数理解析研究所講究録
第 1971 巻 2015 年 121-134

Existence of the Fomin derivative of the invariant measure of a stochastic reaction–diffusion equation

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February 26, 2015

Abstract

We consider a reaction–diffusion equation perturbed by noise (not necessarily white). We prove existence of the Fomin derivative of the corresponding transition semigroup P_t. The main tool is a new estimate for $P_tD\varphi$ in terms of $\|\varphi\|_{L^2(H,\nu)}$, where ν is the invariant measure of P_t.

2010 Mathematics Subject Classification AMS: 60H15, 35R15, 35K57

Key words: Stochastic reaction diffusion equations, invariant measures, Fomin differentiability.

1 Introduction

Let $H=L^2(\mathscr{O})$ where $\mathscr{O}=[0,1]^n$, $n\in\mathbb{N}$ [(1)], and denote by $\partial\mathscr{O}$ the boundary of $\mathscr{O}$. We are concerned with the following stochastic differential equation

$$\begin{cases} dX(t)=[AX(t)+p(X(t))]dt+BdW(t),\\ \\ X(0)=x.\end{cases} \tag{1}$$

where A is the realization of the Laplace operator Δ_ξ equipped with Dirichlet boundary conditions,

$$Ax=\Delta_\xi x,\quad x\in D(A),\quad D(A)=H^2(\mathscr{O})\cap H_0^1(\mathscr{O}),$$

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(1)This choice is made for simplicity, all result below hold for a bounded domain of $\mathscr{O}$ with sufficiently regular boundary (Lipschitz for instance).

p is a decreasing polynomial of odd degree equal to $N > 1$, $B \in L(H)$ and W is an H–valued cylindrical Wiener process on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t>0}, \mathbb{P})$.

It is well known that this equation has unique strong solutions and that the associate transition semigroup possesses a unique invariant measure.

The aim of this article is to derive new properties on this invariant measure. If B is the identity, then the system is gradient and the invariant measure is explicit but this is not the case in general. If B commutes with A and has a bounded inverse, it follows from [BoDaRo96] that the invariant measure has a density which is in a Sobolev space based on the reference gaussian measure associated to the linear equation. It has also been shown in [DaDe04] that under our assumptions, the invariant measure is absolutely continuous with respect to the reference gaussian measure. Otherwise, not much is known on this invariant measure.

For as the operator B is concerned, we shall assume:

Hypothesis 1. *$B = (-A)^{-\gamma/2}$ where $\frac{n}{2} - 1 < \gamma < 1$. Obviously this implies that $n < 4$.*

Remark 2. The assumption $\frac{n}{2} - 1 < \gamma$ implies that the stochastic convolution

$$W_A(t) := \int_0^t (-A)^{-\gamma/2} e^{(t-s)A} dW(s), \quad t \geq 0,$$

is a well defined continuous process see e.g. [DaZa14], whereas under the condition $\gamma < 1$ the Bismut–Elworthy-Li formula (4) below holds and implies strong Feller property on H, see [Ce01]. If $\gamma \geq 1$, we need to work with different topologies.

If $\gamma < \frac{n}{2} - 1$, equation (1) is not expected to have solutions with positive spatial regularity and the equation has to be renormalized. This has been studied in [DaDe03] for $n = 2$ and more recently in [Hai14] and [CaCh14] for $n = 3$.

All following results remain true taking $B = G(-A)^{-\gamma/2}$ with $G \in L(H)$ and $\frac{n}{2} - 1 < \gamma < 1$. We take this form for B for simplicity.

Also, the assumption that p is decreasing is not necessary and could be replaced by: p' is bounded above.

Before explaining the content of the paper, it is convenient to recall some results about problem (1), that we gather from [Da04]. We notice, however, that Reaction–Diffusion equations have been recently the object of several researches, see [DaZa14] and references therein.

We start with the definition of solution of (1).

