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HECKE ALGEBRAS FOR HILBERT MODULAR FORMS AND IDEAL CLASS GROUPS

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ABSTRACT. Using twisting operators defined by characters of order two of $Cl^{+}(F)$, we present a connection between $Cl^{+}(F)/(Cl^{+}(F))^{2}$ and subalgebras of dimension 2 over a certain field in the Hecke algebra acting on spaces of Hilbert modular forms over F. Here $Cl^{+}(F)$ is the ideal class group of F in the narrow sense. In addition, we give examples related to this result. This is a joint work with Yoshio Hiraoka. See [H-O] for further details.

1. PRELIMINARIES

Let F be a totally real algebraic number field of degree g, $\mathfrak{o}$ the maximal order of F, $Cl^{+}(F)$ the ideal class group of F *in the narrow sense*. We assume that

$$\textit{the order of } Cl^{+}(F) \textit{ is even},$$

that is, $Cl^{+}(F)/Cl^{+}(F)^{2}$ is nontrivial. Here $Cl^{+}(F)^{2}:=\{\alpha^{2}\mid\alpha\in Cl^{+}(F)\}$.

Let $\mathfrak{c}$ be an integral ideal of F. Let ψ be a Hecke character of F of finite order such that the nonarchimedean part of its conductor divides $\mathfrak{c}$. Let $k=(k_{1},\ldots,k_{g})\in\mathbf{Z}^{g}$ with $k_{j}>0$ for all j.

Let $\mathcal{S}_{k}(\mathfrak{c},\psi)$ be the space of all (adelic) Hilbert cusp forms (on $GL(2)$) over F of weight k, of level $\mathfrak{c}$, and with character ψ, and $\mathcal{S}_{k}^{0}(\mathfrak{c},\psi)$ the subspace of $\mathcal{S}_{k}(\mathfrak{c},\psi)$ consisting of the newforms. (See [H-O, §2.2 and §2.5] or [S, p. 12 and p. 15] for the definitions of $\mathcal{S}_{k}(\mathfrak{c},\psi)$ and $\mathcal{S}_{k}^{0}(\mathfrak{c},\psi)$.) We put

$$\mathcal{S}:=\mathcal{S}_{k}^{0}(\mathfrak{c},\psi).$$

For the weight $k=(k_{1},\ldots,k_{g})$, we assume, as in [S, (2.38)], that

$$k_{1}\equiv\cdots\equiv k_{g} \pmod{2}.$$

Let $T(\mathfrak{a})$ be the Hecke operator for an integral ideal $\mathfrak{a}$ acting on $\mathcal{S}$. (See [H-O, §2.3] or [S, (2.21)] for the definition of $T(\mathfrak{a})$. Note that $T(\mathfrak{a})$ in [H-O] (and this résumé) is defined as $T'(\mathfrak{a})$ in [S].)

Let $\mathcal{H}_{k}^{0}(\mathfrak{c},\psi;\mathbf{Q})$ be the Hecke algebra for $\mathcal{S}$ with coefficients in $\mathbf{Q}$ (i.e. the subalgebra of $\mathrm{End}_{\mathbf{C}}(\mathcal{S})$ generated over $\mathbf{Q}$ by $T(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$ of F). We put

$$\mathcal{H}:=\mathcal{H}_{k}^{0}(\mathfrak{c},\psi;\mathbf{Q}).$$

We note that $\mathcal{H}$ is a semisimple artinian commutative ring and $\mathbf{Q}(\psi) \subset \mathcal{H}$. Here $\mathbf{Q}(\psi)$ is the subfield of $\mathbf{C}$ generated over $\mathbf{Q}$ by the image of ψ.

2. Action of C on $\mathcal{H}$

Let C be the group of Hecke characters factoring through $Cl^+(F)/Cl^+(F)^2$ (i.e. the group of all Hecke characters χ of F such that $\chi^2 = \mathit{1}$ and the nonarchimedean part of its conductor is equal to $\mathfrak{o}$, where $\mathit{1}$ is the identity Hecke character of F).

