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New Forms for $GU(2,2)$

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1 Introduction

In 2007 [5], Roberts and Schmidt established local New form theory for irreducible admissible generic representation of PGSp(4) ($\simeq$ SO(3,2)) over nonarchimedean field. Following their result, we establish local New form theory for irreducible admissible generic *supercuspidal* representation of PGU(2,2) ($\simeq$ SO(4,2)) over nonarchimedean local field. To begin with, we recall Roberts-Schmidt's theory for PGSp(4). Let $\mathbf{F}$ be a nonarchimedean local field with ring of integers $\mathfrak{o}$. Let

$$\mathrm{GSp}_4(\mathbf{F}) = \{g \in \mathrm{GL}_4(\mathbf{F}) \mid {}^t gJg = aJ, \text{ for some } a \in \mathbf{F}^\times\},$$

where

$$J = \begin{bmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

The mapping $g \to a$ defines a homomorphism to $\mathbf{F}^\times$, called similitude character, and we will write $a = \mu(g)$. Let ψ be a nontrivial additive character on $\mathbf{F}$. The Whittaker functions W on $\mathrm{GSp}_4(\mathbf{F})$ with respect to ψ are smooth functions such that

$$W(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix} g) = \psi(x+y)W(g).$$

We denote by $\mathscr{W}_\psi$ the space of these W. Let $(\pi, V) \subset \mathscr{W}_\psi$ be an irreducible admissible representation of $\mathrm{GSp}_4(\mathbf{F})$ with trivial central character. For $W \in V$, define the Novodovorsky zeta integral

$$Z_N(s,W) = \int_{\mathbf{F}^\times}\int_{\mathbf{F}} W(\begin{bmatrix} a & & & \\ & a & & \\ & x & 1 & \\ & & & 1 \end{bmatrix})dxd^\times a$$

with $s \in \mathbb{C}$, where dx and $d^\times a$ are Haar measures such that $\mathrm{vol}(\mathfrak{o}) = 1$ and $\mathrm{vol}(\mathfrak{o}^\times)$, respectively. Let $\mathfrak{p}$ denote the prime ideal of $\mathfrak{o}$ and $q = |\mathfrak{o}/\mathfrak{p}|$. Let $I(\pi)$ be the $\mathbb{C}$-subspace of $\mathbb{C}[X, X^{-1}]$ spanned by these $Z_N(s,W)$ with $X = q^{-s}$. Since $\mathbb{C}[X,X^{-1}]$ is a principal ideal domain, $I(\pi)$ admits a generator $P(X)$ such that $P(1) = 1$. Denote by $L(s,\pi)$ the generator. Then, there exists $\varepsilon(s,\pi,\psi) = \varepsilon q^{-N_\pi(s-\frac{1}{2})}$ with $N_\pi \in \mathbb{Z}, \varepsilon \in \{\pm 1\}$ such that for arbitrary $W \in V$, it holds that

$$\frac{Z_N(1-s,\pi(u)W)}{L(1-s,\pi)} = \varepsilon(s,\pi,\psi)\frac{Z_N(s,W)}{L(s,\pi)}, \tag{1.1}$$

where

$$u = \begin{bmatrix} & & -1 & \\ & & & -1 \\ 1 & & & \\ & 1 & & \end{bmatrix}.$$

(1.1) is the functional equation of π. The monomial $\varepsilon(s,\pi,\psi)$ of X is called ε-factor of π, and varies according to ψ. For a positive integer n, define the compact subgroup of $\mathrm{GSp}_4(\mathbf{F})$ by

$$K(n) = \{g \in \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-n} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{p}^n & \mathfrak{o} \end{bmatrix} \mid \mu(g) \in \mathfrak{o}^\times\},$$

and call the paramodular group of level n. Now, suppose that $Ker(\psi) = \mathfrak{o}$. Then, N_π is positive. Roberts and Schmidt's *generic main theory* says that there exists uniquely $W \in V$ such that

- W is $K(N_\pi)$-fixed.
- $W(1) = 1$ and $Z(s,W) = (1-q^{-1})L(s,\pi)$.

Inspired by their theory, we obtained a similar result for GU(2,2).

