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Inverse and direct bifurcation problems for nonlinear elliptic equations

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1 Elliptic inverse bifurcation problems

We first consider

$$\begin{aligned} -\Delta u + f(u) &= \lambda u && \text{in } \Omega, \\ u &> 0, && \text{in } \Omega, \\ u(0) &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is an appropriately smooth bounded domain, and $\lambda > 0$ is a parameter. We assume that $f(u)$ is unknown to satisfy the conditions (A.1)–(A.3):

(A.1) $f(u)$ is a function of C^1 for $u \geq 0$ satisfying $f(0) = f'(0) = 0$.

(A.2) $f(u)/u$ is strictly increasing for $u \geq 0$.

(A.3) $f(u)/u \to \infty$ as $u \to \infty$.

The typical examples of $f(u)$ which satisfy (A.1)–(A.3) are as follows.

$$\begin{aligned} f(u) &= u^p \quad (p > 1), \\ f(u) &= u^p + u^m \quad (p > m > 1). \end{aligned}$$

Our first purpose is to study the inverse bifurcation problems in L^q-framework ($1 \leq q \leq \infty$). From mathematical point of view, since (1.1) is regarded as an eigenvalue problem, it seems natural to treat it in L^2-framework. Moreover, from biological point of view, it also seems significant to investigate it in L^1-framework.

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Now we introduce the notion of L^q-bifurcation curve. We know the following fundamental properties of bifurcation diagrams of (1.1).

(1) Let $1 \le q \le \infty$ be fixed. Let $\|\cdot\|_q$ be L^q-norm. For any given $\alpha > 0$, there exists a unique solution pair $(\lambda, u) = (\lambda(q, \alpha), u_\alpha) \in \mathbf{R}_+ \times C^2(\bar{\Omega})$ such that $\|u_\alpha\|_q = \alpha$.

(2) The following set gives all the solutions of (1.1):

$$\{(\lambda(q,\alpha), u_\alpha) : \alpha > 0\} \subset \mathbf{R}_+ \times C^2(\bar{\Omega})$$

(3) $\lambda(q,\alpha) \to \lambda_1$ $(\alpha \to 0,\ \lambda_1:$ the first eigenvalue of $-\Delta_D)$, $\lambda(q,\alpha) \nearrow \infty \quad (\alpha \to \infty)$.

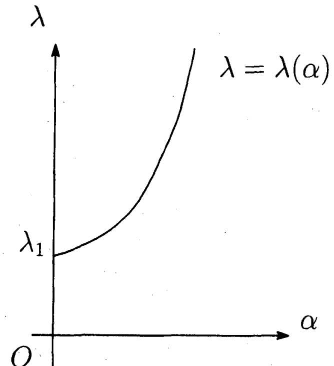

Fig. 1

Let $f(u) = f_1(u)$ and $f(u) = f_2(u)$ be unknown to satisfy (A.1)–(A.3). Furthermore, let

$$F_j(u) := \int_0^u f_j(s)ds \qquad (j = 1, 2).$$

Assume that F_1 and F_2 satisfy the following condition (B.1).

(B.1) Let $W := \{u \ge 0 : F_1(u) = F_2(u)\}$. Then W consists, at most, of the (finite or infinite numbers of) intervals and the points $\{u_n\}_{n=1}^\infty$ whose accumulation point is only ∞.

Theorem 1.1. [14] *Assume that f_1 and f_2 are unknown to satisfy (A.1)–(A.3) and (B.1). Furthermore, if $N \ge 2$, then assume that f_1 and f_2 satisfy the following (A.4).*

(A.4) For $u, v \ge 0$,

$$F_j(u+v) \le C(F_j(u) + F_j(v)) \qquad (j = 1, 2).$$

Suppose $\lambda_1(2,\alpha) = \lambda_2(2,\alpha)$ for any $\alpha > 0$. Here, $\lambda_j(2,\alpha)$ is the L^2-bifurcation curve associated with $f(u) = f_j(u)$ $(j = 1, 2)$. Then $f_1(u) \equiv f_2(u)$ for $u \ge 0$.