For $\chi \in C$ and $f \in \mathcal{S}$, we put

$$(\chi \otimes f)(x) := \chi(\det(x))f(x) \qquad (x \in GL_2(F_{\mathbf{A}})).$$

Then $\chi \otimes f \in \mathcal{S}_k^0(\mathfrak{c}, \chi^2\psi) = \mathcal{S}$ (see [H-O, §3]). So $\mathbf{C}$-linear map $f \mapsto \chi \otimes f$ gives an automorphism on $\mathcal{S}$. Thus we obtain a homomorphism

$$\varrho : C \to GL_{\mathbf{C}}(\mathcal{S})$$

by

$$\varrho(\chi)\,(f) := \chi \otimes f.$$

For $\chi \in C$ and $a \in \mathrm{End}_{\mathbf{C}}(\mathcal{S})$, we put

$$a^{\chi} := \varrho(\chi)^{-1}\, a\, \varrho(\chi) \quad (\in \mathrm{End}_{\mathbf{C}}(\mathcal{S})).$$

Then we have

$$T(\mathfrak{a})^{\chi} = \chi^*(\mathfrak{a})\, T(\mathfrak{a})$$

(see [H-O, (3.2)]). Here χ^* the ideal character associated with a Hecke character χ. Hence $\mathcal{H}^{\chi} = \mathcal{H}$ for every $\chi \in C$.

3. Decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_s$

Let E be the set of all primitive idempotents of $\mathcal{H}$. (Namely E is the set of all idempotents e of $\mathcal{H}$ such that e cannot be written in the sum of orthogonal idempotents.) Then

$$\mathcal{H} = \bigoplus_{e \in E} \mathcal{H}e \qquad \text{and} \qquad \mathcal{H}e \text{ is a field.}$$

We see that $E^{\chi} = E$ for every $\chi \in C$. Let

$$E = \bigsqcup_{\ell=1}^{s} E_\ell$$

be the C-orbit decomposition of E. For each $1 \leq \ell \leq s$, we put

$$\varepsilon_\ell := \sum_{e \in E_\ell} e, \qquad \mathcal{H}_\ell := \mathcal{H}\,\varepsilon_\ell \left(= \bigoplus_{e \in E_\ell} \mathcal{H}e \right), \qquad T(\mathfrak{a})_\ell := T(\mathfrak{a})\varepsilon_\ell.$$

Then

$$\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_s,$$

and

$$(\mathcal{H}_\ell)^\chi = \mathcal{H}_\ell, \qquad T(\mathfrak{a})_\ell^\chi = \chi^*(\mathfrak{a})\,T(\mathfrak{a})_\ell$$

for every $\chi \in C$.

We note that, put

$$\mathcal{S}_\ell := \varepsilon_\ell \mathcal{S},$$

then

$$\mathcal{S} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_s,$$

$\mathcal{H}_\ell$ is the Hecke algebra for $\mathcal{S}_\ell$ with coefficients in $\mathbf{Q}$, and $T(\mathfrak{a})_\ell$ is a Hecke operator on $\mathcal{S}_\ell$.

4. Subalgebras K_ℓ^+ and $\mathcal{H}_\ell^+$ of $\mathcal{H}_\ell$

For an ideal $\mathfrak{a}$ of F, we denote by $[\mathfrak{a}]$ the element of $Cl^+(F)$ containing $\mathfrak{a}$.

We define subalgebras K_ℓ^+ and $\mathcal{H}_\ell^+$ of $\mathcal{H}_\ell$ by

$$K_\ell^+ := \mathbf{Q}(\psi)[\{\, T(\mathfrak{a})_\ell \mid [\mathfrak{a}] \in Cl^+(F)^2 \,\}],$$

$$\mathcal{H}_\ell^+ := \sum_{e \in E_\ell} K_\ell^+ \, e.$$

Then

$$K_\ell^+ \subset \mathcal{H}_\ell^+ \subset \mathcal{H}_\ell,$$

and K_ℓ^+ is a field (see [H-O, Prop. 3.3 (1)]).

