数理解析研究所講究録
第 1963 巻 2015 年 108-113

Generalized Beckner's inequalities and its applications to new geometric properties

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1 Introduction

This note is a survey on [7, 8]. For a Banach space X, let

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X,\ \|x-y\| = \varepsilon\right\}$$

for each $\varepsilon \in (0,2]$, and let

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in S_X\right\}$$

for each $\tau \geq 0$. These constants are, respectively, the moduli of convexity and smoothness of X. Let $1 < p \leq 2 \leq q < \infty$. Then a Banach space X is said to be

(i) *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$,

(ii) *q-uniformly convex* if there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for each $\varepsilon \in (0,2]$,

(iii) *uniformly smooth* if $\lim_{\tau\to 0^+} \rho_X(\tau)/\tau = 0$, and

(iv) *p-uniformly smooth* if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$.

Obviously the implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) hold. These properties are called geometric properties of Banach spaces as well as strict convexity and uniform non-squareness, and play important roles in the study of Banach space geometry. For basic facts of p-uniform smoothness and q-uniform convexity, the readers are referred to [1, 9].

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, normalized if $\|(1,0)\| = \|(0,1)\| = 1$, and symmetric if $\|(x,y)\| = \|(y,x)\|$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Bonsall and Duncan [3] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for each $t \in [0,1]$ (cf.

[6]). The correspondence is given by the equation $\psi(t) = \|(1-t,t)\|$ for each $t \in [0,1]$. Remark that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

We also remark that the norm $\|\cdot\| \in AN_2$ is symmetric if and only if $\psi(1-t) = \psi(t)$ for each $t \in [0,1]$. For example, the function ψ_p corresponding to $\|\cdot\|_p$ is given by

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty, \end{cases}$$

and satisfies $\psi_p(1-t) = \psi_p(t)$ for each $t \in [0,1]$. Let $\Psi_2^S = \{\psi \in \Psi_2 : \psi(1-t) = \psi(1-t) \text{ for each } t \in [0,1]\}$.

The aim of this note is to present generalized Beckner inequalities, and to introduce new geometric properties of Banach spaces that generalize p-uniform smoothness and q-uniform convexity using absolute normalized norms.

2 Generalized Beckner inequalities

We first consider generalized Beckner inequalities. The original Becker inequality is the following: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for each $u, v \in \mathbb{R}$. This was shown in 1975 by Beckner [2]. It is also known that $\gamma_{p,q}$ in the above inequality is the best constant, that is, if $\gamma \in [0,1]$ and the inequality

$$\left(\frac{|u+\gamma v|^q + |u-\gamma v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for each $u, v \in \mathbb{R}$, then we have $\gamma \leq \gamma_{p,q}$. In [10], we constructed an elementary proof of these facts.

Beckner's inequality is easily extended to Banach spaces; see [4, Corollary 1.e.15] for the proof.

Theorem 2.1. *Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality*

$$\left(\frac{\|x+\gamma_{p,q}y\|^q + \|x-\gamma_{p,q}y\|^q}{2}\right)^{1/q} \leq \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p}$$

holds for each $x, y \in X$.

Using the functions ψ_p and ψ_q, Beckner's inequality can be viewed as follows: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality

$$\frac{\|(u+\gamma_{p,q}v, u-\gamma_{p,q}v)\|_q}{2\psi_q(\frac{1}{2})} \leq \frac{\|(u+v, u-v)\|_p}{2\psi_p(\frac{1}{2})}$$

holds for each $u, v \in \mathbb{R}$. From this observation, we considered in [7] generalized Beckner's inequality. Namely, for each $\varphi, \psi \in \Psi_2$, let

$$\Gamma(\varphi, \psi) = \left\{ \gamma \in [0,1] : \frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})} \leq \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})} \text{ for all } u \in [0,1] \right\},$$

and let $\gamma_{\varphi,\psi} = \max \Gamma(\varphi, \psi)$. Then we have the following result. Suppose that X is a Banach space. Then for each ψ the ψ-direct sum of X, denoted by $X \oplus_\psi X$, is the space $X \times X$ equipped with the norm $\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$.

