

Generalized split null point problem governed by widely more generalized widely more hybrid mappings in Hilbert spaces

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Abstract. Generalized split feasibility problem governed by a widely more generalized hybrid mapping is studied. In particular, strong convergence of this algorithm is proved. As tools, resolvents of maximal monotone operators are technically maneuvered to facilitate the argument of the proof to the main result. Applications to iteration methods for various nonlinear mappings and to equilibrium problem are included.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H . A mapping $U : C \rightarrow H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α -inverse strongly monotone. Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2 , respectively. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Then the *split feasibility problem* [6] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Recently, Byrne, Censor, Gibali and Reich [5] considered the following problem: Given set-valued mappings $A_i : H_1 \rightarrow 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \rightarrow 2^{H_2}$, $1 \leq j \leq n$, respectively, and bounded linear operators $T_j : H_1 \rightarrow H_2$, $1 \leq j \leq n$, the *split common null point problem* [5] is to find a point $z \in H_1$ such that

$$z \in \left(\bigcap_{i=1}^m A_i^{-1}0 \right) \cap \left(\bigcap_{j=1}^n T_j^{-1}(B_j^{-1}0) \right),$$

where $A_i^{-1}0$ and $B_j^{-1}0$ are null point sets of A_i and B_j , respectively. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \rightarrow H_1$ is an inverse strongly monotone operator, where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q . Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z,$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D . Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem and generalized split feasibility problems including the split common null point problem; see, for instance, [5, 7, 17, 30]. In the study, they used established results for solving the problems.

In particular, established convergence theorems have been used for finding solutions of the problems. On the other hand, we know many existence and convergence theorems for inverse strongly monotone mappings in Hilbert spaces; see, for instance, [9, 18, 20, 22, 26, 27].

In this article, motivated by the ideas of these problems and results, we consider generalized split feasibility problem and then the problem governed by a widely more generalized hybrid mapping is studied. In particular, strong convergence of this algorithm is proved. As tools, resolvents of maximal monotone operators are technically maneuvered to facilitate the argument of the proof to the main result. Applications to iteration methods for various nonlinear mappings and to equilibrium problem are included.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [25] that

$$\begin{aligned}\|x + y\|^2 &\leq \|x\|^2 + 2\langle y, x + y \rangle; \\ \|\lambda x + (1 - \lambda)y\|^2 &= \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.\end{aligned}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.$$

Let C be a nonempty, closed and convex subset of a Hilbert space H . The nearest point projection of H onto C is denoted by P_C , that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C . We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x \rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [23]. Let $\alpha > 0$ be a given constant. A mapping $A: C \rightarrow H$ is said to be α -inverse strongly monotone if $\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. It is known that $\|Ax - Ay\| \leq (1/\alpha) \|x - y\|$ for all $x, y \in C$ if A is α -inverse-strongly monotone. Let B be a mapping of H into 2^H . The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B on H is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H . For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}: H \rightarrow D(B)$, which is called the resolvent of B for r . Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ is the set of fixed points of J_r . It is also known that $\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu|/\lambda) \|x - J_\lambda x\|$ holds for all $\lambda, \mu > 0$ and $x \in H$; see [23, 10] for more details. As a matter of fact, we know the following lemma [22].

Lemma 2.1. *Let H be a real Hilbert space and let B be a maximal monotone operator on H . For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s} \langle J_s x - J_t x, J_s x - x \rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

We also know the following lemmas:

Lemma 2.2 ([2], [29]). Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$. Suppose that

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.3 ([16]). Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

- (i) $\tau(n_0) \leq \tau(n_0 + 1) \leq \dots$ and $\tau(n) \rightarrow \infty$;
- (ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

From [28], we also have the following lemmas.

Lemma 2.4. Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Then a mapping $A^*(I - T)A : H_1 \rightarrow H_1$ is $\frac{1}{2\|A\|^2}$ -inverse strongly monotone.

Lemma 2.5. Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \rightarrow 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping and let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H$. Then the following are equivalent:

- (i) $z = J_\lambda(I - rA^*(I - T)A)z$;
- (ii) $0 \in A^*(I - T)Az + Bz$;
- (iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

3 Main result

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Then, a mapping $T : C \rightarrow H$ is called generalized hybrid [15] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping (α, β) -generalized hybrid. Notice that the mapping above covers several well-known mappings. For example, an (α, β) -generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. Kawasaki and Takahashi [14] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping S from C into H is said to be

widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha \|Sx - Sy\|^2 + \beta \|x - Sy\|^2 + \gamma \|Sx - y\|^2 + \delta \|x - y\|^2 \\ & + \varepsilon \|x - Sx\|^2 + \zeta \|y - Sy\|^2 + \eta \|(x - Sx) - (y - Sy)\|^2 \leq 0 \end{aligned}$$

for all $x, y \in C$. Such a mapping S is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [15] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = \eta = 0$. A generalized hybrid mapping with a fixed point is quasinonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. We know the following theorem from Kawasaki and Takahashi [14].

Theorem 3.1 ([14]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into itself which satisfies the following conditions (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ and $\varepsilon + \eta \geq 0$.