5. Subgroups C_ℓ and C'_ℓ of C

We define subgroups C_ℓ and C'_ℓ of C by

$$C_\ell := \{\, \chi \in C \mid a^\chi = a \text{ for every } a \in \mathcal{H}_\ell \,\},$$

$$C'_\ell := \{\, \chi \in C \mid e^\chi = e \,\},$$

where $e \in E_\ell$. (We note that C'_ℓ is independent of the choice of an element e of E_ℓ.) Then

$$C \supset C'_\ell \supset C_\ell.$$

6. Subgroups I_ℓ and I'_ℓ of $Cl^+(F)$

We define subgroups I_ℓ and I'_ℓ of $Cl^+(F)$ by

$$I_\ell := \{[\mathfrak{a}] \in Cl^+(F) \,|\, \chi^*(\mathfrak{a}) = 1 \text{ for every } \chi \in C_\ell\},$$
$$I'_\ell := \{[\mathfrak{a}] \in Cl^+(F) \,|\, \chi^*(\mathfrak{a}) = 1 \text{ for every } \chi \in C'_\ell\}.$$

Then

$$Cl^+(F)^2 \subset I'_\ell \subset I_\ell \subset Cl^+(F).$$

We note that $Cl^+(F)/Cl^+(F)^2$ is an abelian group of type $(2, \dots, 2)$ (i.e. isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/2\mathbf{Z}$).

7. Main Theorem

Theorem 7.1. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral ideals of F.*

(1) *Suppose $T(\mathfrak{a})_\ell \neq 0$ and $T(\mathfrak{b})_\ell \neq 0$. Then*

$$[\mathfrak{a}]\, Cl^+(F)^2 = [\mathfrak{b}]\, Cl^+(F)^2 \iff K_\ell^+[T(\mathfrak{a})_\ell] = K_\ell^+[T(\mathfrak{b})_\ell].$$

(2) $[\mathfrak{a}] \in I_\ell \iff$ *there exists an ideal $\mathfrak{a}'$ in $[\mathfrak{a}]\, Cl^+(F)^2$ such that $T(\mathfrak{a}')_\ell \neq 0$.* (*In particular, $[\mathfrak{a}] \notin I_\ell \Longrightarrow T(\mathfrak{a})_\ell = 0$.*)

(3) *Suppose $T(\mathfrak{a})_\ell \neq 0$ and $T(\mathfrak{b})_\ell \neq 0$.*

(i) *If $[\mathfrak{a}] \in I_\ell \setminus I'_\ell$, then*
- $T(\mathfrak{a})_\ell \notin \mathcal{H}_\ell^+,\ T(\mathfrak{a})_\ell^2 \in K_\ell^+$,
- *$K_\ell^+[T(\mathfrak{a})_\ell]$ is a quadratic extension field of K_ℓ^+,*
- $[\mathfrak{a}]\, I'_\ell = [\mathfrak{b}]\, I'_\ell \iff K_\ell^+[T(\mathfrak{a})_\ell] \cong K_\ell^+[T(\mathfrak{b})_\ell]$.

(ii) *If $[\mathfrak{a}] \in I'_\ell \setminus Cl^+(F)^2$, then*
- $T(\mathfrak{a})_\ell \in \mathcal{H}_\ell^+ \setminus K_\ell^+,\ T(\mathfrak{a})_\ell^2 \in \{\, c^2 \,|\, c \in K_\ell^+\}$,
- *$K_\ell^+[T(\mathfrak{a})_\ell] \cong K_\ell^+ \oplus K_\ell^+$ as rings.*

Note that $[\mathfrak{a}] \in Cl^+(F)^2 \Longrightarrow T(\mathfrak{a})_\ell \in K_\ell^+$ (by the definition of K_ℓ^+).

(For the proof of this theorem, see [H-O, §4].)

8. Remark

(1) Suppose $T(\mathfrak{a})_\ell \neq 0$ and $T(\mathfrak{b})_\ell \neq 0$. Then

$$[\mathfrak{a}] \in [\mathfrak{b}]\, Cl^+(F)^2 \iff T(\mathfrak{a})_\ell \in (K_\ell^+)^\times \cdot T(\mathfrak{b})_\ell. \tag{8.1}$$

(2) We have

$$\dim_{K_\ell^+} \mathcal{H}_\ell = [C : C_\ell] = [I_\ell : Cl^+(F)^2], \tag{8.2}$$
$$\dim_{K_\ell^+} \mathcal{H}_\ell^+ = [C : C'_\ell] = [I'_\ell : Cl^+(F)^2]. \tag{8.3}$$

(See [H-O, §5].)