Definition 3. *(i). Let $x \in L^{2N}(\mathscr{O})$; we say that $X \in C_W([0,T];H)$* [(2)] *is a* mild *solution of problem* (1) *if $X(t) \in L^{2N}(\mathscr{O})$ for all $t \geq 0$ and fulfills the following*

[(2)]By $C_W([0,T];H)$ we mean the set of H–valued stochastic processes continuous in mean square and adapted to the filtration $(\mathscr{F}_t)$.

integral equation

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}p(X(s))ds + W_A(t), \quad t \geq 0. \tag{2}$$

(ii). Let $x \in H$; we say that $X \in C_W([0,T];H)$ is a generalized *solution of problem* (1) *if there exists a sequence $(x_n) \subset L^{2N}(\mathscr{O})$, such that*

$$\lim_{n\to\infty} x_n = x \quad \text{in } L^2(\mathscr{O}),$$

and

$$\lim_{n\to\infty} X(\cdot, x_n) = X(\cdot, x) \quad \text{in } C_W([0,T];H).$$

It is convenient to introduce the following approximating problem

$$\begin{cases} dX_\alpha(t) = (AX_\alpha(t) + p_\alpha(X_\alpha(t))dt + (-A)^{-\gamma/2}dW(t), \\ \\ X_\alpha(0) = x \in H, \end{cases} \tag{3}$$

where for any $\alpha > 0$, p_α are the Yosida approximations of p, that is

$$p_\alpha(r) = \frac{1}{\alpha}\,(r - J_\alpha(r)),\ J_\alpha(r) = (1 - \alpha p(\cdot))^{-1}(r), \quad r \in \mathbb{R}.$$

Notice that, since p_α is Lipschitz continuous, then for any $\alpha > 0$, and any $x \in H$, problem (3) has a unique solution $X_\alpha(\cdot, x) \in C_W([0,T];H)$.

The following result is proved in [Da04, Theorem 4.8]

Proposition 4. *Assume that Hypothesis* 1 *holds and let $T > 0$. Then*

(i) If $x \in L^{2N}(\mathscr{O})$, problem (1) *has a unique mild solution $X(\cdot, x)$.*

(ii) If $x \in L^2(\mathscr{O})$, problem (1) *has a unique generalized solution $X(\cdot, x)$.*
In both cases $\lim_{\alpha\to 0} X_\alpha(\cdot, x) = X(\cdot, x)$ in $C_W([0,T];H)$.

We introduce now the transition semigroup P_t

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H)$$

and the approximate transition semigroup P_t^α

$$P_t^\alpha\varphi(x) = \mathbb{E}[\varphi(X_\alpha(t,x))], \quad \varphi \in B_b(H).$$

By $B_b(H)$ we mean the space of all H–valued real mappings that are Borel and bounded.

For P_t^α the following Bismut-Elworthy-Li formula holds, see [Ce01].

$$\langle DP_t^\alpha\varphi(x), h\rangle = \frac{1}{t}\,\mathbb{E}\left[\varphi(X_\alpha(t,x))\int_0^t \langle (-A)^{\frac{\gamma}{2}}\eta_\alpha^h(s,x), dW(s)\rangle\right], \quad h\in H, \tag{4}$$

where for any $h\in H$, $\eta_\alpha^h(t,x) =: D_xX_\alpha(t,x)\cdot h$ is the differential of $X_\alpha(t,x)$ with respect to x in the direction h and is the solution of the equation

$$\frac{d}{dt}\,\eta_\alpha^h(t,x) = A\eta_\alpha^h(t,x) - p_\alpha'(X_\alpha(t,x))\eta_\alpha^h(t,x), \quad \eta_\alpha^h(0,x) = h. \tag{5}$$

The following result is proved in [Da04, Theorem 4.16]

Proposition 5. *Assume that Hypothesis* 1 *holds. Then the semigroup P_t has a unique invariant measure ν. Moreover there exists $c_N > 0$ such that*

$$\int_H |x|_{L^{2N}(\mathcal{O})}^{2N}\nu(dx) \le c_N. \tag{6}$$

Similarly the approximating problem (3) has a unique invariant measure ν_α. It is not difficult to show that ν_α weakly converges to ν and

$$\int_H |x|_{L^{2N}(\mathcal{O})}^{2N}\nu_\alpha(dx) \le c_N. \tag{7}$$

However, since we couldn't find a quotation of this fact, we have added a proof in the Appendix below.

