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ON COMPACTNESS IN L^1

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1. INTRODUCTION

Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space with σ-algebra $\mathcal{A}$ and measure μ and let $\mathcal{F}$ be the family of measurable subsets of Ω with finite measure. Let L^1 and L^∞ be the space of integrable functions defined on Ω and the space of essentially-bounded measurable functions defined on Ω, respectively. We denote by L^∞_{loc} the vector subspace of L^∞ consisting of essentially-bounded measurable functions f defined on Ω for which $\mu\{w \in \Omega : f(w) \neq 0\} < \infty$. In [5], we discussed a method of constructing a separated locally convex topology $\tilde{\tau}$ on L^1 generated by the semi-norms $f \mapsto \int_E |f|\, d\mu$ $(E \in \mathcal{F})$ with the assumption that μ is σ-finite. The topological dual of $(L^1, \tilde{\tau})$ is algebraically isomorphic to L^∞_{loc}. A notion of local uniform integrability for subsets of L^1 was also discussed to obtain a necessary and sufficient condition for a bounded subset of L^1 relative to L^1-norm to be relatively weakly compact in $(L^1, \tilde{\tau})$: Whenever C is a bounded subset of L^1 relative to L^1-norm, C is locally uniformly integrable if and only if C is relatively weakly compact in $(L^1, \tilde{\tau})$. We applied it to show the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on L^1. This result gives an identification of the limit function in almost everywhere convergence of the Cesàro means $n^{-1}\sum_{k=0}^{n-1} T^k f$ of an $f \in L^1$; see [6] for details.

In this paper, we summarize the arguments presented in [7] and [8] about a characterization of compactness for the weak topology of L^1 associated with τ, and then apply similar arguments to discuss some necessary and sufficient conditions of compactness for the topology on L^1 generated by the metric $(f, g) \mapsto \int_\Omega |f-g|\, d\mu$ and the weak topology $\sigma(L^1, L^\infty)$ on L^1 generated by L^∞, respectively. As their applications, (weak) almost periodicity of linear and non-linear operators in L^1 is also discussed.

2. PRELIMINARIES

Throughout the paper, let $\mathbb{N}_+$ and $\mathbb{R}$ denote the set of non-negative integers and the set of real numbers, respectively. Let $\langle E, F\rangle$ be a duality between vector spaces E and F over $\mathbb{R}$. If A is a subset of E, then $A^\circ = \{y \in F : \langle x, y\rangle \leq 1\, (x \in A)\}$ is a subset of F, called the polar of A. For each $y \in F$, we define a linear form f_y on E by

$f_y(x) = \langle x, y \rangle$ $(x \in E)$. Then, $\sigma(E,F)$ denotes the weak topology on E generated by the family $\{f_y : y \in F\}$ and $\tau(E,F)$ denotes the Mackey topology on E with respect to $\langle E, F \rangle$, that is, the topology of uniform convergence on the circled, convex, $\sigma(F,E)$-compact subsets of F. Let $(E, \mathfrak{T})$ is a locally convex space. Then, the topological dual of E is denoted by E'. The bilinear form $(x,f) \mapsto f(x)$ on $E \times E'$ defines a duality $\langle E, E' \rangle$ and the weak topology on E generated by E' is called the weak topology of E (associated with $\mathfrak{T}$ if this distinction is necessary). If E is a Banach space, then the subset $\{x \in E : \|x\| \leq r\}$ of E is called the closed ball with center at 0 and radius r, denoted by $B(r)$. In particular, $B(1)$ is called the closed unit ball in E.