9. Example

Let $F = \mathbf{Q}(\sqrt{42})$. Then $Cl^+(F) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Put $\mathfrak{a}_1 := [2, \theta]$ and $\mathfrak{a}_2 := [3, \theta]$, where $\theta := \sqrt{42}$ and $[\alpha_1, \alpha_2] := \mathbf{Z}\alpha_1 + \mathbf{Z}\alpha_2$. Then

$$Cl^+(F)/Cl^+(F)^2 = Cl^+(F) = \langle [\mathfrak{a}_1], [\mathfrak{a}_2] \rangle = \{[\mathfrak{o}], [\mathfrak{a}_1], [\mathfrak{a}_2], [\mathfrak{a}_1\mathfrak{a}_2]\}.$$

Let $k = (2, 2)$, $\mathfrak{c} = \mathfrak{o}$, and $\psi = \mathit{1}$. We note that $\mathcal{S} = \mathcal{S}^0_{(2,2)}(\mathfrak{o}, \mathit{1}) = \mathcal{S}_{(2,2)}(\mathfrak{o}, \mathit{1})$. From Table 1 in §10 below, we see that

$$\mathcal{H} = K^{[1]} \oplus K^{[2]} \oplus K^{[3]} \oplus K^{[4]} \oplus K^{[5]},$$
$$K^{[1]} \cong \mathbf{Q}(\sqrt{3}, \sqrt{10}), \qquad K^{[2]} \cong K^{[3]} \cong \mathbf{Q}(\sqrt{6}),$$
$$K^{[4]} \cong \mathbf{Q}(\sqrt{2}), \qquad K^{[5]} \cong \mathbf{Q}\big(\sqrt{6 + 2\sqrt{7}}\big).$$

Let χ_i be the element of C such that $\chi_i^*(\mathfrak{a}_j) = (-1)^{\delta_{ij}}$, where δ_{ij} is the Kronecker delta. Then

$$C = \langle \chi_1, \chi_2 \rangle = \{\mathit{1}, \chi_1, \chi_2, \chi_1\chi_2\} \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}.$$

For $E = \{e^{[1]}, e^{[2]}, e^{[3]}, e^{[4]}, e^{[5]}\}$, we have

- $\{e^{[j]}\}$ is a C-orbit for $j = 1, 4, 5$.
- $\{e^{[2]}, e^{[3]}\}$ is a C-orbit. (Because we have

$$e^{[2]} = 2^{-1}\big(T(\mathfrak{o}) + 2^{-1}T(\mathfrak{a}_2)\big)\varepsilon, \quad e^{[3]} = 2^{-1}\big(T(\mathfrak{o}) - 2^{-1}T(\mathfrak{a}_2)\big)\varepsilon$$

with $\varepsilon = e^{[2]} + e^{[3]}$ from the table, and hence we see that $(e^{[2]})^{\chi_2} = e^{[3]}$.)

Thus

$$\varepsilon_1 = e^{[1]}, \quad \varepsilon_2 = e^{[2]} + e^{[3]}, \quad \varepsilon_3 = e^{[4]}, \quad \varepsilon_4 = e^{[5]},$$

and hence

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$
$$\mathcal{H}_1 \cong \mathbf{Q}(\sqrt{3}, \sqrt{10}), \qquad \mathcal{H}_2 \cong \mathbf{Q}(\sqrt{6}) \oplus \mathbf{Q}(\sqrt{6}),$$
$$\mathcal{H}_3 \cong \mathbf{Q}(\sqrt{2}), \qquad \mathcal{H}_4 \cong \mathbf{Q}\big(\sqrt{6 + 2\sqrt{7}}\big).$$

9.1. We identify $\mathcal{H}_1$ with $\mathbf{Q}(\sqrt{3}, \sqrt{10})$. Then we have

$$C_1' = C, \quad C_1 = \{\mathit{1}\},$$

and hence

$$I_1' = \{[\mathfrak{o}]\}, \quad I_1 = Cl^+(F) \quad \text{(by duality)},$$
$$\mathcal{H}_1^+ = K_1^+ = \mathbf{Q}$$