As well known P_t can be uniquely extended to a strongly continuous semigroup of contractions in $L^2(H,\nu)$ (still denoted P_t). We shall denote by $\mathscr{L}$ its infinitesimal generator and by $\mathscr{L}_0$ the differential operator

$$\mathscr{L}_0\varphi \;=\; \frac{1}{2}\,\mathrm{Tr}\,[(-A)^{-\gamma}D^2\varphi] + \langle x, AD\varphi\rangle + \langle p(x), D\varphi\rangle\,, \quad \varphi\in\mathscr{E}_A(H),$$

where $\mathscr{E}_A(H)$ is the linear span of all real parts of functions of the form

$$\varphi_h(x) := e^{i\langle h,x\rangle},\ x\in H,$$

where $h\in D(A)$. We have used the notation $D\varphi$ for the gradient of φ in H.

Similarly, for any $\alpha>0$, P_t^α can be uniquely extended to a strongly continuous semigroup of contractions in $L^2(H,\nu)$ whose infinitesimal generator we denote by $\mathscr{L}^\alpha$. We denote by $\mathscr{L}_0^\alpha$ the differential operator defined by

$$\mathscr{L}_0^\alpha \;=\; \frac{1}{2}\,\mathrm{Tr}\,[(-A)^{-\gamma}D^2\varphi] + \langle x, AD\varphi\rangle + \langle p_\alpha(x), D\varphi\rangle\,, \quad \varphi\in\mathscr{E}_A(H),\ x\in H.$$

Proposition 6. *Assume that Hypothesis 1 holds. Then $\mathscr{L}$ is the closure of $\mathscr{L}_0$ in $L^2(H,\nu)$ and $\mathscr{L}^\alpha$ is the closure of $\mathscr{L}_0^\alpha$ in $L^2(H,\nu_\alpha)$.*

The first assertion of the proposition is proved in [Da04, Theorem 4.23], the proof of the latter is completely similar and so, it is omitted.

Now we are ready to describe the main goal of the paper. First, we shall prove the following integration by parts formula for the invariant measure ν. For any $h \in H$ and any $\varphi \in C_b^1(H)$ there exists a function $v^h \in L^2(H,\nu)$ such that

$$\int_H \langle (-A)^{-1} D\varphi(x), h\rangle \, \nu(dx) = \int_H \varphi(x)\, v^h(x)\, \nu(dx). \tag{8}$$

Then we deduce by (8) the existence of the Fomin derivative of ν in any direction $A^{-1}h$. (3)

A similar result, concerning the Burgers equation driven by white noise, has been proved in [DaDe14]. In the present case the proof of (8) is based, as in [DaDe14], on an estimate of $P_t D\varphi$ depending only on $\|\varphi\|_{L^2(H,\nu)}$. However, the techniques used here are obviously different.

We believe that our method could be used for other SPDEs as: singular dissipative equations and $2D$–Navier–Stokes equations. Both will be the object of future work.

In Section 2 we prove an identity relating $DP_t^\alpha\varphi$ and $P_t^\alpha D\varphi$. Using this identity in Section 3 we prove the estimate

$$\int_H \langle D\varphi(x), h\rangle \, \nu(dx) \le C\|\varphi\|_{L^2(H,\nu)}\, |Ah|_H, \quad \varphi \in L^2(H,\nu). \tag{9}$$

Finally, Section 4 is devoted to show some consequences as the definition of Sobolev space with respect to the measure ν.

2 An identity relating $DP_t^\alpha\varphi$ and $P_t^\alpha D\varphi$

Proposition 7. *For any $\varphi \in C_b^1(H)$, $\alpha > 0$, $h, x \in D(A)$, we have*

$$P_t^\alpha(\langle D\varphi(x), h\rangle) = \langle DP_t^\alpha\varphi(x), h\rangle - \int_0^t P_{t-s}^\alpha(\langle Ah + Dp_\alpha(x)h, DP_s^\alpha\varphi(x)\rangle)ds, \tag{10}$$

where p^α are the Yosida approximations of p.

