数理解析研究所講究録
第 1962 巻 2015 年 68-76

Uniform resolvent estimates for Helmholtz equations in 2D exterior domain and their applications

Kiyoshi Mochizuki
Department of Mathematics, Chuo University

Hideo Nakazawa
Department of Mathematics, Nippon Medical School

April 24, 2015

Abstract

Uniform resolvent estimate for Helmholtz equations in 2D exterior domain is derived. Similar estimates also hold for stationary Schrödinger equations with magnetic fields and stationary dissipative wave equations. As by-product, smoothing estimates for corresponding time dependent problems follow.

1 Problems and results

Consider the following Helmholtz equations in 2D exterior domain with star-shaped boundary:

$$\begin{cases} (-\Delta-\kappa^2)u=f(x), & x\in\Omega,\\ u=0, & x\in\partial\Omega, \end{cases} \tag{1.1}$$

where $u=u(x,\kappa)$, $k=\sigma+i\tau\in\mathbb{C}$ and $\partial\Omega$ is star-shaped with respect to the origin $0\notin\Omega\left(\subsetneq\mathbb{R}^2\right)$, i.e., $(x/r,n)\leqq 0$ for unit outer normal n of $\partial\Omega$. In the following, we exclude the case $\Omega=\mathbb{R}^2$ and assume that $\min\{|x|;x\in\partial\Omega\}>r_0$ for some $r_0>0$. The method described here can be adapted also to stationary Schrödinger equations with magnetic fields or stationary dissipative wave equations (Remark 1.5, or 1.6 below, respectively).

Since the operator $-\Delta$ with domain $\mathcal{D}(-\Delta)=\{u\in H^2(\Omega);u=0 \text{ for } x\in\partial\Omega\}$ is self-adjoint, we may assume that $\Im\kappa\geqq 0$[1].

Our aim is to establish the uniform resolvent estimates for (1.1), which was remained unsolved until the time with the past. The main task is to obtain the positivity of the energy form obtained from Laplacian, which was violated in the existing research (e.g., Ikebe-Saito [2], Mochizuki [11]) due to the term

$$\frac{(N-1)(N-3)}{4r^2}|u|^2$$

(here, N denotes the space dimension and $r=|x|$).

The first key for conquering this difficulty was the Hardy type inequalities related to radiation conditions[2]:

$$||D_r^{\pm}u||<\infty \quad \text{where} \quad D_r^{\pm}u=D^{\pm}u\cdot\frac{x}{r},\quad D^{\pm}u=\nabla u+\left(\frac{N-1}{2r}u\mp i\kappa u\right)\frac{x}{r}\quad (\pm\Im\kappa\geqq 0).$$

In the following, we define the weighted L^2 space by

$$L^2_w=\left\{u\in L^2(\Omega)\,:\,||u||^2_w=||wu||^2_{L^2(\Omega)}<\infty\right\}$$

for some weight $w\geqq 0$.

[1]For stationary dissipative wave equation, the operator which appears does not become self-adjoint. In such a case, we also must treat the case $\Im\kappa\leqq 0$.

[2]The radiation condition is a boundary condition at infinity to ensure the uniqueness of the solution of (1.1).

Proposition 1.1 ([*12*], [*16*]) *Assume that* $\Omega \subseteq \mathbb{R}^N$ *is a general domain with smooth boundary and* $N \geqq 1$. *For any* $v \in C_0^\infty(\Omega)$, $\phi = \phi(r) \in C^\infty((r_0, \infty))$ *and* $a \in (0, 1]$, *the following inequalities hold:*

$$\int_\Omega \phi \left| v_r + \frac{N-1}{2r} v \right|^2 dx \geqq \int_\Omega h_a |v|^2 dx,$$

$$\int_\Omega \phi \left| D_r^\pm v \right|^2 dx \geqq (\pm \Im\kappa) \int_\Omega \frac{2a\phi}{r} |v|^2 dx + \int_\Omega h_a |v|^2 dx, \ (\pm\Im\kappa \geqq 0),$$

where

$$h_a(r) = -\frac{a\phi_r(r)}{r} - \frac{a(a-1)\phi(r)}{r^2}.$$

If these are used, the positivity in the case of $N = 2$ is re-established. However, the duality on the weight function is violated:

