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GRAPHS WITH THREE EIGENVALUES

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1. INTRODUCTION

This note is based on joint work with Xi-Ming Cheng, Alexander Gavrilyuk, and Jack Koolen. We refer the interested reader to our paper [4].

Our objects of study are graphs having precisely three distinct eigenvalues. For graphs with few eigenvalues, the case of three eigenvalues is the first interesting case. Indeed, let Γ be a connected graph on n vertices and let A denote its adjacency matrix. Since the trace of A is zero, if Γ has just one eigenvalue θ_0 then it follows that $\theta_0=0$. Now, if Γ has just two eigenvalues $\theta_0>\theta_1$ then since it is connected, it must have spectrum $\{[\theta_0]^{(1)}, [\theta_1]^{(n-1)}\}$. Since the trace of A is zero and the trace of A^2 is $2e$ where e is the number of edges of Γ, we have the equation

$$n(n-1)\theta_1^2=2e.$$

By interlacing (with an edge), we have $\theta_1 \leqslant -1$ and hence, $\theta_1^2 \geqslant 1$. On the other hand the maximum number of edges in an n vertex graph is $\binom{n}{2}$ and hence $2e \leqslant n(n-1)$. Thus we deduce that Γ is the complete graph.

Regular graphs with three eigenvalues have received a great deal of attention. It is well-known that such graphs are strongly regular graphs. A systematic study of their nonregular counterparts was initiated by Haemers at the BCC in 1995. There he asked for examples of nonregular graphs having precisely three distinct eigenvalues. At that time complete bipartite graphs and a switching of $T(9)$ were the only known examples.

In 1998, papers by Muzychuk and Klin [9] and Van Dam [7] appeared answering Haemers' questions and, in particular, providing many examples of nonregular graphs with three distinct eigenvalues. Most of the currently known examples are so-called **biregular** graphs; graphs that have precisely two distinct valencies. Muzychuk and Klin [9] called such graphs 'strongly biregular graphs'.

An n-vertex graph with a vertex of valency $n-1$ is called a **cone**. Given a graph Γ, the **cone over** Γ is the graph formed by adjoining a vertex adjacent to every vertex of Γ. Examples of families of strongly biregular graphs are complete bipartite graphs and cones over strongly regular graphs. Indeed, a complete bipartite graph $K_{n,m}$ (for $n \geqslant m \geqslant 1$) has spectrum $\{[\sqrt{nm}]^{(1)}, [0]^{(n+m-2)}, [-\sqrt{nm}]^{(1)}\}$. The following result due to Muzychuk and Klin offers a method for finding strongly biregular cones.

Proposition 1.1 (See [9]). *Let Γ be a strongly regular graph with n vertices, valency k, and smallest eigenvalue θ_2. Then the cone over Γ has three distinct eigenvalues if and only if $\theta_2(k-\theta_2)=-n$.*

There are infinitely many strongly regular graphs satisfying the assumption of the proposition and so there are infinitely many cones over strongly regular graphs having three distinct eigenvalues [1, 9]. As well as giving some sporadic examples, using symmetric and affine designs, Van Dam [7] exhibited a couple of infinite families of strongly biregular graphs that are neither cones nor complete bipartite graphs.

There exist some partial classifications of nonregular graphs having three distinct eigenvalues in the following senses. Van Dam [7] classified all such graphs having smallest eigenvalues at least -2 and also classified all such graphs on at most 29 vertices. Chuang and Omidi [5] classified those graphs whose spectral radius is less than 8.

2. Cones and bipartite graphs

For fixed $\theta_0 > \theta_1 > \theta_2$, define the set $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ of connected nonregular graphs having precisely three distinct eigenvalues θ_0, θ_1, and θ_2.

Among graphs with three eigenvalues, complete bipartite graphs are distinguished in the following way.

Theorem 2.1 (Proposition 2 [7]). *Let Γ be a graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ where θ_0 is not an integer. Then Γ is a complete bipartite graph.*

It was shown by Smith [12] that, if the second largest eigenvalue of a connected graph Γ is at most 0 then Γ is a complete r-partite graph with parts of size $p_1, \dots, p_r$, denoted $K_{p_1,\dots,p_r}$. We will see below that complete bipartite graphs are the only nonregular multipartite graphs with precisely three distinct eigenvalues.

Theorem 2.2. *Let Γ be a graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$. If the complement of Γ is disconnected, then Γ is a cone or Γ is complete bipartite.*

Note that if a bipartite graph is not complete bipartite then its diameter must be at least 3 and hence it cannot have fewer than 4 distinct eigenvalues. The next result follows from this observation and from the proof of Theorem 2.2, which we omit.