Throughout the paper, let $(\Omega, \mathcal{A}, \mu)$ denote a positive measure space with σ-algebra $\mathcal{A}$ and measure μ, and let $\mathcal{F}$ denote the family of measurable subsets of Ω with finite measure. Then, $\mathcal{F}$ is ordered by set inclusion in the sense that E is less than F, or $E \leq F$ if and only if $E \subset F\,(E, F \in \mathcal{F})$, so that each finite subset of $\mathcal{F}$ has the least upper bound. Let $E \in \mathcal{A}$. If $\mathcal{A}_E$ denotes the σ-algebra of all intersections of members of $\mathcal{A}$ with E and μ_E denotes the restriction of μ to $\mathcal{A}_E$, then the triple $(E, \mathcal{A}_E, \mu_E)$ is a positive measure space. For $1 \leq p < \infty$, let $\mathcal{L}^p(E)$ be the vector space of measurable functions f defined on E for which $\|f\|_{E,p} = (\int_E |f|^p \, d\mu)^{\frac{1}{p}} < \infty$ and let $\mathcal{L}^\infty(E)$ be the vector space of measurable functions f defined on E for which $\|f\|_{E,\infty} = \inf_N \sup_{w \in E \setminus N} |f(w)| < \infty$, where N ranges over the null subsets of E. If $\mathcal{N}_E$ denotes the set of null functions defined on E and $[f]$ denotes the equivalence class of an $f \in \mathcal{L}^p(E) \bmod \mathcal{N}_E$ $(1 \leq p \leq \infty)$, then $[f] \mapsto \|f\|_{E,p}$ is a norm on the quotient space $\mathcal{L}^p(E)/\mathcal{N}_E$, which thus becomes a Banach space, denoted by $L^p(E)$. In particular, if μ is the counting measure on $\mathbb{N}$, then we write l^1 in place of $L^1(\mathbb{N})$. For each $f \in L^p(\Omega)$, $\|f\|_{\Omega,p}$ is called the L^p-norm of f, simply denoted by $\|f\|_p$. A measurable function f defined on Ω is called essentially-bounded if $\|f\|_\infty < \infty$. Every element of $L^p(E)$ is considered as a measurable function f defined on E with $\|f\|_{E,p} < \infty$, if no confusion will occur. For each $E \in \mathcal{A}$, the bilinear form $(f,h) \mapsto \int_E fh \, d\mu$ on $L^1(E) \times L^\infty(E)$ places $L^1(E)$ and $L^\infty(E)$ in duality. For $E, F \in \mathcal{F}$ with $E \leq F$, let i_{EF} denote the mapping of $L^1(F)$ onto $L^1(E)$ that assigns to each $f \in L^1(F)$ the restriction $f|_E$ of f to E. Then, the canonical imbedding of $L^\infty(E)$ into $L^\infty(F)$ is the adjoint operator of i_{EF}, denoted by j_{FE}.

Let $\mathcal{L}^1_{loc}(\Omega)$ be the vector space of measurable functions f defined on Ω for which $\|f\|_{E,1} < \infty$ for each $E \in \mathcal{F}$ and let $\mathcal{N}_{loc}$ be the vector subspace of $\mathcal{L}^1_{loc}(\Omega)$ consisting of measurable functions f defined on Ω for which $\|f\|_{E,1} = 0$ for each $E \in \mathcal{F}$. If $[f]$ denotes the equivalence class of an $f \in \mathcal{L}^1_{loc}(\Omega) \bmod \mathcal{N}_{loc}$, then $[f] = [g]$ $(f, g \in \mathcal{L}^1_{loc}(\Omega))$ means that for each $E \in \mathcal{F}$, $f(x) = g(x)$ almost everywhere on E. In particular, if μ is σ-finite, then $\mathcal{N}_{loc}$ equals the set $\mathcal{N}_\Omega$ of null functions defined on Ω and hence for $f, g \in \mathcal{L}^1_{loc}(\Omega)$, $[f] = [g]$ if and only if $f(x) = g(x)$ almost everywhere on Ω. For each $E \in \mathcal{F}$, $[f] \mapsto \|f\|_{E,1}$ is a semi-norm on the quotient space $\mathcal{L}^1_{loc}(\Omega)/\mathcal{N}_{loc}$, which becomes a locally convex space, denoted by $L^1_{loc}(\Omega)$, under the separated locally convex topology τ generated by the semi-norms $[f] \mapsto \|f\|_{E,1}$ ($E \in \mathcal{F}$). Every element of $L^1_{loc}(\Omega)$ is considered as a measurable function f defined on Ω for which $\|f\|_{E,1} < \infty$ for each $E \in \mathcal{F}$, if no confusion will occur. If μ is finite, then $L^1_{loc}(\Omega)$ equals $L^1(\Omega)$ and hence τ is just the topology on $L^1(\Omega)$ generated by the metric $(f, g) \mapsto \|f - g\|_1$.

