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# Multicriteria Multipliers of Banach-valued Functions on Locally Compact Abelian Group*

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**Abstract**

Let $G$ be a *locally compact Abelian* $(LCA)$ *group*, $A$ a commutative *Banach algebra*, "$X$" and "$Y$" denote the *Banach spaces* of $A-module$. $L^1(G,A)$ stands for the space of all $A$-valued commutative *Banach algebra* with convolution product. $L^p(G,X)$, $1 \leq p \leq \infty$, for each $p$, is a *Banach space*. In this note, we study the multipliers of $L^1(G,A)$ and the representation of the homomorphism $L^1(G,A)$ module multipliers of $L^1(G,A)$ to $L^p(G,Y)$ which can be identified by $L^1(G,A) \otimes L^q(G,Y^*)^*$ under reasonable conditions, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The multipliers of $L^1(G,A)$ to $C_0(G,X)$ is also subscribed.

**Key words and phrases:** locally compact Abelian (LCA) group, separable Banach space, Radon Nikodym property, multipliers, invariant operator, projective tensor product space.

## 1 Introduction and preliminaries

Let $G$ be a *locally compact Abelian* $(= LCA)$ *group* with *Haar measure* $dt$ and dual group $\widehat{G}$. Let $A$ be a commutative *Banach algebra* with a bounded approximate identity. A continuous linear map $T \in \mathfrak{L}(A) \cong \mathfrak{L}(A,A)$ is called a **multiplier** of $A$ if

$$T(a \cdot b) = a \cdot Tb = (Ta) \cdot b \quad \text{for all } a, b \in A.$$

Denote by $\mathfrak{M}(A)$ the space of all multipliers for $A$.
Clearly, $\mathfrak{M}(A)$ is a *Banach subalgebra* of $\mathfrak{L}(A)$. In particular, if $A = L^1(G)$, a

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commutative group algebra under convolution product, then the multiplier algebra $\mathfrak{M}(L^1(G))$ has the following equivalent statements (i)~(iv) .(See Larsen [7], cf also Lai, Lee and Liu [1]):

**Theorem 1.** *Let $T \in \mathfrak{L}(L^1(G))$. Then the following statements are equivalent.*

*(i) $T$ commutes with convolution (call $T$ a **multiplier**)*

$$T(f*g) = Tf*g = f*T(g), \text{ for all } f,g \in L^1(G)$$

*(ii) $T$ commutes with translation operator $\tau_a$ $(a \in G)$. (call $T$ an **invariant operator**)*

$$T\tau_a = \tau_a T, \tau_a f(t) = f(t-a), \text{ for all } a \in G,$$

*(iii) $\exists!$ a $\mu \in M_b(G)$, space of all bounded regular Borel measures such that,*
$$Tf = \mu * f, \text{ for all } f \in L^1(G).$$

*(iv) there exists a bounded function $\phi$ on $\widehat{G}$ such that*

$$\widehat{Tf} = \phi \hat{f} \text{ or } \phi = \hat{\mu} \in \widehat{M_b(G)} \subsetneqq C^b(\hat{G}).$$

It is remarkable that

(a) the *Fourier transforms* $\widehat{L^1(\hat{G})} = A(\hat{G}) \subsetneqq C_0(\hat{G})$ is dense of 1st category in $C_0(\hat{G})$, the continuous function on $\hat{G}$, vanishing at infinite.

(b)Similarly, it is known that the *Fourier – Stieltjes transforms* :
$\hat{\mu} \in \widehat{M_b(G)} \subsetneqq C^b(\hat{G})$, the space of all bounded continuous functions on $\hat{G}$.

By Theorem 1, we see that the definition of multipliers is in various types. Actually the concept of multiplier comes from *Fourier Series* of a function $f$ by using a bounded sequence $\phi(n)$ multiply the *Fourier coefficient* $c_n$ of $f$, it still approve as a *Fourier coefficient* of another function of $g$. This ideal leads to study for multipliers in harmonic analysis on *locally compact Abelian group G*.

In this Note, we would like to extend the multipliers of $L^1(G)$ to the multipliers of $L^1(G,A)$ as well as multipliers of $L^1(G,X)$ to $L^1(G,Y)$ under module homomorphism of *Banach vetor – valued functions* defined on *LCA* group $G$, and compare the *Banach algebras* $L^1(G,A)$ and $L^1(G)$, do have the same properties as in the Theorem 1? Actually, the invariant operator $T$ in $\mathfrak{L}(L^1(G,A))$ can not be a multiplier of $L^1(G,A)$ provided $dimA > 1$.(See Tewari, Dutta and Vaidya [9]). That is , in Theorem 1, (ii) $\Rightarrow$ (i) is false, the other implications are true.