Corollary 2.3. *Let Γ be a graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$. Then the following are equivalent.*

(1) Γ is bipartite;
(2) Γ is complete bipartite;
(3) $\theta_1 = 0$.

For a graph Γ, let $D(\Gamma)$ denote the number of distinct valencies in Γ. For example, if Γ is regular then $D(\Gamma) = 1$. We also have the following corollary.

Corollary 2.4 (cf. [7]). *Let Γ be a cone in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$. Then $D(\Gamma) \leqslant 3$. Moreover, if $D(\Gamma) = 2$ then Γ is a cone over a strongly regular graph.*

Now we give two remarks.

Remark 2.5. The above corollary generalises a result of Bridges and Mena [1] who studied cones having distinct eigenvalues θ_0, θ_1, and $-\theta_1$. They proved that, except for at most three cones having three valencies, such graphs are cones over strongly regular graphs with parameters (v, k, λ, λ). (Only two of these three exceptional cones have been constructed, it is still an open problem to decide the existence of the largest cone.)

Remark 2.6. Our Theorem 2.2 is reminiscent of Proposition 6.1 (a) given by Muzychuk and Klin [9]. Let $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ and let $W(\Gamma)$ denote its *Weisfeiler-Leman closure* (see [9, Section 6]). We also remark that, by Theorem 2.2 and Corollary 2.4, we see that [9, Proposition 6.1 (a)] says that if $\dim(W(\Gamma)) = 6$ then Γ is biregular with a disconnected complement. Muzychuk and Klin [9] suggest classifying all graphs $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ satisfying $\dim(W(\Gamma)) = 9$, which is the next interesting case after $\dim(W(\Gamma)) = 6$.

We call the 11-vertex cone over the Petersen graph, the **Petersen cone** (see [7, Fig. 1]) and the graph derived from the complement of the Fano plane, the **Fano graph** (see [7, Fig. 2]).

3. Biregular graphs with three eigenvalues

3.1. **Computing feasible parameters.** Let $\Gamma = (V, E)$ be an n-vertex connected graph. Recall that the adjacency matrix A of Γ is an $n \times n$ matrix whose (i, j)th entry, $A_{i,j}$, is 1 if the ith vertex of Γ is adjacent to the jth vertex of Γ and 0 otherwise. By the eigenvalues of Γ we mean the eigenvalues of A. Assume that Γ has precisely three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then Γ has diameter two and since such a graph cannot be complete, it follows by interlacing that $\theta_1 \geqslant 0$ and $\theta_2 \leqslant -\sqrt{2}$.

We write m_i for the multiplicity of eigenvalue θ_i of Γ. If Γ has n vertices then, since $1 + m_1 + m_2 = n$ and $\theta_0 + m_1\theta_1 + m_2\theta_2 = 0$, we have

$$(1) \qquad m_1 = -\frac{(n-1)\theta_2 + \theta_0}{\theta_1 - \theta_2} \quad \text{and} \quad m_2 = \frac{(n-1)\theta_1 + \theta_0}{\theta_1 - \theta_2}.$$

By the Perron-Frobenius Theorem (see, for example, [8]), θ_0 has multiplicity 1 and for any eigenvector for θ_0 all entries have the same sign. This implies that there exists a positive eigenvector α for the eigenvalue θ_0 such that

$$(A - \theta_1 I)(A - \theta_2 I) = \alpha\alpha^t.$$

For a vertex x, denote the entry of α corresponding to x by α_x. This implies that if a vertex x has valency d_x, then $d_x = \alpha_x^2 - \theta_1\theta_2$. Let x and y be vertices of Γ. We write ν_{xy} for the number of common neighbours of x and y. By the above formulae we have

$$\nu_{xy} = (\theta_1 + \theta_2)A_{x,y} + \alpha_x\alpha_y.$$

Let Γ be a graph having r distinct valencies $k_1, \ldots, k_r$ with multiplicities $n_1, \ldots, n_r$, i.e., $n_i := |\{v \in V(\Gamma) : d_v = k_i\}|$. We refer to the valencies k_i and their multiplicities n_i as the **parameters** (or **parameter set**) of Γ. The main result of this section gives us strong restrictions on the parameters of biregular graphs with three eigenvalues.

Van Dam [7] showed that if a graph Γ has precisely three distinct eigenvalues and at most three distinct valencies then the valency partition is equitable. We show a slightly refined version of this result where we assume that Γ has precisely two distinct valencies.