Theorem 1.2 [3] ([*16*]) *For a solution* u *of* (1.1), *the following uniform resolvent estimate holds:*

$$|\kappa|^2 ||u||^2_{r^{-(1+\delta)/2}} \pm \Im\kappa ||u||^2_{r^{-1/2}} + ||u||^2_{r^{-(3+\delta)/2}} + ||D_r^\pm u||^2 \leqq C_3 ||f||^2_{r^{(3+\delta)/2}}. \quad (\pm\Im\kappa \geqq 0) \tag{1.2}$$

Note that (1.2) is a global resolvent estimate with the spectral parameter κ. In this sense, it differs from the local (low or high energy) inequality derived by e.g., Ikebe-Saito [2], Agmon [1] or Kuroda [5], etc. The global resolvent estimate has already been proved by Mochizuki [11]. On the other hand, the uniform estimate is established by Kato-Yajima [4] for Helmholtz equations in the whole space. It is extended by Mochizuki [12] to magnetic Schrödinger equations in an exterior domain in $\mathbb{R}^N$ with $N \geqq 3$.

One key for the duality is a form of the weight function which appears in the Hardy type inequality in a two-dimensional exterior domain, which originally proved by J. Leray [7] (See Lemma 2.1 (2.2), below).

Our results are given in the following form;

Theorem 1.3 ([*13*]) *For a solution* u *of* (1.1) *and for each* $\kappa (\neq 0)$, *the following inequality holds:*

$$||u||^2_{r^{-1}\left(1+\log\frac{r}{r_0}\right)^{-1}} \pm \Im\kappa ||u||^2_{r^{-1/2}\left(1+\log\frac{r}{r_0}\right)^{-1}} \leqq C_1 ||f||^2_{r\left(1+\log\frac{r}{r_0}\right)} \qquad (\pm\Im\kappa \geqq 0) \tag{1.3}$$

for some $C_1 > 0$ *independent of* κ. *Moreover, for vector-valued function* $D^\pm u(x)$ *and for each* $\kappa(\neq 0)$, *the following inequality holds:*

$$||D^\pm u||^2_{\left(4+\log\frac{r}{r_0}\right)^{-1}} \pm \Im\kappa ||D^\pm u||^2_{r^{1/2}\left(4+\log\frac{r}{r_0}\right)^{-1}} \leqq C_2 ||f||^2_{r\left(1+\log\frac{r}{r_0}\right)} \qquad (\pm\Im\kappa \geqq 0). \tag{1.4}$$

where C_2 *is positive constant independent of* κ.

Theorem 1.4 ([*13*]) *Assume that the function* $\varphi(r)$ *is smooth, non-negative and integrable on* $[r_0, \infty)$ *satisfying*

$$2r\varphi_r(r) \leqq \varphi(r) \quad \text{and} \quad \varphi(r) \leqq \frac{1}{\left(4 + \log\frac{r}{r_0}\right)^2}.$$

Then for a solution u *of* (1.1) *and for each* $\kappa(\neq 0)$, *the following inequality holds:*

$$|\kappa|^2 ||u||^2_{\sqrt{\varphi}} + ||\nabla u||^2_{\sqrt{\varphi}} \leqq C_3 ||f||^2_{\sqrt{r^2\left(1+\log\frac{r}{r_0}\right)^2 + \varphi^{-1}}} \tag{1.5}$$

for some $C_3 > 0$ *independent of* κ *where* $||\varphi||_{L^1} = \int_{r_0}^\infty \varphi(s)ds$.

Remark 1.5 *The above two theorems also hold for a solution* u *of the magnetic Schrödinger equation* (*[13]*)

$$\begin{cases} \left(-\sum_{j=1}^{2} \left\{ \partial_j + ib_j(x) \right\}^2 + c(x) - \kappa^2 \right) u = f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.6}$$

[3] The similar result also holds for stationary dissipative wave and Schrödinger equations under suitable conditions.

under the assumption

$$\left\{|\nabla \times b(x)|^2 + |c(x)|^2\right\}^{1/2} \leqq \frac{\varepsilon_0}{r^2\left(1+\log\dfrac{r}{r_0}\right)^2} \quad in \quad \Omega, \tag{1.7}$$

with $0 < \varepsilon_0 < \dfrac{1}{4\sqrt{21}}$, *where* $\partial_j = \partial/\partial_{x_j}(j=1,2)$, $i=\sqrt{-1}$, $b_j(x)$ *is a real-valued* C^1*-function on* $\overline{\Omega} = \Omega \cup \partial\Omega$, *and* $c(x)$ *is real-valued continuous function on* $\overline{\Omega}$.

