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1次元ランダムシュレーディンガー作用素の準位統計について

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Abstract

This note is based on the works [5], [7]. We study the level statistics of one-dimensional Schrödinger operator with random potential decaying like $x^{-\alpha}$ at infinity. We consider the point process ξ_L consisting of the rescaled eigenvalues and show that : (i)(ac spectrum case) for $\alpha>\frac{1}{2}$, ξ_L converges to a clock process, and the fluctuation of the eigenvalue spacing converges to Gaussian. (ii)(critical case) for $\alpha=\frac{1}{2}$, ξ_L converges to the limit of the β-ensemble.

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1 Introduction

1.1 Background

We consider the following Schrödinger operator

$$H:=-\frac{d^{2}}{dt^{2}}+a(t)F(X_{t}) \quad \text{on } L^{2}(\mathbf{R})$$

where $a\in C^{\infty}$ is real valued, $a(-t)=a(t)$, non-increasing for $t\geq 0$, and satisfies

$$C_{1}t^{-\alpha}\leq a(t)\leq C_{2}t^{-\alpha}$$

for some positive constants C_{1}, C_{2} and $\alpha>0$. F is a real-valued, smooth, and non-constant function on a compact Riemannian manifold M such that

$$\langle F\rangle:=\int_{M}F(x)dx=0.$$

$\{X_{t}\}$ is a Brownian motion on M. Since the potential $a(t)F(X_{t})$ is $-\frac{d^{2}}{dt^{2}}$-compact, we have $\sigma_{ess}(H)=[0, \infty)$. Kotani-Ushiroya[3] proved that the

spectrum of H in $[0,\infty)$ is
(1) for $\alpha < \frac{1}{2}$: pure point with exponentially decaying eigenfunctions,
(2) for $\alpha = \frac{1}{2}$: pure point on $[0, E_c]$ and purely singular continuous on $[E_c, \infty)$ with some explicitly computable E_c,
(3) for $\alpha > \frac{1}{2}$: purely absolutely continuous.

In what follows we discuss the level statistics of this operator. For that purpose, let $H_L := H|_{[0,L]}$ be the local Hamiltonian with Dirichlet boundary condition and let $\{E_n(L)\}_{n=1}^{\infty}$ be its eigenvalues in the increasing order. Let $n(L) \in \mathbf{N}$ be s.t. $\{E_n(L)\}_{n \geq n(L)}$ coincides with the set of positive eigenvalues of H_L. We arbitrary take the reference energy $E_0 > 0$ and consider the following point process

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}$$

in order to study the local fluctuation of eigenvalues near E_0. Our aim is to identify the limit of ξ_L as $L \to \infty$ [1].

As for the related works, Killip-Stoiciu [2] studied the CMV matrices whose matrix elements decay like $n^{-\alpha}$. They showed that, ξ_L converges to (i) $\alpha > \frac{1}{2}$: the clock process, (ii) $\alpha = \frac{1}{2}$: the limit of the circular β-ensemble, (iii) $0 < \alpha < \frac{1}{2}$: the Poisson process. Krichevski-Valko-Virag[6] studied the one-dimensional discrete Schrödinger operator with the random potential decaying like $n^{-1/2}$, and proved that ξ_L converges to the Sine$_\beta$-process.

Our aim is to do the analogue of their works for the one-dimensional Schrödinger operator in the continuum.

In subsection 1.2 (resp. subsection 1.3), we state our results for ac-case : $\alpha > \frac{1}{2}$ (resp. critical-case : $\alpha = \frac{1}{2}$)[2].

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[1] Here we consider the scaling of $\sqrt{E_n(L)}$'s instead of $E_n(L)$'s. This corresponds to the unfolding with respect to the density of states.

[2] We have not obtained results for pp-case : $\alpha < \frac{1}{2}$.

1.2 AC-case

Definition 1.1 *Let μ be a probability measure on $[0, \pi)$. We say that ξ is the clock process with spacing π with respect to μ if and only if*

$$\mathbf{E}[e^{-\xi(f)}] = \int_0^\pi d\mu(\phi) \exp\left(-\sum_{n \in \mathbf{Z}} f(n\pi - \phi) \right)$$

where $f \in C_c(\mathbf{R})$ and $\xi(f) := \int_{\mathbf{R}} f d\xi$.