(3) For the definition of Fomin derivative see e.g. [Pu98].

Proof. Let $\varphi \in \mathscr{E}_A(H)$, $u^\alpha(t,x) = P_t^\alpha \varphi(x)$. Then $\varphi \in D(\mathscr{L}^\alpha)$ and so,

$$D_t u^\alpha(t,x) = \mathscr{L}^\alpha u^\alpha(t,x) = P_t^\alpha \left(\frac{1}{2}\mathrm{Tr}\,[(-A)^\gamma D^2\varphi(x)] + \langle Ax + p^\alpha(x), D\varphi(x)\rangle\right)$$

Now let $h \in D(A)$. Then setting $v_h^\alpha(t,x) = \langle DP_t^\alpha \varphi(x), h\rangle$, we have

$$D_t v_h^\alpha(t,x) = \mathscr{L}^\alpha v_h^\alpha(t,x) + \langle Ah + p'_\alpha(x)h, Du^\alpha(t,x)\rangle$$

and by the variation of constants formula we deduce that

$$v_h^\alpha(t,x) = P_t^\alpha v_h^\alpha(0,x) + \int_0^t P_{t-s}^\alpha(\langle Ah + p'_\alpha(x)h, DP_s^\alpha\varphi(x)\rangle)ds$$

which is equivalent to

$$P_t^\alpha(\langle D\varphi(x), h\rangle) = \langle DP_t^\alpha\varphi(x), h\rangle$$

$$-\int_0^t P_{t-s}^\alpha(\langle Ah + p'_\alpha(x)h, DP_s^\alpha\varphi(x)\rangle)ds.$$

for all $\varphi \in \mathscr{E}_A(H)$. Since $\mathscr{E}_A(H)$ is a core for $\mathscr{L}^\alpha$ (Proposition 6), the conclusion follows.

□

Remark 8. Probably identity (10) could be useful also in finite dimensions for SDEs with non degenerate noise. In fact (10) looks simpler than the formula obtained via Maliavin Calculus, even if the latter allows to consider non degenerate equations, see [Ma97], [Sa05].

3 The main result

We first need a lemma.

Lemma 9. *For any $\alpha > 0$, $T > 0$ and any $h \in H$ we have*

$$|\eta_\alpha^h(T,x)| \le |h|, \quad x \in H, \tag{11}$$

and

$$\int_0^T |(-A)^{1/2}\eta_\alpha^h(t,x)|^2 dt \le |h|^2, \quad x \in H. \tag{12}$$

Finally, for any $\beta \in (0, 1/2)$ we have

$$\int_0^T |(-A)^\beta \eta_\alpha^h(t,x)|^2 dt \le C_{T,\beta} T^{1-2\beta}|h|^2, \quad x \in H. \tag{13}$$

Proof. By (5) and $p'_\alpha \le 0$, we have

$$\frac{1}{2}\frac{d}{dt}\,|\eta^h_\alpha(t,x)|^2 + \frac{1}{2}|(-A)^{1/2}\eta^h_\alpha(t,x)|^2 = \int_{\mathscr{O}} p'_\alpha(X_\alpha(t,x))(\eta^h_\alpha(t,x))^2 d\xi \le 0. \tag{14}$$

Integrating in t from 0 to T, yields

$$|\eta^h_\alpha(T,x)|^2 + \int_0^T |(-A)^{1/2}\eta^h_\alpha(t,x)|^2\,dt \le |h|^2.$$

So, (11) and (12) follow. It remains to show (13).