Remark 1.6 *The similar result as in Theorem 1.3 and 1.4 can be also proved for the stationary problem of dissipative wave equation, i.e., Theorem 1.2 above is improved in sharp form under the* $N=2$ *or* $N \geqq 3$. *Using these, we can relax the decay condition of the dissipation given by Mizohata-Mochizuki [9] to established the principle of limiting amplitude. These results are published in forthcoming paper ([14]).*

Noting the above two theorems and the smooth perturbation theory developed by Kato [3], we can establish the smoothing estimates for the corresponding evolution equations.

To state our results for Magnetic Schrödinger equations (1.6), we define the following notations: $\nabla = (\partial_1, \partial_2)$, $\nabla_b = \nabla + ib(x)$, $\Delta_b = \nabla_b \cdot \nabla_b$. The self-adjoint operator L is defined by $L = -\Delta_b + c(x)$ with domain $\mathcal{D}(L) = \left\{u \in L^2(\Omega) \cap H^2_{loc}(\overline{\Omega}); (-\Delta_b + c)u \in L^2(\Omega), u|_{\partial\Omega} = 0\right\}$.

Theorem 1.7 ([*13*]) *Assume* (1.7). *Then the solution operator* e^{-itL} *to the equation*

$$i\frac{\partial u}{\partial t} - Lu = 0, \quad u(0) = f \in L^2(\Omega) \tag{1.8}$$

satisfies

$$\left|\int_0^{\pm}\left\|r^{-1}\left(1+\log\frac{r}{r_0}\right)^{-1}\int_0^t e^{-i(t-\tau)L}h(\tau)d\tau\right\|^2 dt\right| \leqq C_1\left|\int_0^{\pm}\left\|r\left(1+\log\frac{r}{r_0}\right)h(t)\right\|^2 dt\right|.$$

for $h(t)$ *satisfying* $r\left(1+\log\dfrac{r}{r_0}\right)h(t) \in L^2(\mathbb{R}\times\Omega)$, *and*

$$\left|\int_0^{\pm t}\left\|r^{-1}\left(1+\log\frac{r}{r_0}\right)^{-1}e^{-itL}f\right\|^2 dt\right| \leqq 2\sqrt{C_1}||f||^2.$$

Remark 1.8 *The similar result as in Theorem 1.7 can be also proved for the relativistic Schrödinger, Klein-Gordon or wave equation (see [13]).*

In the rest of the paper, proofs of the above theorems are performed. Since the essential part of these comes from the free Laplacian, in the following, we shall only treat this case. For the magnetic Schrödinger operator's case, see our paper [13].

The contents of the present paper is given as following: In section 2, the refined Hardy-type inequalities related to the radiation condition are derived and main theorems (Theorem 1.4 and 1.5) are proved. In the final section, essence of the proof of Theorem 1.7 is given.

2 Refined Hardy type inequalities and the proof of Theorem 1.4, 1.5

We shall start the proof of well-known Hardy-Leray inequality:

Lemma 2.1 *Assume that* $\Omega \subseteq \mathbb{R}^N$ *is a general domain with smooth boundary and* $N \geqq 1$. *For any* $v \in C_0^\infty(\Omega)$, *we have the following inequalities;*

(1) $N \geqq 3$, *then*

$$||v||^2_{\frac{N-2}{2r}} \leqq ||v_r||^2. \tag{2.1}$$

(2) *If* $N=2$ *and* $r>r_0$ *for some* $r_0>0$*, then*

$$||v||^2_{\{2r\log\left(\frac{r}{r_0}\right)\}^{-1}} \leqq ||v_r||^2. \tag{2.2}$$

Here, $v_r = \nabla v \cdot \frac{x}{r}$.

Proof. (see also [16], Lemma 2.2 and the subsequent description) Proofs of (1) and (2) are given in the footnote of text book by Mizohata [8], and by Ladyzenskaya [6], respectively. Here, we give a unified proof.