Theorem 2.2 (Generalized Beckner's inequality [7]). *Let X be a Banach space. Suppose that $\varphi, \psi \in \Psi_2^S$, and that $\gamma \in \Gamma(\varphi, \psi)$. Then the inequality*

$$\frac{\|(x + \gamma y, x - \gamma y)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(x + y, x - y)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for each $x, y \in X$.

We present some conditions that $\gamma_{\varphi,\psi} > 0$; see [7] for details. For each $\psi \in \Psi_2^S$, let ψ_L' denote the left derivative of ψ.

Theorem 2.3. *Let $\varphi, \psi \in \Psi_2^S$. Then the following hold:*

(i) *If $\varphi_L'(1/2) = 0$ and $\psi_L'(1/2) < 0$, then $\gamma_{\varphi,\psi} > 0$.*

(ii) *If $\varphi_L'(1/2) < 0$ and $\psi_L'(1/2) = 0$, then $\gamma_{\varphi,\psi} = 0$.*

(iii) *If $\varphi_L'(1/2) < 0$ and $\psi_L'(1/2) < 0$, then $\gamma_{\varphi,\psi} > 0$.*

In particular, if $\varphi_L'(1/2) < 0$ then

$$\gamma_{\varphi,\psi} \leq \frac{\varphi(\frac{1}{2})\psi_L'(\frac{1}{2})}{\psi(\frac{1}{2})\varphi_L'(\frac{1}{2})}.$$

Theorem 2.4. *Let $\varphi, \psi \in \Psi_2^S$. Suppose that the second derivatives φ'' and ψ'' are continuous on $(\delta, 1-\delta)$ for some $0 \leq \delta < 1/2$. Then the following hold:*

(i) *If $\varphi''(1/2) = 0$ and $\psi''(1/2) > 0$, then $\gamma_{\varphi,\psi} > 0$.*

(ii) *If $\varphi''(1/2) > 0$ and $\psi''(1/2) = 0$, then $\gamma_{\varphi,\psi} = 0$.*

(iii) *If $\varphi''(1/2) > 0$ and $\psi''(1/2) > 0$, then $\gamma_{\varphi,\psi} > 0$.*

In particular, if $\varphi''(1/2) > 0$ then

$$\gamma_{\varphi,\psi} \leq \sqrt{\frac{\varphi(\frac{1}{2})\psi''(\frac{1}{2})}{\psi(\frac{1}{2})\varphi''(\frac{1}{2})}}.$$

Remark 2.5. We remark that

$$\sqrt{\frac{\psi_q(\frac{1}{2})\psi_p''(\frac{1}{2})}{\psi_p(\frac{1}{2})\psi_q''(\frac{1}{2})}} = \sqrt{\frac{p-1}{q-1}} = \gamma_{p,q},$$

where $\gamma_{p,q}$ is the best constant for Beckner's inequality.

For each $\psi \in \Psi_2$, define the function ψ^* by

$$\psi^*(t) = \max_{0\le s\le 1} \frac{(1-s)(1-t)+st}{\psi(s)}$$

for each $t \in [0,1]$. Then $\psi^* \in \Psi_2$ and $(\mathbb{R}^2, \|\cdot\|_\psi)^* = (\mathbb{R}, \|\cdot\|_{\psi^*})$, and so the function ψ^* is called the *dual function* of ψ; see [5]. Clearly, $\psi \in \Psi_2^S$ if and only if $\psi^* \in \Psi_2^S$.

Generalized Beckner inequalities have the following duality property.

Theorem 2.6. *Let $\varphi, \psi \in \Psi_2^S$. Then $\gamma_{\varphi,\psi} = \gamma_{\psi^*,\varphi^*}$.*

3 New geometric properties

We now consider new geometric properties of Banach spaces. First, we present the following characterizations of p-uniform smoothness and q-uniform convexity.