Let us recall a well known estimate from interpolation. For $0 < \beta < 1/2$ we have

$$|(-A)^\beta x| \le |x|^{1-2\beta}\,|(-A)^{1/2}x|^{2\beta}, \quad \forall\, x \in D((-A)^{1/2}). \tag{15}$$

It follows that

$$\int_0^T |(-A)^\beta \eta^h_\alpha(t,x)|^2 dt \le \int_0^T |\eta^h_\alpha(t,x)|^{1-2\beta}\,|(-A)^{1/2}\eta^h_\alpha(t,x)|^{2\beta} dt.$$

Recalling (11) and using Hölder's inequality it follows that

$$\begin{aligned}
&\int_0^T |(-A)^\beta \eta^h_\alpha(t,x)|^2 dt \\
&\le \int_0^T |(-A)^{1/2}\eta^h_\alpha(t,x)|^{2\beta} dt\,|h|^{1-2\beta} \\
&\le T^{1/2-\beta}\left[\int_0^T |(-A)^{1/2}\eta^h_\alpha(t,x)|^2 dt\right]^\beta\,|h|^{1-2\beta}
\end{aligned}$$

Finally, taking into account (12), yields

$$\int_0^T |(-A)^\beta \eta^h_\alpha(t,x)|^2 dt \le T^{1/2-\beta}|h|^2, \tag{16}$$

as claimed.

□

We are now ready to show

Theorem 10. *There exists $C > 0$ such that for all $\varphi \in L^2(H,\nu)$ and all $h \in D(A)$ we have*

$$\int_H \langle D\varphi(x), h\rangle\,\nu(dx) \le C\|\varphi\|_{L^2(H,\nu)}\,|Ah|_H. \tag{17}$$

Proof. Let us integrate identity (10) with respect to ν_α over H. Taking into account the invariance of ν_α, we obtain

$$\begin{aligned}\int_H \langle D\varphi(x), h\rangle \nu_\alpha(dx) &= \int_H \langle DP_t^\alpha \varphi(x), h\rangle \, \nu_\alpha(dx) \\ &\quad - \int_0^t \int_H \langle Ah + p_\alpha'(x)h, DP_s^\alpha \varphi(x)\rangle ds \, \nu_\alpha(dx).\end{aligned} \tag{18}$$

We are going to estimate

$$\int_H \langle DP_t^\alpha \varphi(x), h\rangle \, \nu_\alpha(dx) \tag{19}$$

and

$$\int_0^t \int_H \langle Ah + p_\alpha'(x)h, DP_s^\alpha \varphi(x)\rangle ds \, \nu_\alpha(x), \tag{20}$$

using the Bismut–Elworthy-Li formula (4). First notice that by (4) it follows that

$$\langle DP_t^\alpha \varphi(x), h\rangle^2 \le \frac{1}{t^2} \, \mathbb{E}[\varphi^2(X_\alpha(t,x))] \, \mathbb{E} \int_0^t |(-A)^{\gamma/2} D_x X_\alpha(s,x)h|^2 ds$$

In view of Hypothesis 1 we can choose now $\beta < 1/2$ such that

$$\frac{n}{2} - 1 < \gamma < 2\beta.$$

The operator $(-A)^{(\gamma-2\beta)/2}$ is bounded, set

$$K := \|(-A)^{(\gamma-2\beta)/2}\|. \tag{21}$$

Then, since $(-A)^{\gamma/2} = (-A)^{(\gamma-2\beta)/2}(-A)^\beta$,

$$\mathbb{E} \int_0^t |(-A)^{\gamma/2} D_x X_\alpha(s,x)h|^2 ds \le K^2 \mathbb{E} \int_0^t |(-A)^\beta D_x X_\alpha(s,x)h|^2 ds$$

Taking into account (13) we find

$$\langle DP_t^\alpha \varphi(x), h\rangle^2 \le \frac{K^2}{t^{1+2\beta}} \, P_t^\alpha(\varphi^2)(x)|h|^2.$$

Equivalently

$$\langle DP_t^\alpha \varphi(x), h\rangle \le K t^{-1/2-\beta} \, [P_t^\alpha(\varphi^2)(x)]^{1/2} \, |h|.$$

Integrating with respect to ν_α over H, yields for a function $h \in L^2(H, \nu_\alpha)$,

$$\int_H \langle DP_t^\alpha \varphi(x), h(x)\rangle \nu_\alpha(dx) \le Kt^{-1/2-\beta} \int_H [P_t^\alpha(\varphi^2)(x)]^{1/2} |h(x)| \nu_\alpha(dx)$$

$$\le Kt^{-1/2-\beta} \left(\int_H P_t^\alpha(\varphi^2)\, \nu_\alpha(dx)\right)^{1/2} \left(\int_H |h(x)|^2 \nu_\alpha(dx)\right)^{1/2},$$

that is, taking into account the invariance of ν_α

$$\int_H \langle DP_t^\alpha \varphi(x), h(x)\rangle \nu_\alpha(dx) \le Kt^{-1/2-\beta}\ \|\varphi\|_{L^2(H,\nu_\alpha)}\ \|h\|_{L^2(H,\nu_\alpha)} \tag{22}$$