(by (8.2) and (8.3)). Thus

$$K_1^+[T(\mathfrak{o})_1] = \mathbf{Q}, \qquad K_1^+[T(\mathfrak{a}_1)_1] = \mathbf{Q}(\sqrt{3}),$$
$$K_1^+[T(\mathfrak{a}_2)_1] = \mathbf{Q}(\sqrt{10}), \quad K_1^+[T(\mathfrak{a}_1\mathfrak{a}_2)_1] = \mathbf{Q}(\sqrt{30}).$$

For the extension $\mathcal{H}_1/K_1^+$, see Figure 1.

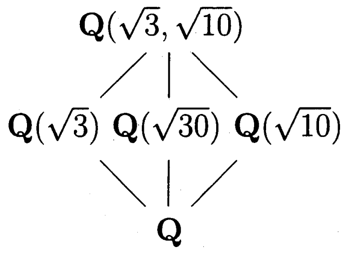

FIGURE 1. $\mathcal{H}_1 = \mathbf{Q}(\sqrt{3}, \sqrt{10})$

Suppose $T(\mathfrak{a})_1 \neq 0$. Then, by Theorem 7.1 and (8.1),

$$\begin{aligned}
\mathfrak{a} \in [\mathfrak{o}] &\iff \mathbf{Q}[T(\mathfrak{a})_1] = \mathbf{Q} &&\iff T(\mathfrak{a})_1 \in \mathbf{Q}^\times,\\
\mathfrak{a} \in [\mathfrak{a}_1] &\iff \mathbf{Q}[T(\mathfrak{a})_1] = \mathbf{Q}(\sqrt{3}) &&\iff T(\mathfrak{a})_1 \in \mathbf{Q}^\times\cdot\sqrt{3},\\
\mathfrak{a} \in [\mathfrak{a}_2] &\iff \mathbf{Q}[T(\mathfrak{a})_1] = \mathbf{Q}(\sqrt{10}) &&\iff T(\mathfrak{a})_1 \in \mathbf{Q}^\times\cdot\sqrt{10},\\
\mathfrak{a} \in [\mathfrak{a}_1\mathfrak{a}_2] &\iff \mathbf{Q}[T(\mathfrak{a})_1] = \mathbf{Q}(\sqrt{30}) &&\iff T(\mathfrak{a})_1 \in \mathbf{Q}^\times\cdot\sqrt{30}.
\end{aligned}$$

9.2. We identify $\mathcal{H}_2$ with $\mathbf{Q}(\sqrt{6}) \oplus \mathbf{Q}(\sqrt{6})$ such as $T(\mathfrak{a}_1)_2$ corresponds to $(\sqrt{6},\ \sqrt{6})$. Then we have

$$C_2' = \{\mathit{1}, \chi_1\}, \quad C_2 = \{\mathit{1}\},$$

and hence

$$I_2' = \{[\mathfrak{o}], [\mathfrak{a}_2]\}, \quad I_2 = Cl^+(F) \quad \text{(by duality)},$$
$$\mathcal{H}_2^+ = \mathbf{Q} \oplus \mathbf{Q}, \qquad K_2^+ = \mathbf{Q}\cdot 1_{\mathcal{H}_2}$$

(by (8.2), (8.3), and [H-O, Prop. 3.3]), where $1_{\mathcal{H}_2} := (1,\ 1)$. Put $\iota := (1,\ -1)$ and $\alpha := (\sqrt{6},\ \sqrt{6})$. Then $T(\mathfrak{a}_2)_2 = 2\iota$, $T(\mathfrak{a}_1)_2 = \alpha$, and

$$\begin{aligned}
K_2^+[T(\mathfrak{o})_2] &= \mathbf{Q}\cdot 1_{\mathcal{H}_2} = \{(a,\ a) \mid a \in \mathbf{Q}\},\\
K_2^+[T(\mathfrak{a}_2)_2] &= \mathbf{Q}[\iota] = \{(a,\ b) \mid a, b \in \mathbf{Q}\} = \mathbf{Q} \oplus \mathbf{Q},\\
K_2^+[T(\mathfrak{a}_1)_2] &= \mathbf{Q}[\alpha] = \{(a + b\sqrt{6},\ a + b\sqrt{6}) \mid a, b \in \mathbf{Q}\},\\
K_2^+[T(\mathfrak{a}_1\mathfrak{a}_2)_2] &= \mathbf{Q}[\alpha\iota] = \{(a + b\sqrt{6},\ a - b\sqrt{6}) \mid a, b \in \mathbf{Q}\}.
\end{aligned}$$