Theorem 3.1. *Let Γ be an n-vertex non-bipartite biregular graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ with valencies $k_1 > k_2$. Then the following conditions hold:*

(i) *The partition* $\{V_1, V_2\}$ *is an equitable partition of* Γ *with quotient matrix* $Q = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, *where*

$$k_{11} = \frac{\alpha_1\theta_0 - \alpha_2 k_1}{\alpha_1 - \alpha_2},\ k_{12} = \alpha_1\frac{k_1 - \theta_0}{\alpha_1 - \alpha_2},\ k_{21} = \alpha_2\frac{\theta_0 - k_2}{\alpha_1 - \alpha_2},\ k_{22} = \frac{\alpha_1 k_2 - \alpha_2\theta_0}{\alpha_1 - \alpha_2}.$$

(ii) *All eigenvalues of* Γ *are integers.*

(iii) *If the matrix* Q *has eigenvalues* θ_0 *and* θ, *then* $\alpha_1\alpha_2 = -\theta(\theta' + 1)$ *where* $\{\theta, \theta'\} = \{\theta_1, \theta_2\}$. *In particular, if* $k_{11} = 0$ *or* $k_{22} = 0$ *then* $\alpha_1\alpha_2 = -\theta_2(\theta_1 + 1)$.

(iv) *We have*

$$n = \frac{(\alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2 - \theta_0 - \theta_1\theta_2)(\theta_0 - \theta_1)(\theta_0 - \theta_2)}{(\theta_0 + \alpha_1\alpha_2 + \theta_1\theta_2)\alpha_1\alpha_2};$$

$$n_1 = \frac{(\theta_0 - \alpha_2^2 + \theta_1\theta_2)(\theta_0 - \theta_1)(\theta_0 - \theta_2)}{(\theta_0 + \alpha_1\alpha_2 + \theta_1\theta_2)(\alpha_1 - \alpha_2)\alpha_1};$$

$$n_2 = \frac{(\alpha_1^2 - \theta_0 - \theta_1\theta_2)(\theta_0 - \theta_1)(\theta_0 - \theta_2)}{(\theta_0 + \alpha_1\alpha_2 + \theta_1\theta_2)(\alpha_1 - \alpha_2)\alpha_2}.$$

(v) *The following conditions are equivalent:*
  (a) $k_{21} = n_1$;
  (b) $k_{12} = n_2$;
  (c) $n_1 = 1$;
  (d) Γ *is a cone over a strongly regular graph.*

(vi) *If* n *is a prime at least* 3, *then* Γ *is a cone over a strongly regular graph.*

(vii) $\alpha_1 - 1 \leqslant (\alpha_1 - \alpha_2)\alpha_2 \leqslant \min\{-(\theta_1 + 1)(\theta_2 + 1), -\theta_1\theta_2\}$, *unless* Γ *is a cone over a strongly regular graph.*

Using Theorem 3.1 we have compiled a table of the feasible parameters for biregular graphs with precisely three distinct eigenvalues, see the appendix of [4].

Remark 3.2. Let Γ be a biregular graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ with the spectrum of Γ fixed. Using Theorem 3.1 one can see that Γ can have at most two possible parameter sets. The problem of determining if the parameters of Γ are determined by its spectrum comes down to Diophantine analysis. So far we do not have any examples of a pair of parameter sets corresponding to graphs with the same spectrum.

We can show that if both a graph and its complement have three eigenvalues then it must be biregular.

Theorem 3.3. *Let* Γ *be an* n*-vertex graph in* $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ *such that its complement* $\overline{\Gamma}$ *is in* $\mathcal{G}(\theta_0', \theta_1', \theta_2')$. *Then* Γ *is biregular with valencies* k_1 *and* k_2 *satisfying*

$$k_1, k_2 = \frac{n + \theta_1 + \theta_2 \pm \sqrt{(n + \theta_1 + \theta_2 + 2\theta_1\theta_2)^2 - 4\theta_2^2(\theta_1 + 1)^2}}{2}.$$

Moreover,

$$n = \frac{(\theta_0 - \theta_1)^2}{\theta_0 - \theta_1 + \theta_1\theta_2 + \theta_2}; \qquad \theta_0 = \frac{n}{2} + \theta_1 \pm \sqrt{n(n + 4\theta_2(\theta_1 + 1))}/2;$$

$$\theta_0' = n - 1 - \theta_0 + \theta_1 - \theta_2, \qquad \theta_1' = -1 - \theta_2, \text{ and } \theta_2' = -1 - \theta_1.$$

But so far we do not have any examples of graphs that satisfy this theorem.

3.2. **Bounding the second largest eigenvalue.** Neumaier [10] showed that for a fixed m, all but finitely many primitive strongly regular graphs with smallest eigenvalue at least $-m$ fall into two infinite families.