Consider the following non-negative inequality:

$$0 \leqq |v_r - gv|^2,$$

where $v \in C_0^\infty(\Omega)$ and $g = g(r) \in C^\infty((r_0,\infty))$. By the direct computation gives

$$0 \leqq |v_r|^2 - \nabla\cdot\left(gv^2\frac{x}{r}\right) - W_g v^2, \tag{2.3}$$

where

$$W_g = -\left(g_r + \frac{N-1}{r}g + g^2\right).$$

(1) If $N \neq 2$, choose $g = ar^{-1}$ with some constant a. Then we have $W_g = a(2-N-a)r^{-2}$. Therefore, we may choose $a = -\frac{N-2}{2}$ to obtain $W_g = \left(\frac{N-2}{2r}\right)^2$. Integrating the both sides of (2.3) over Ω, we have (2.1).

(2) If $N=2$, choose $g = ar^{-1}\left\{\log\left(\frac{r}{r_0}\right)\right\}^{-1}$ with some a. Then we have $W_g = a(1-a)r^{-2}\left\{\log\left(\frac{r}{r_0}\right)\right\}^{-2}$. Therefore, we may choose $a = \dfrac{1}{2}$ to obtain (2.2). □

Now we prepare two identities which comes from (1.1).

Lemma 2.2 ([10], [11], [12], [15], [16]) *Let* u *be a solution of* (1.1). *Assume that two functions* $\varphi = \varphi(r)$ *and* $\psi = \psi(r)$ *are non-negative and satisfy* $\varphi,\psi \in C^\infty((r_0,\infty))$. *Then* u *satisfies the following identities:*

$$|\kappa|^2||u||_{\sqrt{\varphi}} + ||\nabla u||^2_{\sqrt{\varphi}} + \int_\Omega W_1(r)|u|^2dx \pm 2(\Im\kappa)\int_{r_0}^\infty \varphi(R)\left\{\int_{\Omega_R}\left(|\nabla u|^2 + |\kappa|^2|u|^2\right)dx\right\}dR$$
$$= ||D^\pm u||^2_{\sqrt{\varphi}} \mp 2\int_{r_0}^\infty \varphi(R)\left(\int_{\Omega_R} f\overline{i\kappa u}dx\right)dR \quad (\pm\Im\kappa \geqq 0), \tag{2.4}$$

$$||D_r^\pm u||^2_{\sqrt{\pm\Im\kappa\psi + \frac{|\psi_r|}{2}}} + \int_\Omega\left(\frac{\psi}{r} - \psi_r\right)\left(|D^\pm u|^2 - |D_r^\pm u|^2\right)dx$$
$$+\int_{\partial\Omega}\left\{-\psi D^\pm u\overline{D_r^\pm u} + \frac{\psi}{2}|D^\pm u|^2\frac{x}{r}\right\}\cdot n dS + \int_\Omega W_2(r)|u|^2dx = \Re\int_\Omega \psi f\overline{D_r^\pm u}dx \quad (\pm\Im\kappa \geqq 0), \tag{2.5}$$

where

$$W_1(r) = (\pm\Im\kappa)\frac{\varphi}{r} + \frac{\varphi}{4r^2} - \frac{\varphi_r}{2r}, \tag{2.6}$$

$$W_2(r) = \frac{1}{8}\left(\frac{\psi}{r^2}\right)_r - (\pm\Im\kappa)\frac{\psi}{4r^2}, \tag{2.7}$$

and $\Omega_R = \{x\in\Omega \,|\, |x| \leqq R\}$, $S_R = \{x\in\Omega \,|\, |x| = R\}$ *for some large* $R>0$.

Proof. Multiplying the both sides of (1.1) by $-\overline{i\kappa u}$, integrating by parts over Ω_R, and taking the real part, we find

$$\frac{1}{2}\int_{S_R}\left\{|\kappa|^2|u|^2 + |\nabla u|^2 - \left|\nabla u \mp i\kappa u\frac{x}{r}\right|^2\right\}dS \pm (\Im\kappa)\int_{\Omega_R}\left(|\kappa|^2|u|^2 + |\nabla u|^2\right)dx = -\Re\int_{\Omega_R} f\overline{i\kappa u}dx. \tag{2.8}$$