We set

$$(x)_{\pi \mathbf{Z}} := x - [x]_{\pi \mathbf{Z}}, \quad [x]_{\pi \mathbf{Z}} := \max\{y \in \pi \mathbf{Z} \,|\, y \leq x\}.$$

We study the limit of ξ_L under the following assumption

(A)
(1) $\alpha > \frac{1}{2}$,
(2) A sequence $\{L_j\}_{j=1}^\infty$ satisfies $\lim_{j \to \infty} L_j = \infty$ and

$$(\sqrt{E_0} L_j)_{\pi \mathbf{Z}} = \beta + o(1), \quad j \to \infty$$

for some $\beta \in [0, \pi)$.

The condition A(2) ensures that ξ_L converges to a point process. If $a \equiv 0$ for instance, A(2) is indeed necessary.

Theorem 1.1 *Assume (A). Then ξ_{L_j} converges in distribution to the clock process with spacing π with respect to a probability measure μ_β on $[0, \pi)$.*

Remark 1.1 *Let x_t be the solution to the eigenvalue equation : $H_L x_t = \kappa^2 x_t$ ($\kappa > 0$). If we set*

$$\begin{pmatrix} x_t \\ x_t'/\kappa \end{pmatrix} = \begin{pmatrix} r_t \sin \theta_t \\ r_t \cos \theta_t \end{pmatrix}, \quad \theta_t(\kappa) = \kappa t + \tilde{\theta}_t(\kappa),$$

then $\tilde{\theta}_t(\kappa)$ has a limit as t goes to infinity[3] : $\lim_{t \to \infty} \tilde{\theta}_t(\kappa) = \tilde{\theta}_\infty(\kappa)$, a.s. ; μ_β is the distribution of the random variable $(\beta + \tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbf{Z}}$. In some special cases, we can show that $(\tilde{\theta}_\infty(\sqrt{E_0}))_{\pi \mathbf{Z}}$ is not uniformly distributed for large E_0, implying that μ_β really depends on β.

Remark 1.2 *We can consider point processes with respect to two reference energies $E_0, E_0'(E_0 \neq E_0')$ simultaneously : suppose a sequence $\{L_j\}_{j=1}^{\infty}$ satisfies*

$$(\sqrt{E_0}L_j)_{\pi\mathbf{Z}} = \beta + o(1), \quad (\sqrt{E_0'}L_j)_{\pi\mathbf{Z}} = \beta' + o(1), \quad j \to \infty$$

for some $\beta, \beta' \in [0, \pi)$. We set

$$\xi_L := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0})}, \quad \xi_L' := \sum_{n \geq n(L)} \delta_{L(\sqrt{E_n(L)} - \sqrt{E_0'})}.$$

Then the joint distribution of ξ_{L_j}, ξ_{L_j}' converges, for $f, g \in C_c(\mathbf{R})$,

$$\lim_{j \to \infty} \mathbf{E}\left[\exp\left(-\xi_{L_j}(f) - \xi_{L_j}(g)\right)\right]$$
$$= \int_0^{\pi} d\mu(\phi, \phi') \exp\left(-\sum_{n \in \mathbf{Z}} (f(n\pi - \phi) + g(n\pi - \phi'))\right)$$

where $\mu(\phi, \phi')$ is the joint distribution of $(\beta + \tilde{\theta}_{\infty}(\sqrt{E_0}))_{\pi\mathbf{Z}}$ and $(\beta' + \tilde{\theta}_{\infty}(\sqrt{E_0'}))_{\pi\mathbf{Z}}$. We are unable to identify $\mu(\phi, \phi')$ but it may be possible that ϕ and ϕ' are correlated.

Remark 1.3 *Suppose we rearrange eigenvalues near the reference energy E_0 so that*

$$\cdots < E_{-2}'(L) < E_{-1}'(L) < E_0 \leq E_0'(L) < E_1'(L) < E_2'(L) < \cdots.$$

Then an argument similar to the proof of Theorem 2.4 in [4] proves the following fact : for any $n \in \mathbf{Z}$ we have

$$\lim_{L \to \infty} L(\sqrt{E_{n+1}'(L)} - \sqrt{E_n'(L)}) = \pi, \quad a.s. \tag{1.1}$$

which is called the strong clock behavior [1]. We note that the integrated density of states is equal to $\sqrt{E}/\pi$.

We next study the finer structure of the eigenvalue spacing, under the following assumption.