2 Sketch of the Proof of Theorem 1.1

For simplicity, we prove Theorem 1.1 for the case $N = 1$. Let $\Omega = I = (0,1)$. For $j = 1,2$ and $v \in H_0^1(I)$, let

$$\Phi_j(v) := \frac{1}{2}\|v'\|_2^2 + \int_0^1 F_j(v(t))dt. \tag{2.1}$$

For $\alpha > 0$, we put

$$M_\alpha := \{v \in H_0^1(I) : \|v\|_2 = \alpha\}.$$

For $j = 1,2$ and $\alpha > 0$ we put

$$C_j(\alpha) := \min\{\Phi_j(v) : v \in M_\alpha\}. \tag{2.2}$$

By taking a minimizing sequence, Lagrange multiplier theorem and strong maximum principle, there exists a Lagrange multiplier $\lambda_j(\alpha)$ and a unique minimizer $u_{j,\alpha} \in M_\alpha$ which satisfies (1.1) with $f = f_j$. Then by direct calculation, we obtain the following lemma.

Lemma 2.1. *$C_1(\alpha) = C_2(\alpha)$ for $\alpha \geq 0$.*

Now we give the sketch of the proof of Theorem 1.1.

Sketch of the Proof of Theorem 1.1 for $N = 1$.

Clearly, $0 \in W$, where $W := \{u \geq 0 : F_1(u) = F_2(u)\}$. First, assume that $0 \in W$ is contained in the interval $[0, \epsilon]$ for some constant $0 < \epsilon \ll 1$. This implies that for $0 \leq u \leq \epsilon$,

$$F_1(u) = F_2(u).$$

Let K be a connected component of W satisfying $[0, \epsilon] \subset K$. Then $K = [0, u_1]$. If $u_1 < \infty$, then without loss of generality, by (B.1), there exists a constant $0 < \epsilon \ll 1$ such that

$$\begin{aligned} F_1(u) &= F_2(u) \quad (0 \leq u \leq u_1), \\ F_1(u) &< F_2(u), \quad (u_1 < u < u_1 + \epsilon). \end{aligned}$$

Now we choose $\alpha > 0$ satisfying $\|u_{2,\alpha}\|_\infty = u_1 + \epsilon$. Then

$$\begin{aligned} C_1(\alpha) &\leq \Phi_1(u_{2,\alpha}) = \frac{1}{2}\|u_{2,\alpha}'\|_2^2 + \int_0^1 F_1(u_{2,\alpha}(t))dt \\ &< \frac{1}{2}\|u_{2,\alpha}'\|_2^2 + \int_0^1 F_2(u_{2,\alpha}(t))dt \\ &= \Phi_2(u_{2,\alpha}) = C_2(\alpha). \end{aligned}$$

This contradicts Lemma 2.1. Therefore, we see that $u_1 = \infty$ and $K = [0, \infty)$. This implies $F_1(u) \equiv F_2(u)$, and consequently, $f_1(u) \equiv f_2(u)$.

We can also treat the case where $0 \in W$ is an isolated point in W. Thus the proof is complete. ∎

3 L^1-inverse bifurcation problems

It seems that the assumption $\lambda_1(2,\alpha) = \lambda_2(2,\alpha)$ for any $\alpha > 0$ in Theorem 1.1 seems little bit strong. It seems better to consider the problem under more weaker condition

$$\lambda_1(q,\alpha) \approx \lambda_2(q,\alpha) \quad \text{in some sense for} \quad \alpha > \alpha_0, \tag{3.1}$$

where $\alpha_0 > 0$ is a constant. To do this, we consider the following inverse problem.

Let $\lambda_0(1,\alpha)$ be the L^1-bifurcation curve associated with $f(u) = u^p$ $(p > 1)$. Furthermore, let $\lambda(1,\alpha)$ be the L^1-bifurcation curve associated with $f(u) = u^p + g(u)$, where $g(u)$ is an unknown function.

Problem. Assume that for $\alpha \gg 1$

$$\lambda(1,\alpha) \approx \lambda_0(1,\alpha)$$

in some sense. Then can we conclude $g(u) \equiv 0$?

To solve this problem, we assume the following conditions on g.

(B.2) $g(u)$ is C^1 function for $u \geq 0$ with compact support.

We note that $\eta_1(x) = \eta_2(x)$ nearly exponentially for $x \gg 1$ implies that

$$\eta_1(x) = \eta_2(x) + o(x^{-N}) \quad (x \to \infty)$$

for any $N \in \mathbb{N}$.

Theorem 3.1 [16]. *Let $N = 1$ and consider (1.1). Let $p > 1$ be a given constant and assume that $f(u) = u^p + g(u)$ satisfies (A.1)-(A.3) and (B.2), where $g(u)$ is unknown. Suppose $\lambda(1,\alpha) = \lambda_0(1,\alpha)$ nearly exponentially. Then $g(u) \equiv 0$.*

The proof of Theorem 3.1 relies on the fact that the equation (1.1) is ODE, and we treat it in L^1-framework with the aid of the time map.