In the sequel, we shall assume that the measure space $(\Omega, \mathcal{A}, \mu)$ *is* σ-*finite.* The product space $\mathcal{L}$ is the Cartesian product $L = \prod_{E\in\mathcal{F}} L^1(E)$ of the family $\{(L^1(E), \|\cdot\|_{E,1}) : E \in \mathcal{F}\}$ with its product topology. Then, $L^1_{loc}(\Omega)$ is identified as a closed (and hence complete) subspace of $\mathcal{L}$ by the isomorphism $f \mapsto (f|_E)_{E\in\mathcal{F}}$ of $L^1_{loc}(\Omega)$ into $\mathcal{L}$, where $f|_E$ is the restriction of f to E. Let $D = \bigoplus_{E\in\mathcal{F}} L^\infty(E)$ be the direct sum of the family $\{L^\infty(E) : E \in \mathcal{F}\}$. The vector spaces L and D are placed in duality by the bilinear form $(f, g) \mapsto \sum_E \langle f_E, g_E\rangle$ on $L \times D$, where $f = (f_E) \in L, g = (g_E) \in D$ and the sum is taken over at most a finite number of non-zero terms of g. Then, the topological dual of $\mathcal{L}$ is D and the topological dual of $L^1_{loc}(\Omega)$ is the quotient space $D/(L^1_{loc}(\Omega))^\circ$, which is algebraically isomorphic to the vector subspace $L^\infty_{loc}(\Omega)$ of $L^\infty(\Omega)$ consisting of measurable, essentially-bounded functions f defined on Ω for which $\mu\{w \in \Omega : f(w) \neq 0\} < \infty$.

Proposition 1. *$L^1_{loc}(\Omega)$ is a complete locally convex space. The topological dual of $L^1_{loc}(\Omega)$ is algebraically isomorphic to $L^\infty_{loc}(\Omega)$.*

We note that $L^1_{loc}(\Omega)$ is identified as the reduced projective limit $\varprojlim i_{EF} L^1(F)$ of the family $\{(L^1(E), \|\cdot\|_{E,1}) : E \in \mathcal{F}\}$ with respect to the mappings i_{EF} ($E, F \in \mathcal{F}$ and $E \leq F$). If $\mathcal{D} = \bigoplus_{E\in\mathcal{F}} L^\infty(E)$ denotes the locally convex direct sum of the family $\{(L^\infty(E), \tau(L^\infty(E), L^1(E))) : E \in \mathcal{F}\}$, then the quotient space $\mathcal{D}/(L^1_{loc}(\Omega))^\circ$ is the inductive limit $\varinjlim j_{FE} L^\infty(E)$ of the family $\{(L^\infty(E), \tau(L^\infty(E), L^1(E))) : E \in \mathcal{F}\}$ with respect to the mappings j_{FE} ($E, F \in \mathcal{F}$ and $E \leq F$).