(B)

(1) $\frac{1}{2} < \alpha < 1$,
(2) A sequence $\{L_j\}_{j=1}^{\infty}$ satisfies $\lim_{j\to\infty} L_j = \infty$ and

$$\sqrt{E_0}L_j = m_j\pi + \beta + \epsilon_j, \quad j \to \infty$$

for some $\{m_j\}_{j=1}^{\infty}(\subset \mathbf{N})$, $\beta \in [0,\pi)$ and $\{\epsilon_j\}_{j=1}^{\infty}$ with $\lim_{j\to\infty} \epsilon_j = 0$.
(3) $a(t) = t^{-\alpha}(1 + o(1))$, $t \to \infty$.

Roughly speaking, $E_{m_j}(L_j)$ is the eigenvalue closest to E_0. In view of (1.1), we set

$$X_j(n) := \left\{ \left(\sqrt{E_{m_j+n+1}(L_j)} - \sqrt{E_{m_j+n}(L_j)} \right) L_j - \pi \right\} L_j^{\alpha-\frac{1}{2}}, \quad n \in \mathbf{Z}.$$

Theorem 1.2 *Assume (B). Then $\{X_j(n)\}_{n\in\mathbf{Z}}$ converges in distribution to the Gaussian system with covariance*

$$C(n,n') = \frac{C(E_0)}{8E_0} Re \int_0^1 s^{-2\alpha} e^{2i(n-n')\pi s} 2(1 - \cos 2\pi s) ds, \quad n,\ n' \in \mathbf{Z},$$

where $C(E) := \int_M \left| \nabla (L + 2i\sqrt{E})^{-1} F \right|^2 dx$ and L is the generator of (X_t).

Remark 1.4 *By using the results in [2] we have*

$$\sqrt{E_{m_j}(L_j)} = \sqrt{E_0} - \frac{\beta + \tilde{\theta}_\infty(\sqrt{E_0})}{L_j} + Y_j$$

where $Y_j = O(L_j^{-\alpha-\frac{1}{2}+\epsilon}) + O(\epsilon_j L_j^{-1})$, a.s. for any $\epsilon > 0$. Furthermore by the definition of $\{X_j(n)\}$ we have

$$\sqrt{E_{m_j+n}(L_j)} = \begin{cases} \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} + \frac{1}{L_j^{\alpha+\frac{1}{2}}} \sum_{l=0}^{n-1} X_j(l) & (n \geq 1) \\ \sqrt{E_{m_j}(L_j)} + \frac{n\pi}{L_j} - \frac{1}{L_j^{\alpha+\frac{1}{2}}} \sum_{l=n}^{-1} X_j(l) & (n \leq -1) \end{cases}$$

and Theorem 1.2 thus describes the behavior of eigenvalues near $E_{m_j}(L_j)$ in the second order.

Remark 1.5 *Suppose we consider two reference energies $E_0, E_0'(E_0 \neq E_0')$ simultaneously and suppose a sequence $\{L_j\}_{j=1}^{\infty}$ satisfies $\lim_{j\to\infty} L_j = \infty$ and*

$$\sqrt{E_0}L_j = m_j\pi + \beta + o(1), \quad \sqrt{E_0'}L_j = m_j'\pi + \beta' + o(1), \quad j \to \infty$$

for some $m_j, m_j' \in \mathbf{N}$, and $\beta, \beta' \in [0,\pi)$. Then $\{X_j(n)\}_n$ and $\{X_j'(n)\}_n$ converge jointly to the mutually independent Gaussian systems.

1.3 Critical Case

We set the following assumption.

$$\textbf{(C)} \quad a(t) = t^{-\frac{1}{2}}(1 + o(1)), \quad t \to \infty.$$

Theorem 1.3 *Assume (C). Then*

$$\lim_{L\to\infty} \mathbf{E}[e^{-\xi_L(f)}] = \mathbf{E}\left[\int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left(-\sum_{n\in\mathbf{Z}} f(\Psi_1^{-1}(2n\pi + \theta))\right)\right]$$

where $\{\Psi_t(\cdot)\}_{t\geq 0}$ *is the strictly-increasing function valued process such that for any* $c_1, \cdots, c_m \in \mathbf{R}$, $\{\Psi_t(c_j)\}_{j=1}^m$ *is the unique solution of the following SDE :*

$$d\Psi_t(c_j) = 2c_j dt + DRe\left\{(e^{i\Psi_t(c_j)} - 1)\frac{dZ_t}{\sqrt{t}}\right\}$$
$$\Psi_0(c_j) = 0, \quad j = 1, 2, \cdots, m$$

where $C(E_0) := \int_M \left|\nabla(L + 2i\sqrt{E_0})^{-1}F\right|^2 dx$, $D := \sqrt{\frac{C(E_0)}{2E_0}}$ *and* Z_t *is a complex Browninan motion.*

