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Duality of the James constant of Banach spaces

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1 Introduction

This note is based on [14].

Let X be a Banach space, and let B_X and S_X denote the unit ball and unit sphere of X, respectively. Then X is said to be *uniformly non-square* if there exists a positive number δ such that $x, y \in B_X$ and $\|2^{-1}(x+y)\| > 1-\delta$ implies $\|2^{-1}(x+y)\| \leq 1-\delta$. The James constant $J(X)$ of X was defined in 1990 by Gao and Lau [2] as a measure of how "non-square" the unit ball is, namely, the James constant is defined by

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} : x, y \in S_X\}.$$

It is known that $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X, and that X is uniformly non-square if and only if $J(X) < 2$ (cf [2, 4]).

Unlike the von Neumann-Jordan constant $C_{NJ}(X)$, the James constant does not satisfy $J(X^*) = J(X)$ in general. An example of $J(X^*) \neq J(X)$ is given by the Day-James ℓ_2-ℓ_1 space (cf. [4]), where ℓ_2-ℓ_1 is defined to be the space $\mathbb{R}^2$ endowed with the norm

$$\|(x,y)\|_{2,1} = \begin{cases} \|(x,y)\|_2 & \text{if} \quad xy \geq 0, \\ \|(x,y)\|_1 & \text{if} \quad xy \leq 0. \end{cases}$$

See [11] for more computations of the James constant of generalized Day-James spaces. We remark that the norm $\|\cdot\|_{2,1}$ is *symmetric*, that is, $\|(x,y)\|_{2,1} = \|(y,x)\|_{2,1}$ for each (x,y). Moreover, letting $\|(x,y)\|_{2,1}' = \|(x+y, x-y)\|_{2,1}$ for each (x,y) yields an absolute norm on $\mathbb{R}^2$, where a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be *absolute* if $\|(x,y)\| = \|(|x|,|y|)\|$ for each (x,y). Since James constant does not change under isometric isomorphisms, we already have obtained counterexamples of two-dimensional normed spaces that are equipped with either symmetric or absolute norms.

On the other hand, we have some examples of $J(X^*) = J(X)$. The first example is the ℓ_p-space. In fact, the assumption on the dimension is redundant.

Example 1.1 (Gao and Lau [2]). Let $1 \le p, q \le \infty$ with $1/p+1/q = 1$. Then $J(\ell_p^2) = 2^{1/r}$, where $r = \min\{p, q\}$. Consequently, $J((\ell_p^2)^*) = J(\ell_q^2) = J(\ell_p^2)$.

The equation $J(X^*) = J(X)$ can be also satisfied by a polyhedral normed space X. The norms defined in the following example have octagons as the unit balls.

Example 1.2 (Komuro, Saito and Mitani [6, 7]). For each $1/2 < \beta < 1$, let $\|(x, y)\|_\beta = \max\{|x|, |y|, \beta(|x| + |y|)\}$. Then $J((\mathbb{R}^2, \|\cdot\|_\beta)^*) = J((\mathbb{R}^2, \|\cdot\|_\beta))$.

Since the ℓ_p-norms are the best and polyhedral norms are something bad in the geometric sense, we have wide examples of $J(X^*) = J(X)$.

In this note, we consider the following problem: When does the equality $J(X^*) = J(X)$ hold for a Banach space X? It is shown that if the norm of a two-dimensional space is both symmetric and absolute then the James constant of the space coincides with that of its dual space. This provides a global answer to the problem in the two-dimensional case. Moreover, we present some new examples of $J(X^*) \neq J(X)$ by extreme absolute norms.

2 Preliminaries

We recall that a norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be symmetric if $\|(x, y)\| = \|(y, x)\|$ for each (x, y), and absolute if $\|(x, y)\| = \|(|x|, |y|)\|$ for each (x, y). The main result in this note is the following.

Theorem 2.1. *Let X be a two-dimensional real normed space $\mathbb{R}^2$ equipped with a symmetric absolute norm. Then $J(X^*) = J(X)$.*

Since James constant is invariant under scaling, we may assume that the norm $\|\cdot\|$ is also *normalized*, that is, $\|(1, 0)\| = \|(0, 1)\| = 1$. Let AN_2 be the set of all absolute normalized norms on $\mathbb{R}^2$. Then it is known that the set AN_2 is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0, 1]$ satisfying $\max\{1 - t, t\} \le \psi(t) \le 1$ for each $t \in [0, 1]$ (cf. [1, 12]). The correspondence is given by the equation $\psi(t) = \|(1-t, t)\|$ for all $t \in [0, 1]$. Remark that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(x, y)\|_\psi = \begin{cases} (|x| + |y|)\psi\left(\dfrac{|y|}{|x| + |y|}\right) & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

We also remark that the absolute normalized norm $\|\cdot\|_\psi$ on $\mathbb{R}^2$ associated with the function $\psi \in \Psi_2$ is symmetric if and only if $\psi(1 - t) = \psi(t)$ for each $t \in [0, 1]$. Let Ψ_2^S denote the collection of all such elements in Ψ_2. For more information about absolute normalized norms, for example, we refer the readers to [10, 12, 13, 15].