In that process, we use the following two identities:

$$\Re\int_{\Omega_R}\nabla\cdot(\nabla u\overline{i\kappa u})dx = \Re\int_{S_R}\frac{x}{r}\cdot\nabla u\overline{i\kappa u}dS = \frac{1}{2}\int_{S_R}\left\{|\kappa|^2|u|^2 + |\nabla u|^2 - \left|\nabla u \mp i\kappa u\frac{x}{r}\right|^2\right\}dS.$$

$$-\left|\nabla u \mp i\kappa u\frac{x}{r}\right|^2 = -\left|\nabla u + \left(\frac{u}{2r} - (\pm i\kappa)u\right)\frac{x}{r}\right|^2 + \frac{|u|^2}{4r^2} \pm (\Im\kappa)\frac{|u|^2}{r} + \nabla\cdot\left\{\frac{|u|^2}{2r}\frac{x}{r}\right\}.$$

Multiplying the both sides of (2.8) by φ and integration over (r_0, ∞), we obtain (2.4). □

Next, we shall derive the second identity (2.5). Put $v = e^\rho u$ and $g = e^\rho f$, where $\rho = \mp i\kappa r + \frac{N-1}{2}\log r$ $(\pm\Im\kappa \geqq 0)$. Then v satisfies the equation

$$-\Delta v + 2\rho_r v_r + \frac{(N-1)(N-3)}{4r^2}v = g. \tag{2.9}$$

Consider (2.9) $\times\ \psi\overline{v_r}$ and integration by parts over Ω. Moreover, represent the resulting identity by the original u and f. Taking the real part of the both sides, we have (2.5) since $N = 2$. □

From these identities, we derive some inequalities. In (2.5), choose $\psi(r) = r$. Then the second term of the l.h.s of (2.5) vanishes, and the third term of l.h.s of (2.5) becomes non-negative by the boundary condition. By (2.7), the weight function W_2 becomes

$$W_2 = -\frac{1}{4r^2}\left\{(\pm\Im\kappa)r + \frac{1}{2}\right\}.$$

Since the weight function of the first term of the l.h.s of (2.5) becomes $(\pm\Im\kappa)r + \frac{1}{2}$, we can pack these terms to obtain

Lemma 2.3 *Let u be a solution of* (1.1). *Then u satisfies*

$$\int_\Omega \left\{(\pm\Im\kappa)r + \frac{1}{2}\right\}\left(|D^\pm u|^2 - \frac{|u|^2}{4r^2}\right)dx \leqq \int_\Omega r\left|f\overline{Dr^\pm u}\right|dx. \tag{2.10}$$

Using Schwarz inequality in the r.h.s of the above inequality, we obtain

$$[\text{r.h.s of}\,(2.10)] \leqq \frac{1}{4\varepsilon}||f||^2_{r\left(1+\log\frac{r}{r_0}\right)} + \int_\Omega \frac{\varepsilon}{\left(1+\log\frac{r}{r_0}\right)^2}\left(|D^\pm u|^2 - \frac{|u|^2}{4r^2}\right)dx + \varepsilon||u||^2_{\left\{2r\left(1+\log\frac{r}{r_0}\right)\right\}^{-1}}. \tag{2.11}$$

If moving the second term on the r.h.s of (2.11) to the l.h.s of (2.10), we have the following

Proposition 2.4 *Under the same assumptions as in the preceding proposition, it holds that for any $\varepsilon > 0$*

$$\int_\Omega \left\{(\pm\Im\kappa)r + \frac{1}{2} - \frac{\varepsilon}{\left(1+\log\frac{r}{r_0}\right)^2}\right\}\left(|D^\pm u|^2 - \frac{|u|^2}{4r^2}\right)dx \leqq \frac{1}{4\varepsilon}||f||^2_{r\left(1+\log\frac{r}{r_0}\right)} + \varepsilon||u||^2_{\left\{2r\left(1+\log\frac{r}{r_0}\right)\right\}^{-1}}. \tag{2.12}$$

We regard (2.12) as an estimate of the term involving

$$|D^\pm u|^2 - \frac{|u|^2}{4r^2}$$

from above. Conversely, we need some estimates from below a same portion.