Then S has a fixed point if and only if there exists $z \in C$ such that $\{S^n z : n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of S is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

The following lemmas for widely more generalized hybrid mappings are essential for proving our main theorem.

Lemma 3.2 ([14]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into itself such that $F(S) \neq \emptyset$ and it satisfies the conditions (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\zeta + \eta \geq 0$ and $\alpha + \beta > 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\varepsilon + \eta \geq 0$ and $\alpha + \gamma > 0$.

Then S is quasi-nonexpansive.

Lemma 3.3 ([12]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H . Let $S : C \rightarrow H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping. Suppose that it satisfies the following conditions (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta + \zeta + \eta > 0$.

If $x_n \rightarrow z$ and $x_n - Sx_n \rightarrow 0$, then $z \in F(S)$.

In this section, we solve generalized split feasibility problem governed by a widely more generalized hybrid mapping in Hilbert spaces.

Theorem 3.4 ([13]). *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1 . Let $B : H_1 \rightarrow 2^{H_1}$ be a maximal monotone mapping such that $D(B) \subset C$ and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into C which satisfies the conditions (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.

Let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator.

Suppose that $B^{-1}0 \cap F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)S J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap F(S) \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0 \cap F(S) \cap A^{-1}F(T)}u$.

4 Applications

Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x \rangle \leq f(y), \quad \forall y \in H\}$$

for all $x \in H$. By Rockafellar [19], it is shown that ∂f is maximal monotone. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C , i.e.,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Then $i_C : H \rightarrow (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and hence ∂i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$ as follows:

$$J_\lambda x = (I + \lambda \partial i_C)^{-1}x, \quad \forall x \in H, \lambda > 0.$$

Putting $B = \partial i_C$ in Theorem 3.4, we have $J_\lambda = P_C$. Thus we obtain the following theorem from Theorem 3.4.

Theorem 4.1. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty, closed and convex subset of H_1 . Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into C which satisfies the conditions (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.

Let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)S P_C(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S) \cap A^{-1}F(T)}u$.

Replacing a widely more generalized hybrid mapping in Theorem 3.4 by a generalized hybrid mapping, we have the following theorem.

Theorem 4.2. Let H_1 and H_2 be Hilbert spaces and let C be a nonempty closed convex subset of H_1 . Let $B : H_1 \rightarrow 2^{H_1}$ be a maximal monotone mapping such that $D(B) \subset C$ and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let S be a generalized hybrid mapping from C into C . Let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)S J_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in B^{-1}0 \cap F(S) \cap A^{-1}F(T)$, where $z_0 = P_{B^{-1}0 \cap F(S) \cap A^{-1}F(T)}u$.

We also get the following theorem from Theorem 4.2.

Theorem 4.3. Let H_1 and H_2 be Hilbert spaces and let C be a nonempty closed convex subset of H_1 . Let $S : C \rightarrow C$ be a nonexpansive mapping and let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. Let $x_1 = x \in C$ and let $\{x_n\}$ be a sequence in C generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)P_C(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point z_0 of $F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S) \cap A^{-1}F(T)}u$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H , let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem: Find $z \in C$ such that

$$f(z, y) \geq 0, \quad \forall y \in C.$$

The set of such $z \in C$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{z \in C : f(z, y) \geq 0, \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

- (A1) $f(x, x) = 0$ for all $x \in C$;
 - (A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
 - (A3) for all $x, y, z \in C$,
- $$\limsup_{t \rightarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

- (A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [4] and [8].

Lemma 4.4 ([4]). *Let C be a nonempty closed convex subset of H , let f be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0$$

for all $y \in C$.

Lemma 4.5 ([8]). *For $r > 0$ and $x \in H$, define the resolvent $T_r : H \rightarrow C$ of f for $r > 0$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

- (i) T_r is single-valued;
- (ii) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (iii) $F(T_r) = EP(f)$;
- (iv) $EP(f)$ is closed and convex.

Takahashi, Takahashi and Toyoda [22] showed the following. See [1] for a more general result.

Lemma 4.6 ([22]). *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_f as follows:*

$$A_f(x) = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z \rangle, \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone with the domain of A_f in C . Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall r > 0.$$

We obtain the following theorem from Theorem 3.4.

Theorem 4.7. *Let H_1 and H_2 be Hilbert spaces and let C be a nonempty closed convex subset of H_1 . Let $f : C \times C \rightarrow \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_{λ_n} be the resolvent of A_f for $\lambda_n > 0$ in Lemma 4.6. Let S be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into C which satisfies the conditions (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ and $\varepsilon + \eta \geq 0$.

Let $T : H_2 \rightarrow H_2$ be a nonexpansive mapping. Let $A : H_1 \rightarrow H_2$ be a bounded linear operator. Suppose that $EP(f) \cap F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \rightarrow u$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)(\alpha_n u_n + (1 - \alpha_n)ST_{\lambda_n}(I - \lambda_n A^*(I - T)A)x_n)$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$ and $\{\alpha_n\} \subset (0, 1)$ satisfy

$$0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2}, \quad 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n \rightarrow \infty} \alpha_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in EP(f) \cap F(S) \cap A^{-1}F(T)$, where $z_0 = P_{EP(f) \cap F(S) \cap A^{-1}F(T)}u$.

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