A subset A of $L^1_{loc}(\Omega)$ is said to be locally uniformly integrable if for each $E \in \mathcal{F}$, the set $\{f|_E : f \in A\}$ of the restrictions $f|_E$ of the functions f in A to E is uniformly integrable in $L^1(E)$, that is, for each $E \in \mathcal{F}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that for each $F \in \mathcal{A}$ with $F \subset E$ and $\mu(F) < \delta$, $\sup_{f\in A} \int_F |f|\, d\mu < \epsilon$. It follows from the theorem of Tychonoff that if A is a locally uniformly integrable, bounded subset of $L^1_{loc}(\Omega)$, then A is relatively weakly compact, since $L^1_{loc}(\Omega)$ is a complete subspace of $\mathcal{L}$. The converse holds.

Proposition 2. *A subset C of $L^1_{loc}(\Omega)$ is relatively weakly compact if and only if C is bounded and locally uniformly integrable.*

Remark 1. The arguments discussed so far is applicable for σ-compact topological spaces X. In this case, we choose as $\mathcal{A}$ the σ-algebra of

Borel sets of X and as μ a Borel measure on X such that $\mu(K) < \infty$ for each compact subset K of X. For example, let $X = \mathbb{R}$, let μ be Lebesgue measure on $\mathbb{R}$, let $\mathcal{K}$ be the family of compact subsets of $\mathbb{R}$ and let $L^1_{loc}(\mathbb{R})$ be the space of Borel measurable functions f defined on $\mathbb{R}$ for which $\|f\|_{K,1} = \int_K |f|\, d\mu < \infty$ $(K \in \mathcal{K})$, endowed with the separated locally convex topology generated by the semi-norms $f \mapsto \|f\|_{K,1}$ $(K \in \mathcal{K})$. Then, $L^1_{loc}(\mathbb{R})$ contains the space $C(\mathbb{R})$ of continuous (not necessarily bounded) functions defined on $\mathbb{R}$. If a subset $\mathcal{C}$ of $C(\mathbb{R})$ is uniformly bounded on the compact subsets of $\mathbb{R}$, that is, $\sup_{f\in\mathcal{C}} \sup_{x\in K} |f(x)| < \infty$ $(K \in \mathcal{K})$, then $\mathcal{C}$ is relatively weakly compact in $L^1_{loc}(\mathbb{R})$.

We recall that whenever E is a metrizable locally convex space, then a subset C of E is weakly compact if and only if C is sequentially weakly compact.

Proposition 3. *A subset C of $L^1_{loc}(\Omega)$ is weakly compact if and only if C is sequentially weakly compact.*

3. On weak compactness in a separated locally convex topology on L^1

In this section, $L^1(\Omega)$ shall be considered as a locally convex space under the separated locally convex topology $\tilde{\tau}$ generated by the semi-norms $f \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$, if $L^1(\Omega)$ is not specified explicitly as a Banach space with the norm $f \mapsto \|f\|_1$, and we show a necessary and sufficient condition for a subset of $L^1(\Omega)$ to be relatively weakly compact. It is clear that $\tilde{\tau}$ is the relative topology of τ on $L^1_{loc}(\Omega)$ to $L^1(\Omega)$, since $L^1(\Omega)$ is a subspace of $L^1_{loc}(\Omega)$. The topological dual of $L^1(\Omega)$ is algebraically isomorphic to $L^\infty_{loc}(\Omega)$. The result concerning completeness of $L^1(\Omega)$ follows immediately from the separation theorem.

Proposition 4. *The completion of $(L^1(\Omega), \tilde{\tau})$ is $L^1_{loc}(\Omega)$.*

We showed a sufficient condition for a subset of $L^1(\Omega)$ to be relatively weakly compact to obtain the existence of the mean values for commutative semigroups of Dunford-Schwartz operators on L^1; see [6].