For this aim, we note the following calculations (cf. the proof of Lemma 2.1). Assume that $f = f(r)$, $g = g(r) \in C^\infty((r_0, \infty))$ with $f \geqq 0$. Then

$$\begin{aligned} 0 &\leqq f\left|D_r^\pm u - \left(\frac{1}{2r} + g\right)u\right|^2 \\ &= f|D_r^\pm u|^2 - \nabla\cdot\left(f\left(\frac{1}{2r}+g\right)|u|^2\frac{x}{r}\right) - \frac{f}{4r^2}|u|^2 + (\pm\Im\kappa)W_3|u|^2 + W_4|u|^2 + W_5|u|^2, \end{aligned} \tag{2.13}$$

where

$$W_3(r) = 2f\left(\frac{1}{2r} + g\right), \tag{2.14}$$

$$W_4(r) = -f\left(g_r + \frac{g}{r} + g^2\right), \tag{2.15}$$

$$W_5(r) = -f_r\left(\frac{1}{2r} + g\right). \tag{2.16}$$

Integrating the both sides of (2.13) over Ω, we have

Lemma 2.5 *For any $f = f(r)$, $g = g(r) \in C^\infty((r_0,\infty))$ with $f \geqq 0$, it holds that*

$$\int_\Omega f\left(|D_r^\pm u|^2 - \frac{|u|^2}{4r^2}\right)dx \geqq (\pm\Im\kappa)\int_\Omega W_3|u|^2dx + \int_\Omega (W_4+W_5)\,|u|^2dx \qquad (\pm\Im\kappa \geqq 0),$$

where W_3, W_4 and W_5 are defined by (2.14), (2.15) *and* (2.16), *respectively.*

Comparing Proposition 2.4 and Lemma 2.5, we choose f and g as

$$\begin{aligned} f(r) &= (\pm\Im\kappa)r + \frac{1}{2} - \frac{\varepsilon}{\left(1+\log\frac{r}{r_0}\right)^2}, \qquad (\pm\Im\kappa \geqq 0) \\ g(r) &= \frac{1}{2r\left(1+\log\frac{r}{r_0}\right)}. \end{aligned}$$

We then have as in the proof of Lemma 2.1 (2),

$$W_4 = \frac{f}{4r^2\left(1+\log\frac{r}{r_0}\right)^2}.$$

Moreover, easy computations give

$$(\pm\Im\kappa)W_3 + W_5 = \left(\frac{1}{2r}+g\right)(2(\pm\Im\kappa)f - f_r),$$

where

$$\begin{aligned} 2(\pm\Im\kappa)f - f_r &= 2(\pm\Im\kappa)^2 r - \frac{2\varepsilon(\pm\Im\kappa)}{\left(1+\log\frac{r}{r_0}\right)^2} - \frac{2\varepsilon}{r\left(1+\log\frac{r}{r_0}\right)^3} \\ &= 2r\left[\left\{(\pm\Im\kappa) - \frac{\varepsilon}{2r\left(1+\log\frac{r}{r_0}\right)^2}\right\}^2 - \frac{\varepsilon^2}{4r^2\left(1+\log\frac{r}{r_0}\right)^4}\right] - \frac{2\varepsilon}{r\left(1+\log\frac{r}{r_0}\right)^3} \end{aligned}$$

Noting the definition of f, we have

$$0 \leqq \frac{1}{2r} + g \leqq \frac{1}{r},$$

which gives

$$(\pm\Im\kappa)W_3 + W_4 \geqq -\frac{\varepsilon^2}{2r^2\left(1+\log\frac{r}{r_0}\right)^2} - \frac{2\varepsilon}{r^2\left(1+\log\frac{r}{r_0}\right)^2} \qquad (\pm\Im\kappa \geqq 0).$$

Therefore the weight function of $||u||^2$ can be estimated from below if we choose ε so small as $2\varepsilon^2 \leqq \varepsilon$;

$$(\pm\Im\kappa)W_3 + W_4 + W_5 \geqq \frac{1}{4r^2\left(1+\log\frac{r}{r_0}\right)^2}\left\{(\pm\Im\kappa)r + \frac{1}{2} - 10\varepsilon\right\} \qquad (\pm\Im\kappa \geqq 0).$$