See Figure 2 for their relation. We note that $\mathbf{Q}[\alpha]$ and $\mathbf{Q}[\alpha\iota]$ are *distinct fields*, which are isomorphic to $\mathbf{Q}(\sqrt{6})$. (Note that $[\mathfrak{a}_1]\, I_2' = [\mathfrak{a}_1\mathfrak{a}_2]\, I_2'$.)

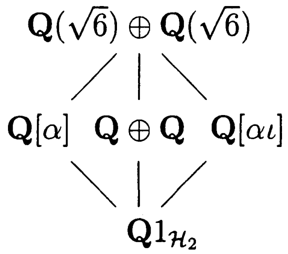

FIGURE 2. $\mathcal{H}_2 = \mathbf{Q}(\sqrt{6}) \oplus \mathbf{Q}(\sqrt{6})$

Suppose $T(\mathfrak{a})_2 \neq 0$. Then, by Theorem 7.1 and (8.1), we have

$$\begin{aligned}
\mathfrak{a} \in [\mathfrak{o}] \quad &\Longleftrightarrow \mathbf{Q}[T(\mathfrak{a})_2] = \mathbf{Q}\cdot 1_{\mathcal{H}_2} \\
&\Longleftrightarrow T(\mathfrak{a})_2 \in \mathbf{Q}^{\times}\cdot 1_{\mathcal{H}_2} = \{(a,\, a) \mid a \in \mathbf{Q}^{\times}\}, \\
\mathfrak{a} \in [\mathfrak{a}_2] \quad &\Longleftrightarrow \mathbf{Q}[T(\mathfrak{a})_2] = \mathbf{Q} \oplus \mathbf{Q} \\
&\Longleftrightarrow T(\mathfrak{a})_2 \in \mathbf{Q}^{\times}\cdot \iota = \{(a,\, -a) \mid a \in \mathbf{Q}^{\times}\}, \\
\mathfrak{a} \in [\mathfrak{a}_1] \quad &\Longleftrightarrow \mathbf{Q}[T(\mathfrak{a})_2] = \mathbf{Q}[\alpha] \\
&\Longleftrightarrow T(\mathfrak{a})_2 \in \mathbf{Q}^{\times}\cdot \alpha = \{(a\sqrt{6},\, a\sqrt{6}) \mid a \in \mathbf{Q}^{\times}\}, \\
\mathfrak{a} \in [\mathfrak{a}_1\mathfrak{a}_2] &\Longleftrightarrow \mathbf{Q}[T(\mathfrak{a})_2] = \mathbf{Q}[\alpha\iota] \\
&\Longleftrightarrow \ T(\mathfrak{a})_2 \in \mathbf{Q}^{\times}\cdot \alpha\iota = \{(a\sqrt{6},\, -a\sqrt{6}) \mid a \in \mathbf{Q}^{\times}\}.
\end{aligned}$$

9.3. We identify $\mathcal{H}_3$ with $\mathbf{Q}(\sqrt{2})$. Then

$$\begin{aligned}
&C_3' = C, \quad C_3 = \{\mathit{1}, \chi_2\}, \\
&I_3' = \{[\mathfrak{o}]\}, \quad I_3 = \{[\mathfrak{o}], [\mathfrak{a}_1]\}, \\
&\mathcal{H}_3^+ = K_3^+ = \mathbf{Q} \quad \text{(by (8.2) and (8.3))}.
\end{aligned}$$