## 2 Multipliers of Banach algebra .

Let $A$ be a commutative *Banach algebra*, we say that a *Banach space* $X$ is $A-module$ if

$$AX \subset X, \text{and } \| a\cdot x \|_X \leqslant \| a \|_A \ \| x \|_X \text{ for each } a \in A, x \in X.$$

and $X$ is said to be an $essential\ A-module$ if

$$AX = X, \text{ and } \| ax \|_X \leqslant \| a \|_A \ \| x \|_X, \text{ for each } a \in A,\ x \in X.$$

For convenience, we give following Theorem to check that an $A-module$ *Banach space* to be essential.

**Theorem 2.** *Let A be a commutative Banach algebra with uniform bounded approximate identity. Then any* $A-module$ *Banach space is essential.*

For example, the group algebra $L^1(G)$ has bounded approximate identity: $\{e_\alpha\}$, where $e_\alpha$ is $e_\alpha = \frac{\chi_{V_\alpha}}{|V_\alpha|}$ , where $\{V_\alpha\}$ is defined by an open neighborhood system of the identity $\theta \in G$ with ordered by $\alpha \prec \beta$ if $V_\beta \subset V_\alpha$, then $\| e_\alpha \|_1 = \int_G \frac{\chi_{V_\alpha}}{|V_\alpha|} dt = 1$. Thus by Theorem 2, directly we get easily that

$$L^1(G) * L^p(G) = L^p(G), \text{ if } 1 < p < \infty$$

if $p = \infty$ , we choose $C_0(G)$, the space of continuous functions vanishing at infinite on $G$, we also have

$$L^1(G) * C_0(G) = C_0(G)$$

**Remark1** It is remarkable that not every *Banach algebra* has a bounded approximate identity. For example, the space

$$A^p(G) = \{f \in L^1(G) \mid \hat{f} \in L^p(\hat{G}), 1 < p \leqslant \infty\} \ \ (\subset L^1(G))$$

with norm defined by $\| f \|_{A^p} = \| f \|_1 + \| \hat{f} \|_p$ is a *commutative Banach algebra* for each $p$, $1 \le p < \infty$. But there is an approximate identity $\{e_\alpha\}$ in $L^1(G)$ with *Fourier transform* $\widehat{e_\alpha}$ having compact support in $\hat{G}$ for each $\alpha$, then $\widehat{e_\alpha} \in L^p(\hat{G})$ shows that $\{e_\alpha\}$ is also an approximate identity of $A^p(G)$, but this system $\{e_\alpha\}$ of approximate identity is not uniform bounded in $A^p(G)$. ( cf. Lai [2, p254])

## 3 Multipliers of Banach Module Homomorphism .

Let $A$ be a *commutative Banach algebra* and $X, Y$ $A-$*module Banach spaces*. A bounded linear operator $T \in \mathcal{L}(X,Y)$ satisfying

(3.1) $T(ax) = a(Tx)$ for all $a \in A$, $x \in X$,

is called a **multiplier** of $X$ to $Y$ under $A$-module. The space of such multipliers is $A-$*module homomorphisms* from $X$ to $Y$ and is denoted by

(3.2) $\mathfrak{M}_A(X,Y) = Hom_A(X,Y) = \{T \in \mathcal{L}(X,Y) | \ T(ax) = a(Tx), a \in A, x \in X\}$.

It is a closed subalgebra of $\mathcal{L}(X,Y)$, the space of all bounded linear mappings of $X$ into $Y$. In particular, if $A = X = Y = L^1(G)$, then the multiplier space $\mathfrak{M}(L^1(G))$ coincides with the expression of isometrically isomorphic relations "$\cong$" as follows.

(3.3) $\mathfrak{M}(L^1(G)) = Hom_{L^1(G)}(L^1(G), L^1(G)) \cong (L^1(G), L^1(G)) \cong M_b(G)$.

where $(E(G), F(G))$ stands for the space of all invariant operators commute with translation operator $\tau_a$ on the function spaces of $E(G)$ to $F(G)$.

