by Proposition (2.2)-(iii) yields

$$\lim_{t\to\infty} t\frac{x''(t)}{x'(t)} = 0.$$

Thus, $x'(t) \in \mathrm{SV}$ and by application of Proposition 2.2 we get

$$x(t) \sim \int_T^t x'(s)\, ds \sim tx'(t) \in \mathrm{RV}(1), \quad t \to \infty$$

implying that $x(t) \in \mathrm{RV}(1)$.

Combining Theorem 4.5 with Theorems 4.6-4.9 we have the following results for sublinear equation (A).

Theorem 4.10 *Let $q(t) \in \mathrm{RV}(\sigma), \sigma \in \mathbb{R}$. Then, all increasing solutions $x(t)$ of sublinear equation* (A) *such that $x(t)/t \to \infty$ as $t \to \infty$ are:*

1. *Regularly varying of index $\rho > 1$ with $\rho = \dfrac{\sigma+2}{1-\gamma}$, if and only if $\sigma > -\gamma - 1$, in which case any such solution has one and the same asymptotic behavior given by* (4.13).
2. *Nontrivial regularly varying of index* 1 *if and only if* (4.16) *holds, in which case any such solution has one and the same asymptotic behavior given by* (4.17).

Theorem 4.11 *Let $q(t) \in \mathrm{RV}(\sigma), \sigma \in \mathbb{R}$. Then, all decreasing solutions $x(t)$ of sublinear equation* (A) *such that $x(t) \to 0$ as $t \to \infty$ are:*

1. *Regularly varying of index $\rho < 0$ with $\rho = \dfrac{\sigma+2}{1-\gamma}$, if and only if $\sigma < -2$, in which case any such solution has one and the same asymptotic behavior given by* (4.13).
2. *Nontrivial slowly varying if and only if* (4.14) *holds, in which case any such solution has one and the same asymptotic behavior given by* (4.15).

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