Suppose $T(\mathfrak{a})_3 \neq 0$. Then $[\mathfrak{a}] \in \{[\mathfrak{o}], [\mathfrak{a}_1]\}$ by Theorem 7.1 (3). Moreover, by Theorem 7.1 and (8.1), we see that

$$\begin{aligned}
\mathfrak{a} \in [\mathfrak{o}] \ &\Longleftrightarrow \ \mathbf{Q}[T(\mathfrak{a})_3] = \mathbf{Q} \qquad \Longleftrightarrow \ T(\mathfrak{a})_3 \in \mathbf{Q}^{\times}, \\
\mathfrak{a} \in [\mathfrak{a}_1] \ &\Longleftrightarrow \ \mathbf{Q}[T(\mathfrak{a})_3] = \mathbf{Q}(\sqrt{2}) \ \Longleftrightarrow \ T(\mathfrak{a})_3 \in \mathbf{Q}^{\times}\cdot\sqrt{2}.
\end{aligned}$$

9.4. We identify $\mathcal{H}_4$ with $\mathbf{Q}\big(\sqrt{6 + 2\sqrt{7}}\big)$. Then

$$\begin{aligned}
&C_4' = C, \quad C_4 = \{\mathit{1}, \chi_1\}, \\
&I_4' = \{[\mathfrak{o}]\}, \quad I_4 = \{[\mathfrak{o}], [\mathfrak{a}_2]\}, \\
&\mathcal{H}_4^+ = K_4^+ = \mathbf{Q}(\sqrt{7}) \quad \text{(by (8.2) and (8.3))}.
\end{aligned}$$

Suppose $T(\mathfrak{a})_4 \neq 0$. Then $[\mathfrak{a}] \in \{[\mathfrak{o}], [\mathfrak{a}_2]\}$ by Theorem 7.1 (3). Moreover, by Theorem 7.1 and (8.1), we see that

$$\begin{aligned} \mathfrak{a} \in [\mathfrak{o}] &\iff \mathbf{Q}(\sqrt{7})\,[T(\mathfrak{a})_4] = \mathbf{Q}(\sqrt{7}) \\ &\iff T(\mathfrak{a})_4 \in \mathbf{Q}(\sqrt{7})^{\times}, \\ \mathfrak{a} \in [\mathfrak{a}_2] &\iff \mathbf{Q}(\sqrt{7})\,[T(\mathfrak{a})_4] = \mathbf{Q}(\sqrt{6+2\sqrt{7}}) \\ &\iff T(\mathfrak{a})_4 \in \mathbf{Q}(\sqrt{7})^{\times}\cdot\sqrt{6+2\sqrt{7}}. \end{aligned}$$

10. Table

Let $F = \mathbf{Q}(\sqrt{42})$. Then the class number of F in the narrow sense is 4. (The class number of F in the wide sense is 2.) Put $\theta := \sqrt{42}$, $\mathfrak{a}_1 := [2, \theta]$, and $\mathfrak{a}_2 := [3, \theta]$. Then $Cl^+(F)/Cl^+(F)^2 = Cl^+(F) = \langle [\mathfrak{a}_1], [\mathfrak{a}_2] \rangle \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$.

Let $k = (2,2)$, $\mathfrak{c} = \mathfrak{o}$, and $\psi = \mathit{1}$. Then $\mathcal{S} = \mathcal{S}^0_{(2,2)}(\mathfrak{o}, \mathit{1}) = \mathcal{S}_{(2,2)}(\mathfrak{o}, \mathit{1})$ and $\dim_{\mathbf{C}} \mathcal{S} = 14$.

Table 1: The characteristic polynomials of $T(\mathfrak{p})$ on $\mathcal{S}_{(2,2)}(\mathfrak{o}, \mathit{1})$ for $F = \mathbf{Q}(\sqrt{42})$