In what follows, we denote the normed space $(\mathbb{R}^2, \|\cdot\|_\psi)$ by X_ψ for short. For each $\psi \in \Psi_2$, let ψ^* be the function on $[0, 1]$ given by

$$\psi^*(s) = \max_{0 \le t \le 1} \frac{(1 - s)(1 - t) + st}{\psi(t)}$$

for each s. Then it follows that $\psi^* \in \Psi_2$ and $X_\psi^* = X_{\psi^*}$, and so $\psi^{**} = \psi$; see [9]. The function ψ^* is called the *dual function* of ψ. If $\psi \in \Psi_2^S$, then $\psi^* \in \Psi_2^S$ and the behavior of ψ^* is given by

$$\psi^*(s) = \max_{0 \le t \le 1/2} \frac{(1-s)(1-t)+st}{\psi(t)}$$

for each $s \in [0, 1/2]$; see [8] for details. Under these settings, the main result is translated as follows:

Theorem 2.1'. *Let $\psi \in \Psi_2^S$. Then $J(X_{\psi^*}) = J(X_\psi)$.*

3 Proof of the main theorem

We shall begin with the definition of piecewise linear functions. A finite sequence $(t_i)_{i=0}^n$ of real numbers is said to be a *partition* of the interval $[0, 1/2]$ if $0 = t_0 < t_1 < \cdots < t_n = 1/2$. Any finite subset P of $[0, 1/2]$ including 0 and 1/2 can be viewed as a partition of $[0, 1/2]$ by taking strictly increasing rearrangement, and so we identify the partition $(t_i)_{i=0}^n$ with the set $\{t_i : 0 \le i \le n\}$. A function ψ on the interval $[0, 1/2]$ is said to be *piecewise linear* if its graph is a broken line. More precisely, ψ is piecewise linear if there exist a partition $(t_i)_{i=0}^n$ of $[0, 1/2]$ and a finite sequence $(a_i)_{i=0}^n$ of real numbers such that

$$\psi(t) = \frac{a_i - a_{i-1}}{t_i - t_{i-1}} t + \frac{a_{i-1}t_i - a_i t_{i-1}}{t_i - t_{i-1}} \tag{1}$$

for each $t \in [t_{i-1}, t_i]$. Letting

$$\alpha_i = \frac{a_i - a_{i-1}}{t_i - t_{i-1}} \quad \text{and} \quad \beta_i = \frac{a_{i-1}t_i - a_i t_{i-1}}{t_i - t_{i-1}},$$

one has that $\psi(t) = \alpha_i t + \beta_i$ for each $t \in [t_{i-1}, t_i]$, and that $\psi(t_i) = a_i$ for each $0 \le i \le n$.

We have two key lemmas to prove the main theorem.

Lemma 3.1. *The function $\psi \mapsto J(X_\psi)$ is continuous on Ψ_2^S.*

Lemma 3.2. *Let $\psi \in \Psi_2^S$. Then there exists a sequence (ψ_n) of strictly convex functions in Ψ_2^S such that $\|\psi_n - \psi\|_\infty \to 0$ and $\|\psi_n^* - \psi^*\|_\infty \to 0$ as $n \to \infty$.*

Sketch of Proof. The proof proceeds as follows:

1. Establish the inequality $J(X_\psi) \le J(X_{\psi^*})$ for piecewise linear functions $\psi \in \Psi_2^S$.

2. Approximate each strictly convex function in Ψ_2^S by piecewise linear functions. This and Lemma 3.1 together show that $J(X_\psi) \le J(X_{\psi^*})$ for each strictly convex element $\psi \in \Psi_2^S$.

3. Use Lemma 3.2 to approximate each elements in Ψ_2^S by strictly convex functions in Ψ_2^S. Applying Lemma 3.1 again shows that $J(X_\psi) \le J(X_{\psi^*})$ for each element $\psi \in \Psi_2^S$.

4. Observe that $J(X_{\psi^*}) \le J(X_{\psi^{**}}) = J(X_\psi)$ by $\psi^{**} = \psi$. This completes the proof.

4 New examples of $J(X^*) \neq J(X)$

We conclude this paper with new examples of $J(X^*) \neq J(X)$. Remark that both the sets AN_2 and Ψ_2 are convex, and that the correspondence preserves the convex structure. Namely, the following hold:

(i) If $\|\cdot\|, \|\cdot\|' \in AN_2$, then $\lambda\|\cdot\| + (1-\lambda)\|\cdot\|' \in AN_2$ for all $\lambda \in (0,1)$.

(ii) If $\psi, \psi' \in \Psi_2$, then $\lambda\psi + (1-\lambda)\psi' \in \Psi_2$ for all $\lambda \in (0,1)$.