Proposition 5. *Let C be a bounded subset of $L^1(\Omega)$ relative to L^1-norm, that is, $\sup_{f\in C} \|f\|_1 < \infty$. Then, C is relatively weakly compact in $(L^1(\Omega), \tilde{\tau})$ if and only if C is locally uniformly integrable.*

Example 1. Let $\Omega = \mathbb{R}$, let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$, let μ be Lebesgue measure on $\mathbb{R}$, let $\mathcal{F}$ be the family of Lebesgue measurable subsets of $\mathbb{R}$ with finite measure. Let $L^1(\mathbb{R})$ be endowed with the separated locally convex topology $\tilde{\tau}$ generated by the semi-norms $f \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$ and let $f \in L^1(\mathbb{R})$. For each $y \in \mathbb{R}$, f_y is the translate of f, that is, $f_y(x) = f(x-y)$ $(x \in \mathbb{R})$. Then, $\{f_y : y \in \mathbb{R}\}$ is relatively weakly compact in $(L^1(\mathbb{R}), \tilde{\tau})$. For example, for $f(x) = e^{-|x|}$ ($x \in \mathbb{R}$), $\{f_y : y \in \mathbb{R}\}$ is not relatively weakly compact in $(L^1(\mathbb{R}), \|\cdot\|_1)$, but relatively weak compact in $(L^1(\mathbb{R}), \tilde{\tau})$; see also Remark 3.

Example 2. The closed unit ball in l^1 is weakly compact in the topology of pointwise convergence, due to Fatou's lemma.

Example 3. Let $\Omega, \mathcal{A}, \mu, \mathcal{F}$ and $\tilde{\tau}$ be as in Example 1. Let f_n ($n \in \mathbb{N}$) be a characteristic function on $[n, 2n)$. Then, $\{f_n\}$ is not bounded relative to L^1-norm, but it converges to a null function in the topology $\tilde{\tau}$ on $L^1(\mathbb{R})$. Thus, it is relatively compact and hence relatively weakly compact in $(L^1(\mathbb{R}), \tilde{\tau})$.

A subset C of $L^1(\Omega)$ is said to be *locally bounded* if C is a bounded subset of $L^1(\Omega)$, that is, for each $E \in \mathcal{F}$, $\sup_{f\in C} \|f\|_{E,1} < \infty$.

We show a necessary and sufficient condition for a subset of $L^1(\Omega)$ to be relatively weakly compact in $(L^1(\Omega), \tilde{\tau})$.

Theorem 1. *A subset C of $L^1(\Omega)$ is relatively weakly compact in $(L^1(\Omega), \tilde{\tau})$ if and only if C is locally uniformly integrable, locally bounded and for each sequence $\{f_n\}$ in C,*

$$\sup_{E\in\mathcal{F}} \liminf_{n\to\infty} \left| \int_E f_n \, d\mu \right| < \infty.$$

4. On weak compactness in L^1

In the sequel, $L^1(\Omega)$ shall be considered as a Banach space under the norm $f \mapsto \|f\|_1$. From Theorem 1, it is natural to ask a question of under which conditions every locally uniformly integrable, locally bounded subset of $L^1(\Omega)$ is relatively weakly compact in $(L^1(\Omega), \|\cdot\|_1)$.

The following theorem is due to Grothendieck.

Theorem 2. *A subset C of a Banach space E is relatively weakly compact if and only if for each $\epsilon > 0$, there exists a weakly compact subset D of E such that $C \subset D + B(\epsilon)$.*

Motivated by his result, we introduce a notion of the type of uniform integrability to obtain a necessary and sufficient condition for a subset of $L^1(\Omega)$ to be relatively weakly compact. We call a subset C of $L^1(\Omega)$ *uniformly integrable at infinity* if for each $\epsilon > 0$, there exists an $E \in \mathcal{F}$ such that

$$\sup_{f\in C} \int_{\Omega\setminus E} |f| \, d\mu < \epsilon \quad \left(\text{or } \lim_{E\in\mathcal{F}} \sup_{f\in C} \int_{\Omega\setminus E} |f| \, d\mu = 0 \right).$$

Theorem 3. *Let C be a subset of $L^1(\Omega)$. Then, the following are equivalent:*