Now we can estimate (19) and (20). As for (19) we have by (22)

$$\int_H \langle DP_t^\alpha \varphi(x), h\rangle \nu_\alpha(dx) \le Kt^{-1/2-\beta}\ \|\varphi\|_{L^2(H,\nu_\alpha)}\ |h| \tag{23}$$

and as for (20)

$$\begin{aligned}&\int_0^t \int_H \langle Ah + p'_\alpha(x)h, DP_s^\alpha \varphi(x)\rangle ds\, d\nu_\alpha \\ &\le K \int_0^t s^{-1/2-\beta}\, ds\, \|\varphi\|_{L^2(H,\nu)}\, (|Ah| + \|p'_\alpha h\|_{L^2(H,\nu_\alpha)})\end{aligned} \tag{24}$$

Now, using (22) and (23) we deduce (recall that $1/2 + \beta < 1$)

$$\begin{aligned}&\int_H \langle D\varphi(x), h\rangle \nu_\alpha(dx) \le Kt^{-1/2-\beta}\ \|\varphi\|_{L^2(H,\nu_\alpha)}\ |h| \\ &+ \frac{2Kt^{1/2-\beta}}{1-2\beta}\ \|\varphi\|_{L^2(H,\nu)}\, (|Ah| + \|p'_\alpha h\|_{L^2(H,\nu_\alpha)}).\end{aligned} \tag{25}$$

Setting $t = 1$ in (25) and letting $\alpha \to 0$, we arrive, recalling Proposition 14 below, at

$$\int_H \langle D\varphi(x), h\rangle\, \nu(dx) \le C\|\varphi\|_{L^2(H,\nu)}(|Ah| + \|p'\, h\|_{L^2(H,\nu)}). \tag{26}$$

Thanks to Sobolev embedding, we choose $r > 2$ such that $D(A) \subset L^r(\mathscr{O})$. Then thanks to Hölder inequality and (7), there exists $C_r > 0, C'_r > 0$ such that

$$\|p'\, h\|_{L^2(H,\nu)} \le C_r |h|_{L^r(\mathscr{O})} \le C'_r |Ah|, \quad \forall\, h \in D(A).$$

Thus the conclusion follows.

□

4 Some consequences of Theorem 10

Proposition 11. *Assume that estimate* (17) *is fulfilled. Then the linear operator*

$$\varphi \in C_b^1(H) \mapsto (-A)^{-1}D\varphi \in C_b(H;H),$$

is closable in $L^2(H,\nu)$.

Proof. Step 1. For any $h \in H$ the linear operator

$$\varphi \in C_b^1(H) \mapsto \langle (-A)^{-1}D\varphi(x), h\rangle \in C_b(H)$$

is closable in $L^2(H,\nu)$.

In fact, let $(\varphi_n) \subset C_b^1(H)$ and $f \in L^2(H,\nu)$ be such that

$$\begin{cases} \varphi_n \to 0 \ \text{ in } L^2(H,\nu), \\ \\ \langle (-A)^{-1}D\varphi_n(x), h\rangle \to f \ \text{ in } L^2(H,\nu). \end{cases}$$

We claim that $f = 0$.

Take $\psi \in C_b^1(H)$, then replacing in (17) φ by $\psi\varphi_n$, yields

$$\left| \int_H [\psi(x)((-A)^{-1}D\varphi_n(x)\cdot h) + \varphi_n(x)((-A)^{-1}D\psi(x)\cdot h)]\,\nu(dx) \right|$$

$$\le \|\varphi_n\psi\|_{L^2(H,\nu)}\,|h|_H \le \|\psi\|_\infty\,\|\varphi_n\|_{L^2(H,\nu)}\,|h|_H.$$

Letting $n \to \infty$, we have

$$\int_H \psi(x)f(x)\,\nu(dx) = 0,$$

which yields $f = 0$ by the arbitrariness of ψ, thereby proving the claim.