Definition 1.2 *For* $\beta > 0$, *the circular* β*-ensemble with n-points is given by*

$$\mathbf{E}_n^\beta[G] := \frac{1}{Z_{n,\beta}} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \cdots \int_{-\pi}^{\pi} \frac{d\theta_n}{2\pi} G(\theta_1, \cdots, \theta_n) |\triangle(e^{i\theta_1}, \cdots, e^{i\theta_n})|^\beta$$

where $Z_{n,\beta}$ *is the normalization constant,* $G \in C(\mathbf{T}^n)$ *is bounded and* $\triangle$ *is the Vandermonde determinant. The limit* ξ_β *of the circular* β*-ensemble is defined*

$$\mathbf{E}[e^{-\xi_\beta(f)}] = \lim_{n\to\infty} \mathbf{E}_n^\beta\left[\exp\left(-\sum_{j=1}^n f(n\theta_j)\right)\right], \quad f \in C_c^+(\mathbf{R})$$

whose existence and characterization is given by [2]. The result in [2] together with Theorem 1.3 imply the limit of ξ_L coincides with that of the circular β-emsemble modulo a scaling.

Corollary 1.4 *Assume (C). Writing* $\xi_\beta = \sum_n \delta_{\lambda_n}$, *let* $\xi'_\beta := \sum_n \delta_{\lambda_n/2}$. *Then* $\xi_L \xrightarrow{d} \xi'_\beta$ *with* $\beta = \beta(E_0) := \frac{8E_0}{C(E_0)}$.

Remark 1.6 *The corresponding* $\beta = \beta(E_0) = \frac{8E_0}{C(E_0)}$ *depends on the reference energy* E_0, *so that the spacing distribution may change if we look at the different region in the spectrum. To see how* β *changes, we recall some results in [3]. Let* $\sigma_F(\lambda)$ *be the spectral measure of the generator* L *of* $\{X_t\}$ *with respect to* F. *Then*

$$\gamma(E) := -\frac{1}{4E}\int_{-\infty}^{0} \frac{\lambda}{\lambda^2 + 4E}\, d\sigma_F(\lambda), \quad E > 0$$

is the Lyapunov exponent in the sense that any generalized eigenfunction ψ_E *of* H *satisfies*

$$\lim_{|t|\to\infty} (\log t)^{-1} \log \left\{\psi_E(t)^2 + \psi'_E(t)^2\right\}^{1/2} = -\gamma(E), \quad a.s.$$

Moreover $E < E_c$ *(resp.* $E > E_c$*) if and only if* $\gamma(E) > \frac{1}{2}$ *(resp.* $\gamma(E) < \frac{1}{2}$*) and* $\gamma(E_c) = \frac{1}{2}$. *Since* $C(E) = 8E \cdot \gamma(E)$, *we have* $\beta(E) = \frac{1}{\gamma(E)}$. *It then follows that* $E < E_c$ *(resp.* $E > E_c$*) if and only if* $\beta(E) < 2$ *(resp.* $\beta(E) > 2$*) and* $\beta(E_c) = 2$. *This is consistent with our general belief that in the point spectrum (resp. in the continuous spectrum) the level repulsion is weak (resp. strong). We also note that if* $\beta = 2$, *the circular* β*-ensemble with* n*-points coincides with the eigenvalue distribution of the unitary ensemble with the Haar measure on* $U(n)$.

Remark 1.7 *If we consider two reference energies* $E_0, E'_0 (E_0 \neq E'_0)$, *then the corresponding point process* ξ_L, ξ'_L *converges jointly to the independent* $\xi_\beta, \xi'_{\beta'}$.

Remark 1.8 *We can also prove that* ξ_L *converges to the Sine*$_\beta$*-process [7], which is the bulk scaling limit of the Gaussian beta ensemble [8]. Together with Corollary 1.4, we have that the scaling limits of these two beta-ensembles coincide.*

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