Theorem 3.4 (See [10]). *Let $m \geqslant 1$ be a fixed integer. Then there exists a constant $C(m)$ such that any connected and coconnected strongly regular graph Γ with smallest eigenvalue $-m$ having more than $C(m)$ vertices has the following parameters (given in the form $\mathrm{srg}(n,k,\lambda,\mu)$).*

(1) $\mathrm{srg}(n, sm, s-1+(m-1)^2, m^2)$ *where $s \in \mathbb{N}$ and $n = (s+1)(s(m-1)+m)/m$; or*

(2) $\mathrm{srg}((s+1)^2, sm, s-1+(m-2)(m-1), m(m-1))$ *where $s \in \mathbb{N}$.*

In the next result we show that for fixed $\theta \neq 0$ there are only finitely many cones over strongly regular graphs with exactly three distinct eigenvalues and one of them equal to θ.

Lemma 3.5. *Let $\theta \neq 0$ be a fixed algebraic integer and let $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ be a cone over a strongly regular graph with $\theta \in \{\theta_0, \theta_1, \theta_2\}$. Then Γ is one of a finite number of graphs.*

Our main result of this section is that there are only finitely many biregular connected graphs with three distinct eigenvalues and bounded second largest eigenvalue.

Theorem 3.6. *Let Γ be an n-vertex biregular graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ and let t be a positive integer. Then there exists a constant $C(t)$ such that if $0 < \theta_1 \leqslant t$, then $n \leqslant C(t)$.*

Note that the above result is not true for connected graphs with exactly 4 distinct eigenvalues and exactly two distinct valencies. Indeed, the friendship graphs, i.e., cones over a disjoint union of copies of K_2, can have unbounded number of vertices and all but two of the eigenvalues are equal to ± 1.

3.3. **Second largest eigenvalue 1.** In this section we will classify the connected biregular graphs with three distinct eigenvalues and second largest eigenvalue 1. First we classify the cones of strongly regular graphs with second largest eigenvalue 1. Seidel [11] (see also [2, Thm 3.12.4 (i)]) classified the strongly regular graphs with smallest eigenvalue -2.

Theorem 3.7 ([11]). *Let Γ be a connected strongly regular graph with smallest eigenvalue -2. Then Γ is either a triangular graph $T(m)$ for $m \geqslant 5$; an $(m \times m)$-grid for $m \geqslant 3$; the Petersen graph; the Shrikhande graph; the Clebsch graph; the Schläfli graph; or one of the three Chang graphs.*

Lemma 3.8. *Let Γ be a cone over a strongly regular graph in $\mathcal{G}(\theta_0, 1, \theta_2)$. Then Γ is the Petersen cone.*

Proof. The strongly regular graphs with second largest eigenvalue 1 are exactly the complements of the graphs in Theorem 3.7. Checking whether each graph satisfies the condition of Proposition 1.1 gives the lemma. □

More generally we have the following.

Proposition 3.9. *Let Γ be a biregular graph in $\mathcal{G}(\theta_0, 1, -t)$. Then $t = 2$, and Γ is the Petersen cone or Fano graph.*

3.4. **Graphs in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ with $\theta_1 + \theta_2 = -1$.** In this section we examine graphs in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ where $\theta_1 + \theta_2 = -1$. We find that such graphs have properties which make it convenient to use the so-called 'star complement method' to attempt to construct them. Compare Eq. (2) to the equation in Theorem 3.12. We denote by $\mathbf{j}$ the 'all ones' (column) vector and we define the matrix $J := \mathbf{j}\mathbf{j}^\top$.

Now we give a structural result for graphs with a certain spectrum.

Lemma 3.10. *Let Γ be a connected graph with spectrum $\{[\theta_0]^1, [\theta]^n, [-\theta-1]^{n-1}\}$, so that its adjacency matrix A under the valency partition has block form:*

$$A = \begin{pmatrix} A_1 & B^\top \\ B & A_2 \end{pmatrix} \text{ with quotient matrix } \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

where A_1 and A_2 are both $n \times n$ matrices, i.e., $n = n_1 = n_2$, where n_i is the number of vertices with valency k_i.

Then the matrices A_1 and $J - I - A_2$ are cospectral.

We can also show the converse.

Lemma 3.11. *Let Γ be a graph in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$, so that its adjacency matrix A under the valency partition has block form:*

$$A = \begin{pmatrix} A_1 & B^\top \\ B & A_2 \end{pmatrix} \text{ with quotient matrix } \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

where A_1 and $J - I - A_2$ are cospectral. Assume $n = n_1 = n_2$, where n_i is the number of vertices with valency k_i.