By the above mentioned argument, we have

Proposition 2.6 *For a solution u of* (1.1) *and for small $\varepsilon > 0$, it holds that*

$$\int_\Omega \left\{(\pm\Im\kappa)r + \frac{1}{2} - \frac{\varepsilon}{\left(1+\log\frac{r}{r_0}\right)^2}\right\}\left(|D_r^\pm u|^2 - \frac{|u|^2}{4r^2}\right)dx \geqq ||u||^2_{\sqrt{W_6}} \qquad (\pm\Im\kappa \geqq 0),$$

where

$$W_6(r) = \frac{1}{4r^2\left(1+\log\frac{r}{r_0}\right)^2}\left\{(\pm\Im\kappa)r + \frac{1}{2} - 10\varepsilon\right\}.$$

[Proof of Theorem 1.4 (1.3)] Combining Proposition 2.4 and 2.6, we have

$$||u||^2_{\sqrt{W_6}} \leqq C(\varepsilon)||f||^2_{r\left(1+\log\frac{r}{r_0}\right)} + \varepsilon||u||^2_{\left\{2r\left(1+\log\frac{r}{r_0}\right)\right\}^{-1}}.$$

Moving the last term of r.h.s of the above equation to the other side, we obtain

$$||u||^2_{\sqrt{W_7}} \leqq C(\varepsilon)||f||^2_{r^2\left(1+\log\frac{r}{r_0}\right)^2},$$

where

$$W_7 = \left\{(\pm\Im\kappa)r + \frac{1}{2} - 11\varepsilon\right\}\frac{1}{4r^2\left(1+\log\frac{r}{r_0}\right)^2}.$$

If we choose ε so small, the desired inequality (1.3) holds. □

[Proof of Theorem 1.4 (1.4).] Put

$$\psi(r) = \frac{r}{\left(4+\log\frac{r}{r_0}\right)^2}$$

in Lemma 2.2 (2.5). Then $\frac{\psi}{r} - \psi_r \geqq 0$ holds to neglect the second term in the l.h.s of (2.5). As in the proof of Lemma 2.3, the third terms in the l.h.s. of (2.5) becomes non-negative. As for the weight function of the first term of l.h.s of (2.5), we have

$$(\pm\Im\kappa)\psi + \frac{\psi_r}{2} \geqq \frac{1}{\left(4+\log\frac{r}{r_0}\right)^2}\left\{(\pm\Im\kappa)r + 1\right\}.$$

Moreover we obtain

$$-W_2 \leqq \frac{C\left\{(\pm\Im\kappa)r+1\right\}}{4r^2\left(4+\log\frac{r}{r_0}\right)^2}$$

to conclude

$$\int_\Omega (-W_2)|u|^2 dx \leqq C||f||^2_{r\left(4+\log\frac{r}{r_0}\right)}$$

by (1.3) for some $C > 0$ independent of κ. Similar estimate as in (2.11) gives

$$\int_\Omega r|f\overline{D_r^\pm}|dx \leqq \varepsilon||D_r^\pm||^2_{\left(4+\log\frac{r}{r_0}\right)^{-1}} + \frac{1}{4\varepsilon}||f||^2_{r\left(4+\log\frac{r}{r_0}\right)}.$$

Using these two inequalities in (2.5), we obtain (1.4). □

[Proof of Theorem 1.5 (1.5)] In Lemma 2.2, (2.4), let the function φ satisfies the assumptions in Theorem 1.5:

$$\varphi \in L^1((r_0,\infty)), \qquad \frac{\varphi_r}{\varphi} \leqq \frac{1}{2r}, \qquad 0 \leqq \varphi \leqq \frac{1}{\left(4+\log\frac{r}{r_0}\right)^2}.$$

Then by (2.6), we find $W_1 \geqq 0$. Therefore we can neglect the third and fourth terms in the l.h.s. of (2.4). For the first term of the r.h.s of (2.4), we can utilize Theorem 1.3 (1.4) to obtain

$$||D^\pm u||^2_{\sqrt{\varphi}} \leqq C||f||^2_{r\left(4+\log\frac{r}{r_0}\right)}$$

for some $C > 0$. By the Schwarz inequality,

$$2\int_{\Omega_R} |f\overline{i\kappa u}|dx \leqq 4||f||^2_{(\sqrt{\varphi})^{-1}} + \frac{1}{2}|\kappa|^2||u||^2_{\sqrt{\varphi}}$$