(1) *C is relatively weakly compact;*

(2) *for each $\epsilon > 0$, there exists an $E \in \mathcal{F}$ such that $C_E = \{f|_E : f \in C\}$ is uniformly integrable, bounded in $L^1(E)$ and $C \subset C_E + B(\epsilon)$;*
(3) *C is locally bounded, locally uniformly integrable and uniformly integrable at infinity;*
(4) *C is bounded, uniformly integrable and uniformly integrable at infinity;*
(5) *$|C| = \{|f| : f \in C\}$ is relatively weakly compact, where $|f|(x) = |f(x)|$ $(x \in \Omega)$;*
(6) *C is bounded and for each decreasing sequence $\{E_n\}$ in $\mathcal{A}$ with empty intersection, $\int_{E_n} f\,d\mu$ converges to 0 uniformly in $f \in C$;*
(7) *C is bounded and there exists an $f \in L^1(\Omega)$ such that for each $\epsilon > 0$, there exists a $\delta > 0$ such that for each $E \in \mathcal{A}$ with $\int_E |f|\,d\mu < \delta$, $\sup_{g \in C} |\int_E g\,d\mu| < \epsilon$.*

Remark 2. The equivalence (1) $\Leftrightarrow$ (6) is due to Dunford and Pettis, according to [3] and (1) $\Leftrightarrow$ (7) is obtained as in Bartle, Dunford and Schwartz [1]. The latter implies that Theorem 3, Theorem 4 and Corollary 1 hold without the assumption that μ is σ-finite, since every function in a weakly compact subset of $L^1(\Omega)$ vanishes on the complement of a σ-finite set.

Remark 3. Let $\Omega, \mathcal{A}$ and μ be as in Example 1. Then, the subset of $L^1(\mathbb{R})$ consisting of the translates of $f(x) = e^{-|x|}$ $(x \in \mathbb{R})$ is not uniformly integrable at infinity and hence is not weakly compact in $L^1(\mathbb{R})$.

Corollary 1. *Every order interval in $L^1(\Omega)$ is weakly compact, where an order interval is a subset of the form $\{h \in L^1(\Omega) : f(x) \le h(x) \le g(x)$ almost everywhere on $\Omega\}$ $(f, g \in L^1(\Omega))$.*

The following theorem is due to Theorem 3 and the convergence theorem of Vitali.

Theorem 4. *Let C be a weakly compact subset of $L^1(\Omega)$, let $\{f_n\}$ be a sequence in C and let $f \in C$. If $f_n(x)$ converges to $f(x)$ almost everywhere on Ω, then $\|f_n - f\|_1 \to 0$ as $n \to \infty$.*

Corollary 2. *Every weakly convergent sequence in l^1 is strongly convergent.*

5. On strong compactness in L^1

Note that in the sequel, $L^1(\Omega)$ shall be considered as a Banach space under the norm $f \mapsto \|f\|_1$. As in the arguments in the previous sections, $L^\infty(\Omega)$ is considered as a locally convex space under the separated locally convex topology $\hat{\tau}$ generated by the semi-norms $f \mapsto \|f\|_{E,1}$ $(E \in \mathcal{F})$. It is clear that $\hat{\tau}$ is the relative topology of τ on $L^1_{loc}(\Omega)$ to $L^\infty(\Omega)$, since $L^\infty(\Omega)$ is a subspace of $L^1_{loc}(\Omega)$. The topological dual of $(L^\infty(\Omega), \hat{\tau})$ is algebraically isomorphic to $L^\infty_{loc}(\Omega)$. The result concerning completeness of $(L^\infty(\Omega), \hat{\tau})$ also follows immediately from the separation theorem.