In general, the multiplier space $Hom_A(X, Y^*)$ was characterized by Rieffel [8] as the following dual space of the module tensor product $X \otimes_A Y$:

(3.4) $Hom_A(X, Y^*) \cong (X \otimes_A Y)^*$,

where $\otimes_A$ denotes the $A-$*module tensor product* defined by $X \otimes_A Y = X \hat{\otimes}_\gamma \ Y/K$. $K$ is the closed linear subspace of the complete projective tensor product space $X \hat{\otimes}_\gamma Y$ generating by elements: $ax \otimes y - x \otimes ay$, for $a \in A, x \in X, y \in Y$
Here $\hat{\otimes}_\gamma$ is the completion of the algebra tensor $X \otimes Y$ under the largest reasonable cross norm $\gamma$, and

$$X \otimes Y = \{u = \sum_i x_i \otimes y_i \ | \ \sum_i \| x_i \|_X \ \| y_i \|_Y < \infty\}$$

with norm $\gamma(u) \equiv |||u||| = \inf_u \sum_i \| x_i \otimes y_i \| = \inf_u \sum_i \| x_i \|_x \| y_i \|_y$, $\inf_u$ means that the infimum is taken by all representations of $u = \sum_i x_i \otimes y_i$ in $X \otimes Y$.

The *reasonable crossnorm* means that

$$u \in X \otimes Y,\ u = x \otimes y \text{ implies } \| u \| = \| x \otimes y \| = \| x \|_X \| y \|_Y;$$
$$\text{and } u = \sum_i x_i \otimes y_i,\ \ \| u \| = inf \sum_i \| x_i \|_X \| y_i \|_Y .$$

Note that a bounded linear operator $T \in Hom_A(X, Y^*)$ in (3.4) corresponding a continuous linear functional $\psi$ on $X \otimes_A Y$ is given by

$$(Tx)(y) = \psi(x \otimes y) \quad \text{for all } x \in X, y \in Y.$$

Here $Hom_A(X,Y^*) = \mathcal{M}_A(X,Y^*)$ is the space of all $A-module\ homomorphisms$ from $X$ to $Y^*$, the topological dual of $Y$, that is , each $T \in Hom_A(X,Y^*)$satisfies

$$T(ax) = a(Tx) \quad \text{for all } a \in A,\ x \in X,\ Tx \in Y^*.$$

where $T$ is a bounded linear operator from $X$ to $Y^*$; $X \otimes_A Y$ denotes the $A-module$ tensor product space of $X$ and $Y$.

There are some known results in scalar-valued function space of $L^1(G)-module$ by convolution. We state three typical $L^1(G)-module$ multiplier problems as follows.

**Theorem 3.** *(i)* $Hom_G(L^1(G), L^1(G)) \cong M_b(G)$, *(by Theorem 1, (iii)* $\Longleftrightarrow$ *(i))* *where* $Hom_G = Hom_{L^1(G)}$ , *and* $M_b(G)$ *is the space of all bounded regular Borel measures on* $G$.

*(ii)* $Hom_G(L^1(G), L^p(G)) \cong (L^1(G) \otimes_G L^q(G))^* = (L^q(G))^* = L^p(G)$, *for* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ *where* $\otimes_G = \otimes_{L^1(G)}$.

*(iii)* $Hom_G(L^P(G), L^P(G)) \cong (L^p(G) \otimes_G L^q(G))^* \cong S_p(G)^*$, *where* $S_p(G)$ *is a Banach algebra generated by*

$$\{u = \sum_i^\infty f_i g_i : f_i \in L^p(G), g_i \in L^q(G), \sum_i^\infty \| f_i \|_p \| g_i \|_q < \infty\}$$

*under pointwise product and the norm is defined by (cf. Larsen [7])*

$$|||u||| = \inf\{\sum_i^\infty \| f_i \|_p \| g_i \|_q;\ u = \sum_i^\infty f_i \cdot g_i \in S_p(G)\}.$$

# 4 Multipliers of Banach-valued Functions on $G$.

Let $A$ be a *commutative semi − simple Banach algebra* with bounded approximate identity . Assume $X$ is on $A$ − *module Banach space.* It is not hard to prove that $L^1(G,A) = L^1(G)\widehat{\otimes}_\gamma A$. Since both $L^1(G)$ and $A$ have bounded approximate identity, thus $L^1(G,A)$ is a *commutative Banach algebra* with bounded approximate identity. By Theorem 2

$$L^1(G,A) * L^p(G,X) = L^p(G,X), \quad 1 < p < \infty$$

Denote by

$$L^1(G,A) = \{f : G \longrightarrow A \mid f \text{ is measurable and is Bochner integrable on } G\}$$

Then $L^1(G,A)$ is a *commutative Banach algebra,* under convolution.
Actually

$$| f * g(t) |_A \leqslant \int_G | f(s-t) |_A | g(s) |_A \, ds = \| g \|_1 \int_G | f(s-t) |_A \, ds = \| g \|_1 \| f \|_1,$$

$$\| f * g \|_1 = \int_G | f * g(t) |_A \, dt \leqslant \| g \|_1 \int_G | f(s-t) |_A \, dt \leqslant \| g \|_1 \| f \|_1 .$$