holds. By the above mentioned inequalities, we have (1.5). □

3 Essence of proof of Theorem 1.8

For the sake of simplicity, we shall consider the Helmholtz equation case (1.1). Magnetic Schrödinger equation (1.8) case also can be treated by the similar arguments. We have only to prove the following inequality by the smooth perturbation theory developed by Kato [3] (see also [12]):

$$||A(-\Delta-\kappa^2)^{-1}A^*f|| \leqq C||f|| \tag{3.1}$$

for any $f \in L^2(\Omega)$ and for some $C > 0$ with $\Im\kappa \neq 0$ and

$$A = r^{-1}\left(1+\log\frac{r}{r_0}\right)^{-1}.$$

We regard A as an operator in $L^2(\Omega)$. By this definition, $A^* = A$ holds. To show (3.1), put $u = (-\Delta-\kappa^2)^{-1}A^*f$. Then u satisfies Helmholtz equation

$$\begin{cases} (-\Delta-\kappa^2)u = A^*f(x), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

Then by Theorem 1.3 (1.3), we have

$$[\text{l.h.s of}(3.1)] = ||Au|| = ||u||_{r^{-1}\left(1+\log\frac{r}{r_0}\right)^{-1}} \leqq C||A^*f||_{r\left(1+\log\frac{r}{r_0}\right)} = C||f||$$

to obtain the desired inequality. □

References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, Ann. Scuola. Norm. Sup. pisa(4). vol. 2 (1975), pp. 151–218.

[2] T. Ikebe and Y. Saito, Limiting absorption method and absolute continuity for the Schrödinger operators, J. Math. Kyoto Univ. vol. 12 (1972), pp. 513–542.

[3] T. Kato, Wave operators and similarity for some non-selfadjoint operators, Math. Ann. vol. 162 (1966), pp. 258–279.

[4] T. Kato and K. Yajima, Some examples of smooth operators and the associated smoothing effect, Rev. Math. Phys. 1 (1989), no. 4, pp. 481-496.

[5] S.T. Kuroda, An Introduction to scattering theory, Lecture Notes Series, no. 51, Aarhus Univ., (1978).

[6] O. Ladzenskaya, The mathematical theory of viscous incompressible flow, Revised English edition. Translated from the Russian by Richard A. Silverman Gordon and Breach Science Publishers, New York-London 1963 xiv+184 pp.

[7] J. Leray, Etude de diverses équations integrales non linéaires et de quelques problèmes que pose l'hydrodynamique, Journal de mathematiques pures et appliquees 9e serie, tome 12 (1933), pp. 1-82.

[8] S. Mizohata, The theory of partial differential equations, Translated from the Japanese by Katsumi Miyahara. Cambridge University Press, New York, 1973. xii+490 pp.

[9] S. Mizohata and K. Mochizuki, On the principle of limiting amplitude of dissipative wave equations, Jour. Math. Kyoto Univ., 6 (1966), pp. 109-127.

[10] K. Mochizuki, Spectral and Scattering Theory for Second Order Elliptic Differential Operators in an Exterior Domain, Lecture Notes Univ. Utah. Winter and Spring (1972).

[11] K. Mochizuki, Scattering theory for wave equations with dissipative terms, Publ. RIMS, Kyoto Univ. 12 (1976), pp. 383-390.

[12] K. Mochizuki, Uniform resolvent estimates for magnetic Schrödinger operators and smoothing effects for related evolution equations, Publ. RIMS, Kyoto Univ. 46 (2010), no. 4, pp. 741-754.

[13] K. Mochizuki and H. Nakazawa, Uniform resolvent estimates for magnetic Schrödinger operators in 2D exterior domain and their applications to related evolution equations, to appear in Publ. RIMS, Kyoto Univ.

[14] K. Mochizuki and H. Nakazawa, Uniform resolvent estimates for stationary problems of dissipative wave equations in an exterior domain and their applications to the principle of limiting amplitude, in preparation.

[15] H. Nakazawa, The principle of limiting absorption for non-selfadjoint Schrödinger operator with energy dependent potential, Tokyo J. Math., 23 (2000), 519-536.

[16] H. Nakazawa, Uniform resolvent estimates for Schrödinger equations in an exterior domain in $\mathbb{R}^2$ and their applications to scattering problems, Bull. Lib. Arts & Sci. Nippon Med. Sch., No. 42 (2013), pp. 1-12.