(iii) $\|\cdot\|_{\lambda\psi+(1-\lambda)\psi'} = \lambda\|\cdot\|_\psi + (1-\lambda)\|\cdot\|_{\psi'}$ for each $\psi, \psi' \in \Psi_2$ and all $\lambda \in (0,1)$.

By (iii), the extreme points of AN_2 and Ψ_2 are essentially the same. Moreover, we have the following result.

Theorem 4.1 (Grząślewicz [3]; Komuro, Saito and Mitani [5]). *For each* $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$, *define the function* $\psi_{\alpha,\beta}$ *by*

$$\psi_{\alpha,\beta} = \begin{cases} 1-t & \text{if } 0 \leq t \leq \alpha, \\ \dfrac{(\alpha+\beta-1)t+\beta-2\alpha\beta}{\beta-\alpha} & \text{if } \alpha \leq t \leq \beta, \\ t & \text{if } \beta \leq t \leq 1. \end{cases}$$

Then $\mathrm{ext}(\Psi_2) = \{\psi_{\alpha,\beta} : 0 \leq \alpha \leq 1/2 \leq \beta \leq 1\}$.

The James constant of $X_{\psi_{\alpha,\beta}}$ is completely determined by Komuro, Saito and Mitani [6]; see also [7].

Theorem 4.2 (Komuro, Saito and Mitani [6]). *Let* $0 \leq \alpha \leq 1/2 \leq \beta \leq 1$ *with* $\alpha < 1-\beta$.

(i) If $\psi_{\alpha,\beta}(1/2) \leq 1/2(1-\alpha)$, *then*

$$J(X_{\psi_{\alpha,\beta}}) = \frac{1}{\psi_{\alpha,\beta}(1/2)}.$$

(ii) If $1/2(1-\alpha) \leq \psi_{\alpha,\beta}(1/2) \leq c(\alpha,\beta)$, *then*

$$J(X_{\psi_{\alpha,\beta}}) = 1 + \frac{1}{\psi_{\alpha,\beta}(1/2) + (2\beta-1)/(\beta-\alpha)}.$$

(iii) If $c(\alpha,\beta) \leq \psi_{\alpha,\beta}(1/2)$, *then*

$$J(X_{\psi_{\alpha,\beta}}) = 2\psi_{\alpha,\beta}(1/2),$$

where

$$c(\alpha,\beta) = \frac{1}{4}\left(1 - \frac{2\beta-1}{\beta-\alpha} + \sqrt{\left(1+\frac{2\beta-1}{\beta-\alpha}\right)^2 + 4}\right).$$

Using this result, we can provide new examples of $J(X^*) \neq J(X)$, where X is the space $\mathbb{R}^2$ endowed with an extreme absolute normalized norm on $\mathbb{R}^2$.

Example 4.3. The computation is based on Theorem 4.2. For each $\beta \in (1/2, 1)$, let ψ_β be an asymmetric element of Ψ_2 given by

$$\psi_\beta(t) = \psi_{0,\beta}(t) = \begin{cases} \dfrac{\beta - 1}{\beta} t + 1 & \text{if} \quad t \in [0, \beta], \\ t & \text{if} \quad t \in [\beta, 1], \end{cases}$$

and let

$$c(\beta) = c(0, \beta) = \frac{1}{4} \left(\frac{1 - \beta}{\beta} + \sqrt{\left(1 + \frac{2\beta - 1}{\beta}\right)^2 + 4} \right).$$

Then it follows that $\psi_\beta(1/2) \geq c(\beta)$ if and only if $\beta \geq 2/3$. Hence, by Theorem 4.2, we have

$$J(X_{\psi_\beta}) = \begin{cases} \dfrac{6\beta - 2}{5\beta - 2} & \text{if} \quad \beta \in (1/2, 2/3], \\ \dfrac{3\beta - 1}{\beta} & \text{if} \quad \beta \in [2/3, 1). \end{cases}$$

We next consider the dual function of ψ_β. After an easy computation, we obtain

$$\psi_\beta^*(t) = \begin{cases} 1 - t & \text{if} \quad t \in [0, (2\beta - 1)/(3\beta - 1)], \\ \dfrac{2\beta - 1}{\beta} t + \dfrac{1 - \beta}{\beta} & \text{if} \quad t \in [(2\beta - 1)/(3\beta - 1), 1]. \end{cases}$$

From this, we have

$$J(X_{\psi_\beta^*}) = \begin{cases} \dfrac{1}{\beta} & \text{if} \quad \beta \in (1/2, 2/3], \\ \dfrac{2}{2 - \beta} & \text{if} \quad \beta \in [2/3, 1). \end{cases}$$

Thus, consequently, we obtain $J(X_{\psi_\beta^*}) \neq J(X_{\psi_\beta})$ whenever $\beta \neq 2/3$.

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