Step 2. Conclusion.

Let $(\varphi_n) \subset C_b^1(H)$ and $F \in C_b(H;H)$ such that

$$\begin{cases} \varphi_n \to 0 \ \text{ in } L^2(H,\nu), \\ \\ (-A)^{-1}D\varphi_n \to F \ \text{ in } L^2(H,\nu;H). \end{cases}$$

We claim that $F = 0$.

Let (e_k) be an orthonormal basis on H consisting of eigenvectors of A and let (α_k) such that

$$Ae_k = -\alpha_k e_k, \quad k \in \mathbb{N}.$$

Then for any $k \in \mathbb{N}$ we have

$$\langle (-A)^{-1} D_k \varphi_n(x), e_k \rangle \to \alpha_k^{-1} \langle F(x), e_k \rangle \quad \text{in } L^2(H, \nu).$$

By Step 1 taking $h = e_k$ we see that $D_k = D_{e_k}$ is a closable operator on $L^2(H, \nu)$ for any $k \in \mathbb{N}$.

So,

$$\langle F(\cdot), e_k \rangle = 0, \quad \forall\, k \in \mathbb{N}$$

which yields $F = 0$ as required.

□

4.1 The Sobolev space and the integration by parts formula

Let us denote by $W_A^{1,2}(H, \nu)$ the domain of the closure of $(-A)^{-1}D$. Denoting by M^* the adjoint of $(-A)^{-1}D$ we have

$$\int_H ((-A)^{-1} D\varphi(x) \cdot F(x))\, \nu(dx) = \int_H \varphi(x)\, M^*(F)(x)\, \nu(dx). \tag{27}$$

Let now $h \in H$, set $F^h(x) = h$, $\forall\, x \in H$. By Theorem 10 we obtain

$$\int_H ((-A)^{-1} D\varphi(x) \cdot F^h(x))\, \nu(dx) \leq C \|\varphi\|_{L^2(H,\nu)}\, |Ah|_H,$$

so that F^h belongs to the domain of M^*.

Setting $M^*(F^h) = v^h$, we obtain the following integration by part formula.

Proposition 12. *For any $h \in H$ and any $\varphi \in W_A^{1,2}(H, \nu)$ there exists a function $v^h \in L^2(H, \nu)$ such that*

$$\int_H \langle (-A)^{-1} D\varphi(x), h \rangle\, \nu(dx) = \int_H \varphi(x)\, v^h(x)\, \nu(dx). \tag{28}$$

Therefore if $h \in H$ there exists the Fomin derivative of ν in the direction of $A^{-1}h$.

Remark 13. Assume that $p = 0$. Then $\mu = N_Q$, where $Q = -\frac{1}{2}\, A^{-1}$. Setting $v^h(x) = \sqrt{2} \langle Q^{-1/2}x, h \rangle$ (27) reduces to the usual integration by parts formula for the Gaussian measure μ. Notice, however, that in this case we can take $h \in D((-A)^{1/2})$

A Convergence of ν_α to ν

Proposition 14. *For all $\varphi \in C_b(H)$ we have*

$$\lim_{\alpha\to 0}\int_H \varphi\, d\nu_\alpha = \int_H \varphi\, d\nu. \tag{29}$$

Proof. Let $\varphi \in C_b(H)$. Fix $\alpha \in (0,1]$, $x \in H$ and write

$$\begin{aligned}\left|\int_H \varphi\, d\nu - \int_H \varphi\, d\nu_\alpha\right| &\le \left|\int_H \varphi\, d\nu - P_t\varphi(x)\right| \\ &+ |P_t\varphi(x) - P_t^\alpha\varphi(x)| + \left|P_t^\alpha\varphi(x) - \int_H \varphi\, d\nu_\alpha\right|.\end{aligned} \tag{30}$$