Denote by

$$L^p_X = \{f : G \longrightarrow X \mid f \text{ is measurable and } | f(\cdot) |_X \in L^p(G)\}, 1 \leqslant p < \infty,$$

$$\| f \|_p = \left(\int_G | f(t) |_X^p \, dt\right)^{\frac{1}{p}}, \text{ for } f \in L^p_X, 1 \leqslant p < \infty \qquad (2.1)$$

$$\text{and for } p = \infty, \quad \| f \|_\infty = \underset{t \in G}{esssup} \, | f(t) |_X \ \text{ for } f \in L^\infty_X \qquad (2.2)$$

Show that $L^p_X, 1 \leqslant p \leqslant \infty$ are *Banach spaces* with the norm $\| f \|_p, 1 \leqslant p < \infty$, as (2.1) and if $p = \infty$, the norm is taken $\| \cdot \|_\infty$ as (2.2). If $X = \mathbb{C}$, the complex numbers, then

$$L^p_X = L^p = L^p(G), 1 \leqslant p \leqslant \infty.$$

If $X$ and $Y$ are $A$ − *module* Bananch space, the multiplier space of $X$ to $Y$ is given by

$$Hom_A(X,Y) = \{T \in \mathfrak{L}(X,Y) \mid T(ax) = aT(x), \ a \in A, \ x \in X\}.$$

Recall [8], Rieffel characterized the homomorphism module multiplier is represented by the dual space of module tensor product as the following form:

$$Hom_A(X,Y^*) \cong (X \otimes_A Y)^* \text{ or } Hom_A(X,Y) \cong (X \otimes_A Y^*)^*.(\text{ if } Y \text{ is reflexive})$$

where $\otimes_A$ is namely module tensor product of $X$ into $Y^*$ or of $X$ into $Y$ and $Z^*$ denotes the dual space of the Banach space $Z$. The space $\otimes_A$ is the complete projective tensor product $X\widehat{\otimes}_\gamma Y^*$ quotients by $K$, that is , $X \otimes_A Y = X\widehat{\otimes}_\gamma Y/K$.
Here $K$ is the closed linear subspace of the projective tensor product space $X\widehat{\otimes}_\gamma Y$ generated by the elements $ax \otimes y - x \otimes ay$; for $a \in A, x \in X, y \in Y$
and $X \otimes_\gamma Y$ is the completion of the algebra tensor $x \otimes y$ under the $\gamma-$norm, and

$$X \otimes Y = \{u = \sum_i x_i \otimes y_i \mid x_i \in X, y_i \in Y, \sum_i \| x_i \| \| y_i \| < \infty\}$$

$$\begin{aligned}\gamma(u) &= \inf_u \{\sum_i \| xi \| \| y_i \| | \ u = \sum x_i \otimes y_i \in Y\} \\ &= |||u||| = \inf_u \sum_i \| x_i \otimes y_i, x_i \in X, y_i \| = \inf_u \sum_i \| x_i \|_X \| y_i \|_Y\end{aligned}$$

where $\inf_u$ means that the infimum is taken by all representations of $u = \sum_i x_i \otimes y_i$ in $X \otimes Y$, and the tensor norm. We state the following Theorem for the characterization of the invariant operators. For detail, we consult Lai [3,4] and [6] cf. also Lai [5].

**Theorem 4.** *Let $X$ and $Y$ be Banach spaces. Then the following two statements are equivalent.*

*(a) $T \in (L^1(G,Y), L^1(G,X))$ is an invariant operator.*

*(b) There exists a unique continuous linear map $L \in \mathcal{L}(Y, M_b(G,X))$ such that $T(f \otimes y) = f * L_y$ for all $f \in L^1(G)$, $y \in Y$.*

*Moreover, $(L^1(G,Y), L^1(G,X)) \cong \mathcal{L}(Y, M_b(G,X))$.*

**Theorem 5.** *Let $A$ be a commutative semi-simple Banach algebra (not necessarily with identity) and $X$ a Banach $A$-module. Then*

(5.1) $Hom_{L^1(G,A)}(L^1(G,A), L^1(G,X)) \cong Hom_A(A, M_b(G,X))$.

In Lai [6], he showed that an invariant operator is also a multiplier if and only if the $A$ in $L^1(G,A)$ must be scalar space $\mathbb{C}$.

**Theorem 6.** *Let A be a commutative Banach algebra with identity of norm 1. X be a unit linked, order-free, Banach-module and A a faithful representation on X, then each invariant operator $T: L^1(G,A) \to F(G,X)$ is a multiplier if and only if $A \cong C$. Here $F(G,X) = L^p(G,X)$ for each p, $1 \leq p \leq \infty$, or $F(G,X) = C_0(G,X)$.*

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