Then $k_{12} = k_{21}$ and Γ has spectrum $\{[\theta_0]^1, [\theta]^n, [-\theta-1]^{n-1}\}$, where $\theta \in \{\theta_1, \theta_2\}$ is an eigenvalue of the quotient matrix of the valency partition of Γ. Moreover, $BA_1 = (J - I - A_2)B$, and

(2) $B^\top B = (\theta I - A_1)(\theta I - (J - I - A_1))$, *and* $BB^\top = (\theta I - A_2)(\theta I - (J - I - A_2))$.

In Table 1 in [4] we observe that, apart from the Petersen cone, all feasible parameter sets for graphs in $\mathcal{G}(\theta_0, \theta_1, \theta_2)$ with $\theta_1 + \theta_2 = -1$ have $n_1 = n_2$. It is an interesting problem to decide whether this property follows from the spectrum in general except from the Petersen cone.

Let θ be an eigenvalue of an n-vertex graph Γ and suppose that θ has multiplicity m. Define a **star set** for θ to be a subset $X \subset V(\Gamma)$ such that $|X| = m$ and θ is not an eigenvalue of $\Gamma - X$. Now we can state the *Reconstruction Theorem* (See [6, Theorems 7.4.1 and 7.4.4]).

Theorem 3.12. *Let X be a subset of vertices of a graph Γ and suppose that Γ has adjacency matrix*

$$\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for θ if and only if θ is not an eigenvalue of C and $\theta I - A_X = B^\top(\theta I - C)^{-1}B$.

The graph Λ induced by $\Gamma - X$ (having adjacency matrix C in Theorem 3.12) is called the **star complement** of θ. Star sets and star complements exist for any eigenvalue and any graph and moreover, for $\theta \notin \{0, 1\}$, it can be shown that Λ-neighbourhoods of the vertices of X are non-empty and distinct [6, Chapter 7]. For vectors in $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^{n-m}$, define the bilinear map $\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^\top(\theta I - C)^{-1}\mathbf{w}$. Let V be the set of vectors $\mathbf{v} \in \{0, 1\}^{n-m}$ satisfying $\langle \mathbf{v}, \mathbf{v} \rangle = \theta$. Form a graph having V

as its vertex set where two vectors $\mathbf{v}$ and $\mathbf{w}$ are adjacent if $\langle \mathbf{v}, \mathbf{w} \rangle \in \{0, -1\}$. This graph is known as the **compatibility graph** for Λ. Cliques in the compatibility graph then give the columns of the matrix B as in Theorem 3.12.

We use this technique to obtain the following theorem.

Theorem 3.13. *There exist at least* 21 *graphs having parameters* $n = 30$; $n_1 = 15$; $n_2 = 15$; $k_1 = 14$; $k_2 = 8$; *spectrum* $\{[12]^1, [2]^{15}, [-3]^{14}\}$.

We remark that, by combining the above theorem with Van Dam's classification [7, Section 7], we now have a complete understanding of graphs with three distinct eigenvalues on at most 30 vertices.

A graph in $\mathcal{G}(20, 2, -3)$ having 3 distinct valencies was constructed in [3]. We show that (see Theorem 3.14) there exists precisely one more graph having the same parameters.

Theorem 3.14. *The graphs* Γ_1 *and* Γ_2 *are the only graphs having parameters* $n = 36$; $k_1 = 24$; $k_2 = 14$; $k_3 = 8$; *spectrum* $\{[20]^1, [2]^{17}, [-3]^{18}\}$.

Using similar techniques, we also obtain the following nonexistence result.

Theorem 3.15. *There do not exist any graphs having parameters* $n = 44$; $n_1 = 22$; $n_2 = 22$; $k_1 = 22$; $k_2 = 7$; *spectrum* $\{[19]^1, [2]^{22}, [-3]^{21}\}$.

4. Open problems

Finally we give some open problems.

(1) Find more examples. Currently we have infinitely many graphs $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ with $D(\Gamma) = 2$, finitely many with $D(\Gamma) = 3$, and no examples with $D(\Gamma) \geqslant 4$.
(2) Does there exist Γ such that $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ and $\overline{\Gamma} \in \mathcal{G}(\theta_0', \theta_1', \theta_2')$?
(3) Except for the Petersen cone, if $\Gamma \in \mathcal{G}(\theta_0, \theta, -\theta + 1)$ is biregular, does it imply that the valency-partition of Γ divides that vertices equally into two sets?
(4) Investigate the graphs $\Gamma \in \mathcal{G}(\theta_0, \theta_1, \theta_2)$ with $\dim(W(\Gamma)) = 9$ (See Remark 2.6).

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