Now we give the brief sketch of the proof of Theorem 3.1. Without loss of generality, we assume that supp $g \subset [a,b]$ $(0 \leq a < b)$. C denotes arbitrary positive constants independent of $\lambda \gg 1$.

We know that $(\lambda, u_\lambda) \in \mathbf{R}_+ \times C^2(\bar{I})$: the solution of (1.1) for given $\lambda > \pi^2$. Therefore, $\alpha = \|u_\lambda\|_1$. We write $\lambda = \lambda(\alpha)$ for simplicity. Let

$$G(u) := \int_0^u g(s)ds.$$

For two functions $X(\lambda)$ and $Y(\lambda)$,

$$X(\lambda) \sim Y(\lambda)$$

implies

$$C^{-1}Y(\lambda) \le X(\lambda) \le CY(\lambda) \quad (\lambda \gg 1). \tag{3.2}$$

It is well known that for $\lambda \gg 1$,

$$\|u_\lambda\|_\infty^{p-1} = \lambda\left(1 + O(e^{-C\sqrt{\lambda}})\right). \tag{3.3}$$

We know that for $\lambda > \pi^2$

$$u_\lambda(t) = u_\lambda(1-t), \quad 0 \le t \le 1, \tag{3.4}$$

$$u_\lambda\left(\frac{1}{2}\right) = \max_{0\le t\le 1} u_\lambda(t) = \|u_\lambda\|_\infty, \tag{3.5}$$

$$u'_\lambda(t) > 0, \quad 0 \le t < \frac{1}{2}. \tag{3.6}$$

For $\lambda > \pi^2$ and $0 \le s \le 1$, let

$$L_\lambda(s) := 1 - s^2 - \frac{2}{p+1}(1 - s^{p+1}), \tag{3.7}$$

$$\begin{aligned} M_\lambda(s) &:= 1 - s^2 - \frac{2}{p+1}\frac{\|u_\lambda\|_\infty}{\lambda}(1 - s^{p+1}) \\ &\quad - \frac{2}{\lambda\|u_\lambda\|_\infty^2}\left(G(\|u_\lambda\|_\infty) - G(\|u_\lambda\|_\infty s)\right), \end{aligned} \tag{3.8}$$

$$\begin{aligned} U_\lambda &:= \frac{2(\|u_\lambda\|_\infty - \lambda)}{(p+1)\lambda}\int_0^1 \frac{(1-s)(1-s^{p+1})}{\sqrt{M_\lambda(s)}\sqrt{L_\lambda(s)}(\sqrt{M_\lambda(s)} + \sqrt{L_\lambda(s)})}ds, \\ V_\lambda &:= \frac{2}{\lambda\|u_\lambda\|_\infty^2}\int_0^1 \frac{(1-s)(G(\|u_\lambda\|_\infty) - G(\|u_\lambda\|_\infty s))}{\sqrt{M_\lambda(s)}\sqrt{L_\lambda(s)}(\sqrt{M_\lambda(s)} + \sqrt{L_\lambda(s)})}ds. \end{aligned}$$

Lemma 3.2. *For* $\lambda \gg 1$

$$\|u_\lambda\|_\infty - \|u_\lambda\|_1 = \frac{1}{\sqrt{\lambda}}\|u_\lambda\|_\infty(C(1) + U_\lambda + V_\lambda), \tag{3.9}$$

where $C(1)$ is a constant determined explicitly.

Lemma 3.3. *For* $\lambda \gg 1$

$$|U_\lambda| \le C\sqrt{\lambda}e^{-C\sqrt{\lambda}}. \tag{3.10}$$

Proposition 3.4. *Assume that $V_\lambda = 0$ for $\lambda \gg 1$. That is,*

$$\|u_\lambda\|_\infty - \|u_\lambda\|_1 = \frac{1}{\sqrt{\lambda}}\|u_\lambda\|_\infty(C(1) + U_\lambda). \tag{3.11}$$

Then for $\alpha \gg 1$,

$$\lambda(\alpha) = \alpha^{p-1} + C_1\alpha^{(p-1)/2} + \sum_{k=0}^{N} a_k\alpha^{k(1-p)/2} + o(\alpha^{N(1-p)/2}), \tag{3.12}$$

where C_1, $\{a_j\}_{j=0}^{N}$ are constants determined explicitly.

To prove Proposition 3.3, we would like to calculate V_λ precisely.

Lemma 3.5. *Let $H(\theta) := G(b) - G(\theta)$. Then, for $\lambda \gg 1$,*

$$V_\lambda \sim \sum_{k=0}^{\infty}\left(C_k\int_0^b H(\theta)\theta^k d\theta\right)\|u_\lambda\|_\infty^{-(p+2+k)},$$

where $C_k \neq 0$ ($k \in \mathbf{N}_0 := \mathbf{N}\bigcup\{0\}$) is a constant.