$\mathfrak{p}$	class	$\Phi^{[1]}_{\mathfrak{p}}(X)$	$\Phi^{[2]}_{\mathfrak{p}}(X)$	$\Phi^{[3]}_{\mathfrak{p}}(X)$	$\Phi^{[4]}_{\mathfrak{p}}(X)$	$\Phi^{[5]}_{\mathfrak{p}}(X)$
$[2,\theta]\ (=\mathfrak{a}_1)$	$(1,0)$	$(X^2-3)^2$	X^2-6	X^2-6	X^2-8	X^4
$[3,\theta]\ (=\mathfrak{a}_2)$	$(0,1)$	$(X^2-10)^2$	$(X-2)^2$	$(X+2)^2$	X^2	X^4-12X^2+8
$(5)_F$	$(0,0)$	$(X-8)^4$	$(X-8)^2$	$(X-8)^2$	$(X+2)^2$	$(X^2-28)^2$
$[7,\theta]$	$(0,0)$	$(X-2)^4$	$(X+4)^2$	$(X+4)^2$	$(X-2)^2$	$(X^2-28)^2$
$[11,\theta+8]$	$(1,0)$	$(X^2-12)^2$	X^2-6	X^2-6	X^2-32	X^4
$[13,\theta+9]$	$(0,1)$	$(X^2-10)^2$	$(X+2)^2$	$(X-2)^2$	X^2	X^4-52X^2+648
$[17,\theta+12]$	$(1,1)$	X^4	X^2-54	X^2-54	X^2	X^4
$[19,\theta+17]$	$(0,1)$	$(X^2-10)^2$	$(X-4)^2$	$(X+4)^2$	X^2	X^4-76X^2+72
$[29,\theta+19]$	$(1,0)$	$(X^2-48)^2$	X^2-24	X^2-24	X^2-8	X^4
$[41,\theta+1]$	$(1,1)$	$(X^2-120)^2$	X^2-6	X^2-6	X^2	X^4

In Table 1,

- $\mathfrak{p}$ indicates a prime ideal of F.
- $(p)_F := p\mathfrak{o}$.
- For fixed generators $[\mathfrak{a}_1]$, $[\mathfrak{a}_2]$ of $Cl^+(F)$, the "class" (i_1, i_2) for $\mathfrak{p}$ indicates $[\mathfrak{p}] = [\mathfrak{a}_1]^{i_1}[\mathfrak{a}_2]^{i_2}$.
- $\Phi^{[j]}_{\mathfrak{p}}(X)$ indicates the characteristic polynomial of the Hecke operator $T(\mathfrak{p})$ on $e^{[j]}\mathcal{S}$. Here $e^{[1]}, \cdots, e^{[5]}$ is all primitive idempotents of the Hecke algebra $\mathcal{H}$ for $\mathcal{S}$ with coefficients in $\mathbf{Q}$.

We note that, put

$$K^{[j]} := \mathcal{H}\, e^{[j]},$$

then $K^{[j]}$ is a field,

$$\mathcal{S} = e^{[1]}\mathcal{S} \oplus \cdots \oplus e^{[5]}\mathcal{S}, \qquad \mathcal{H} = K^{[1]} \oplus \cdots \oplus K^{[5]},$$

and $\Phi_{\mathfrak{p}}(X) := \Phi_{\mathfrak{p}}^{[1]}(X) \cdots \Phi_{\mathfrak{p}}^{[5]}(X)$ is the characteristic polynomial of $T(\mathfrak{p})$ on $\mathcal{S}$.

The table above was given by Y. Hiraoka by computing the trace formula (see [O, §2] for the formula), and PARI/GP ([P]) was used to compute some factors of the formula. This table is used in §9 above.

11. Information about another examples and tables

Some examles and tables in the following cases (with $\mathfrak{c} = \mathfrak{o}$, $\psi = \mathit{1}$) are given in [H-O, §6 and §7].

- $F = \mathbf{Q}(\sqrt{30})$, $k = (2,2)$
- $F = \mathbf{Q}(\sqrt{35})$, $k = (2,2)$
- $F = \mathbf{Q}(\sqrt{39})$, $k = (2,2)$

($Cl^+(F)/Cl^+(F)^2 \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ in these case.)

- $F = \mathbf{Q}(\theta)$ with $\theta^3 - 12\theta + 10 = 0$, $k = (2,2,2)$
- $F = \mathbf{Q}(\theta)$ with $\theta^3 - 10\theta - 6 = 0$, $k = (2,2,2)$
- $F = \mathbf{Q}(\theta)$ with $\theta^3 - 4\theta - 1 = 0$, $k = (4,4,4)$

(F are totally real *non-abelian* cubic fields and $Cl^+(F)/Cl^+(F)^2 \cong \mathbf{Z}/2\mathbf{Z}$ in these case.)

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