Proposition 6. *The completion of $(L^\infty(\Omega), \hat{\tau})$ is $L^1_{loc}(\Omega)$.*

The weak topology $\sigma(L^\infty(\Omega), L^1(\Omega))$, simply denoted by $\sigma(L^\infty, L^1)$, is finer than the weak topology of $L^\infty(\Omega)$ associated with $\hat{\tau}$, from which we directly deduce the following result concerning sequential compactness in $\sigma(L^\infty, L^1)$ on $L^\infty(\Omega)$.

Proposition 7. *The closed unit ball in $L^\infty(\Omega)$ is sequentially compact relative to the weak topology $\sigma(L^\infty, L^1)$.*

Remark 4. If E is a reflexive or smooth Banach space, then every closed unit ball in E' is sequentially compact relative to the weak topology $\sigma(E', E)$; see [2] for more details.

A subset C of a Banach space E is said to be *limited* if for each sequence $\{x'_n\}$ in E' converging to 0 in the weak topology $\sigma(E', E)$, $\lim_{n\to\infty} |\langle x, x'_n\rangle|$ converges to 0 uniformly in $x \in C$.

Using similar arguments to [9], we obtain a characterization of strong compactness in Banach spaces E for which the closed unit ball in E' is sequentially compact relative to the weak topology $\sigma(E', E)$.

Proposition 8. *Let E be a Banach space. Whenever the closed unit ball in E' is sequentially compact relative to the weak topology $\sigma(E', E)$, a subset C of E is relatively compact if and only if C is bounded and limited.*

Remark 5. According to [2], Proposition 8 is due to Gelfand.

Corollary 3. *Whenever E is a reflexive or smooth Banach space, a subset C of E is relatively compact if and only if C is bounded and limited.*

The following theorem is due to Proposition 7 and Proposition 8.

Theorem 5. *A subset C of $L^1(\Omega)$ is relatively compact if and only if C is bounded and limited.*

Remark 6. It is clear that Theorem 5 holds without the assumption that μ is σ-finite.

6. Miscellaneous Applications

In this section, we apply the results about weak and strong compactness in $L^1(\Omega)$ to obtain some characterizations of (weak) almost periodicity for linear and non-linear operators in $L^1(\Omega)$.

Let T be a linear contraction on $L^1(\Omega)$, that is, T is a linear operator on $L^1(\Omega)$ such that $\|Tf\|_1 \leq \|f\|_1$ $(f \in L^1(\Omega))$. In addition, if $\|Tf\|_\infty \leq \|f\|_\infty$ $(f \in L^1(\Omega) \cap L^\infty(\Omega))$, then T is said to be a

Dunford-Schwartz operator on $L^1(\Omega)$. If for each $f \in L^1(\Omega)$, the orbit $\{T^n f : n = 0, 1, 2, \cdots\}$ of f under T is relatively (weakly) compact, then T is said to be (weakly) almost periodic.

Proposition 9. *Let T be a Dunford-Schwartz operator on $L^1(\Omega)$. Then, T is weakly almost periodic if and only if for each $f \in L^1(\Omega)$, the orbit of f under T is uniformly integrable at infinity.*

Proposition 10. *Let T be a linear contraction on $L^1(\Omega)$. Then, T is almost periodic if and only if for each $f \in L^1(\Omega)$, the orbit of f under T is limited.*

Let C be a closed convex subset of $L^1(\Omega)$ and let T be a nonexpansive operator on C, that is, T is a mapping of C into itself such that $\|Tf - Tg\|_1 \leq \|f - g\|_1$ $(f, g \in C)$. Then, T is said to be *almost periodic* if for each $f \in C$, the orbit $\{T^n f : n = 0, 1, 2, \cdots\}$ of f under T is relatively compact. It is known that if a nonexpansive operator T on C is almost periodic, then T has the mean values on C; see also [4] for more details.

Proposition 11. *Let C be a closed convex subset of $L^1(\Omega)$ and let T be a nonexpansive operator on C. Whenever T has a fixed point in C, T is almost periodic if and only if for each $f \in C$, the orbit of f under T is limited.*

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