It should be mentioned that, to prove Lemma 3.5, we need the condition $q = 1$.

By using Lemma 3.5 and the assumption that $\lambda(1,\alpha) = \lambda_0(1,\alpha)$ nearly exponentially, we obtain the following Lemma 3.6.

Lemma 3.6. *Let $H(\theta) := G(b) - G(\theta)$. Then for any non-negative integer n,*

$$\int_0^b H(\theta)\theta^n d\theta = 0. \tag{3.13}$$

We can prove Lemma 3.6, since we treat it in L^1-framework. Theorem 3.1 follows from Lemma 3.6. Thus the proof is complete. ∎

4 Direct problems

We consider the semilinear non-autonomous logistic equation of population dynamics

$$\begin{aligned} -u''(t) + k(t)u(t)^p &= \lambda u(t), \quad t \in I := (-1/2, 1/2), && (4.1)\\ u(t) &> 0 \quad t \in I, && (4.2)\\ u(-1/2) &= u(1/2) = 0, && (4.3)\end{aligned}$$

where $p > 1$ is a given constant, and $\lambda > 0$ is a parameter. We assume that $k(t) \in C^2(\bar{I})$ satisfies the following conditions.

$$k(t) > 0, \quad k(t) = k(-t), \quad t \in \bar{I}, \tag{4.4}$$

$$k'(t) \geq 0, \quad 0 \leq t \leq 1/2. \tag{4.5}$$

The local and global structure of the bifurcation diagrams of (4.1)–(4.3) have been investigated by many authors in L^∞-framework. Especially, the following basic properties are well known.

(**a**) For each $\lambda > \pi^2$, there exists a unique solution $u_\lambda \in C^2(\bar{I})$ such that (λ, u_λ) satisfies (4.1)–(4.3).

(**b**) The set $\{(\lambda, u_\lambda) : \lambda > \pi^2\}$ gives all the solutions of (1.1)–(1.3) and is a continuous unbounded curve in $\mathbb{R}_+ \times C(\bar{I})$ emanating from $(\pi^2, 0)$.

(**c**) $\pi^2 < \mu < \lambda$ holds if and only if $u_\mu < u_\lambda$ in I.

For a given $\alpha > 0$, we denote by $(\lambda(q,\alpha), u_\alpha) \in \{\lambda > \pi^2\} \times C^2(\bar{I})$ the solution pair of (4.1)–(4.3) with $\|k^{1/(p-1)}u_\alpha\|_q = \alpha$, which uniquely exists by (**c**) above. We call the graph $\lambda = \lambda(q,\alpha)$ $(\alpha > 0)$ the L^q-bifurcation diagram of (4.1)–(4.3). Then we know that

(**d**) $\lambda(q,\alpha)$ is increasing for $\alpha > 0$ and $\lambda(q,\alpha) \to \infty$ as $\alpha \to \infty$.

We assume the following condition.

(**H**) Assume that $k(t)$ satisfies (1.4) and (1.5). Furthermore, $K'(t)/K(t)$ and $K''(t)/K(t)$ are non-increasing for $0 \le t \le 1/2$, where $K(t) := k(t)^{-1/(p-1)}$.

Comparing to the autonomous case, however, there are no works which obtain precise asymptotic formula in non-autonomous case. By the terms which come from k, k', k'' and u', the tools for autonomous case are not useful any more in non-autonomous problems. To overcome this difficulty, we adopt a new parameter $\|k^{1/(p-1)}u_\alpha\|_q = \alpha$ to parameterize the bifurcation curve $\lambda(q,\alpha)$. By the new idea above, the tools for autonomous problems can be available to our non-autonomous case.

Theorem 4.1 [15]. *Let $p > 1$ and $q \ge 1$ be fixed. Assume that k is a given function which satisfies* (**H**). *Then as $\alpha \to \infty$,*

$$\lambda(q,\alpha) \ge \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + m_0 - r_{p,q} + o(1), \tag{4.6}$$

$$\lambda(q,\alpha) \le \alpha^{p-1} + C_1\alpha^{(p-1)/2} + a_0 + M_0 + o(1), \tag{4.7}$$

where $C_1, C_2, C(q), a_0, M_0, M_1, m_0, r_{p,q}, w_{p,q}$ are constants determined explicitly.

The proof of Theorem 4.1 depends on the precise calculation of the time map. ∎

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