Proposition 3.1. *Let X be a Banach space, and let $1 < p \le 2$. Then X is p-uniformly smooth if and only if there exists $M > 0$ such that $\rho_X(\tau) \le \|(1, M\tau)\|_p - 1$ for each $\tau \in [0,1]$.*

Proof. Suppose that X is p-uniformly smooth. Then there exists a $K > 0$ satisfying $\rho_X(\tau) \le K\tau^p$ for each $\tau > 0$. Since the function f on $[0,1]$ given by

$$f(\tau) = 1 + pK(1+K)^{p-1}\tau^p - (1 + K\tau^p)^p$$

is nondecreasing, it follows that $f \ge 0$. Putting $M = p^{1/p}K^{1/p}(1+K)^{1-1/p}$ we have

$$\begin{aligned}\rho_X(\tau) &\le 1 + K\tau^p - 1\\ &\le (1 + pK(1+K)^{p-1}\tau^p)^{1/p} - 1\\ &= \|(1, M\tau)\|_p - 1\end{aligned}$$

for each $\tau \in [0,1]$.

Conversely, let M be a positive real number such that

$$\rho_X(\tau) \le \|(1, M\tau)\|_p - 1$$

for each $\tau \in [0,1]$. Then for each $\tau \in [0,1]$ one has

$$\rho_X(\tau) \le \|(1, M\tau)\|_p - 1 = (1 + M^p\tau^p)^{1/p} - 1 \le 1 + \frac{1}{p}M^p\tau^p - 1 = \frac{1}{p}M^p\tau^p.$$

On the other hand, if $\tau \ge 1$ then $\rho_X(\tau) \le \tau \le \tau^p$. Hence we obtain

$$\rho_X(\tau) \le \max\{M^p/p, 1\}\tau^p$$

for each $\tau \ge 0$, that is, the space X is p-uniformly smooth. $\square$

Proposition 3.2. *Let $2 \le q < \infty$. Then a Banach space X is q-uniformly convex if and only if it is $K > 0$ such that $\|(1 - \delta_X(\varepsilon), K\varepsilon)\|_q \le 1$ for each $\varepsilon \in [0,2]$.*

Proof. Suppose that X is q-uniformly convex. Then there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for each $\varepsilon \in [0,2]$. One can easily check that

$$(1-x)^q \leq 1 - \frac{x}{2}$$

for each $x \in [0,1]$. Hence, by $0 \leq C\varepsilon^q \leq \delta_X(\varepsilon) \leq 1$, we have

$$(1-\delta_X(\varepsilon))^q \leq (1-C\varepsilon^q)^q \leq 1 - \frac{C\varepsilon^q}{2}.$$

Putting $K = (C/2)^{1/q}$, we obtain $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_q = ((1-\delta_X(\varepsilon))^q + K^q\varepsilon^q)^{1/q} \leq 1$ for each $\varepsilon \in [0,2]$.

Conversely, assume that there exists $K > 0$ such that $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_q \leq 1$ for each $\varepsilon \in [0,2]$. Then $(1-\delta_X(\varepsilon))^q \leq 1 - K^q\varepsilon^q$, and so

$$1 - \delta_X(\varepsilon) \leq (1 - K^q\varepsilon^q)^{1/q} \leq 1 - \frac{1}{q}K^q\varepsilon^q.$$

Thus, for $C = K^q/q$, we have $\delta_X(\varepsilon) \geq C\varepsilon^q$ for each $\varepsilon \in [0,2]$. This shows X is q-uniformly convex. □

These propositions allows us to consider new geometric properties using absolute normalized norms. We now introduce ψ-uniform smoothness and ψ^*-uniform convexity as follows: Let $\psi \in \Psi_2$. Then a Banach space X is said to be

(i) *ψ-uniformly smooth* if there exists $M > 0$ such that $\rho_X(\tau) \leq \|(1, M\tau)\|_\psi - 1$ for each $\tau \in [0,1]$.

(ii) *ψ^*-uniformly convex* if there exists $K > 0$ such that $\|(1-\delta_X(\varepsilon), K\varepsilon)\|_{\psi^*} \leq 1$ for each $\varepsilon \in [0,2]$.

Then Propositions 3.1 and 3.2 guarantee that a Banach space X is

(a) p-uniformly smooth if and only if it is ψ_p-uniformly smooth, and

(b) q-uniformly convex if and only if it is ψ_q-uniformly smooth.

Naturally, one has $\psi_q = (\psi_p)^*$ provided that $1/p + 1/q = 1$. Hence the above new geometric properties are natural generalizations of that of p-uniform smoothness and q-uniform convexity.

For further results in this direction, the readers are referred to [8].

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