Now choosing β such that $\frac{1}{n} < \gamma < 2\beta < 1$, and taking into account (13), we have

$$\int_0^T |(-A)^{\gamma/2}\eta_\alpha^h(t,x)|^2 dt \le KC_{T,\beta}T^{1-2\beta}|h|^2, \quad x \in H, \tag{31}$$

where K is defined in (21). It follows by Bismut-Elworthy formula that for a suitable constant $C > 0$ we have

$$\begin{aligned}\left|P_t^\alpha\varphi(x) - \int_H \varphi\, d\nu_\alpha\right| &= \left|\int_H [P_t^\alpha\varphi(x) - P_t^\alpha\varphi(y)]\, \nu_\alpha(dy)\right| \\ &\le Ct^{-\beta}\|\varphi\|_\infty \int_H |x-y|\,\nu_\alpha(dy) \le Ct^{-\beta}\|\varphi\|_\infty\left((|x| + \int_H |y|\,\nu_\alpha(dy)\right).\end{aligned} \tag{32}$$

Claim. There exists $M > 0$ such that

$$\int_H |y|\,\nu_\alpha(dy) \le M, \quad \forall\, \alpha \in (0,1]. \tag{33}$$

Once the claim is proved the conclusion follows easily from (30) and (32) and [Da04, Theorem 4.16]. (4)

To prove the claim it is enough to show

$$\mathbb{E}|X_\alpha(t,x)| \le M, \quad \forall\, \alpha \in (0,1]. \tag{34}$$

(4) Since p_α is dissipative the proof that $\lim_{\alpha\to 0} P_t^\alpha\varphi(x) = \int_H \varphi\, d\nu_\alpha$ is exactly the same as that in [Da04, Theorem 4.16].

This can be proved as the estimate (4.13) in [Da04] taking into account that for any $m \in \mathbb{N}$ there is $K_m > 0$ such that

$$\mathbb{E}\int_{\mathscr{O}} |W_A(t,\xi)|^{2m} d\xi \le K_m, \quad \forall\, t \ge 0. \tag{35}$$

Here is finally the proof of (35). We start from the identity

$$W_A(t,\xi) = \sum_{k\in\mathbb{N}^n} \int_0^t e^{-\pi^2|k|^2(t-s)}(\pi^2|k|^2)^{-\gamma/2} e_k(\xi)dW_k(s),$$

where

$$e_k(\xi) = (2\pi)^{n/2}\sin(k_1\xi_1)\cdots\sin(k_n\xi_n), \quad k = (k_1,...,k_n).$$

Therefore for each $(t,\xi) \in [0,+\infty)\times\mathscr{O}$, $W_A(t,\xi)$ is a real Gaussian variable with mean 0 and covariance $q(t,\xi)$ given by

$$q(t,\xi) = \sum_{k\in\mathbb{N}^n} \int_0^t e^{-2\pi^2|k|^2(t-s)}(\pi^2|k|^2)^{-\gamma}|e_k(\xi)|^2 ds. \tag{36}$$

Since $|e_k(\xi)|^2 \le (2/\pi)^n$ and $\gamma > \frac{n}{2}-1$, we find

$$q(t,\xi) \le C(n,\gamma)\sum_{k\in\mathbb{N}^n} \frac{1}{|k|^{2+2\gamma}} = C_1(n,\gamma) < \infty, \quad \forall\, t \ge 0. \tag{37}$$

Therefore

$$\mathbb{E}(W_A(t,\xi))^{2m} \le C_2(n,\gamma,m), \quad \forall\, t \ge 0. \tag{38}$$

Finally, integrating in ξ over $\mathscr{O}$, yields

$$\mathbb{E}\int_{\mathscr{O}} (W_A(t,\xi))^{2m} d\xi \le C_2(n,\gamma,m) \text{ meas. } (\mathscr{O}), \quad \forall\, t \ge 0,$$

and the conclusion follows.

Remark 15. *We have used $p' \le 0$ in (32). If we assume only that p' is bounded above, the differential of the transition semigroup may grow in time. However, using classical arguments (see for instance [De13]), convergence to the invariant measure can be proved under this more general assumption.*

□

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