2 Functional equation

Let $\mathbf{F}$ be a field and $\mathbf{E} = \mathbf{F}(\sqrt{d})$ be a quadratic extension of $\mathbf{F}$ with $d \in \mathfrak{o}^\times \setminus (\mathfrak{o}^\times)^2$. Let ψ be a nontrivial additive character on $\mathbf{F}$ and define the additive character $\psi_{\mathbf{E}}$ on $\mathbf{E}$ by $\psi_E(a + b\sqrt{d}) = \psi(b)$. Let $\mathrm{Gal}(\mathbf{E}/\mathbf{F}) = \{1, c\}$, and

$$\mathbf{G} = \mathrm{GU}_{2,2}(\mathbf{E}) = \{g \in \mathrm{GL}_4(\mathbf{E}) \mid {}^t g^c J g = \mu(g) J, \mu(g) \in \mathbf{F}^\times\}.$$

We say a smooth function W on $\mathbf{G}$ is a Whittaker functions with respect to ψ, if

$$W(\begin{bmatrix} 1 & x & * & * \\ & 1 & y & * \\ & & 1 & -x^c \\ & & & 1 \end{bmatrix} g) = \psi_{\mathbf{E}}(x)\psi(y)W(g),$$

and denote by $\mathscr{W}_\psi$ the space of such functions. First of all, we should give a functional equation of irreducible admissible generic representation of $\mathbf{G}$ over a local nonarchimedena filed. However, before that, let us consider the global situation. Let $\mathbf{F}$ be a global number field and $\mathbf{E}$ be a quadratic extension of $\mathbf{F}$. Furusawa and Morimoto [3] defined a zeta integral for global Whittaker function W with respect to $\psi_{\mathbf{E}}$ (here $\psi_{\mathbf{E}}$ is an additive character on $\mathbb{A}_{\mathbf{E}}$), and Schwartz-Bruhat function ϕ of $\mathbb{A}_{\mathbf{F}}^2$ by

$$Z(s, W, \phi) = \int_{GL_2(\mathbb{A}_F)} \int_{\mathbb{A}_F} W(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} i(h))\phi([0,1]h)|\det(h)|^s dxdh$$

where i is the embedding such that

$$i(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{bmatrix}.$$

When the global Whittaker function W is obtained by taking a integral of an automorphic cusp form, the zeta integral is written by this cusp form and the Eisenstein series $E(h, \phi, s)$ (defined in the standard way). Therefore, by using the well-known formula $E(h, \phi, s) = E({}^t h^{-1}, \phi^\sharp, 1-s)$, we obtain a imcomplete global functional equation

$$Z(s, W, \phi) = Z(1-s, W', \phi^\sharp) \tag{2.1}$$

where $\phi^\sharp$ is the Fourier transformation of ϕ with respect to ψ, and

$$W'(g) = W(\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} {}^t g^{-1} \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}).$$

Now, let $\mathbf{F}$ be a local nonarchimedean field. Let (π, V) be an irreducible, admissible generic representation of $\mathbf{G}$, i.e., $V \subset \mathscr{W}_\psi$. For $x, y \in \mathbf{E}$, and $z \in \mathbf{F}$, let

$$\mathbf{n}(x,y,z) = \begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x^c \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & y & z \\ & 1 & & y^c \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

This is an element of $\mathbf{G}$. Let U denote the subgroup consisting of these elements. Let $U_{\mathbf{F}} = \{\mathbf{n}(x,y,z) \mid x,y,z \in \mathbf{F}\}$. We define *'Klingen type'* parabolic?? subgroup

$$Q' = \mathbf{E}^\times\{\begin{bmatrix} \det(g) & & \\ & g & \\ & & 1 \end{bmatrix} \mid g \in \mathrm{GL}_2(\mathbf{F})\}U \subset \mathbf{G}.$$

Then $U_{\mathbf{F}}$ is a normal subgroup of Q'. The following sets U_3, P_3 are subgroups of $\mathrm{GL}_3(\mathbf{F})$:

$$U_3 = \{\begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix} \in \mathrm{GL}_3(\mathbf{F})\} \subset P_3 = \{\begin{bmatrix} * & * & * \\ * & * & * \\ & & 1 \end{bmatrix} \in \mathrm{GL}_3(\mathbf{F})\}.$$

Then, we have

$$Q'/U_F \simeq \mathbf{E}^\times \times P_3(\mathbf{F})$$

via the homomorphism $i : Q' \to P_3$ defined by

$$i\left(t\begin{bmatrix} \det(g) & & \\ & g & \\ & & 1 \end{bmatrix}\mathbf{n}(x,y,z)\right) = \left(t, \begin{bmatrix} g & \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & & \frac{y-y^c}{2\sqrt{d}} \\ & 1 & \frac{x-x^c}{2\sqrt{d}} \\ & & 1 \end{bmatrix}\right). \tag{2.2}$$

Put

$$\overline{V} = V/\langle \pi(u)v - v \mid u \in U_F \rangle.$$

Via (2.2), $\overline{V}$ is a $E^\times \times P_3$-module. Denote by $\overline{\pi}$ the action of $E^\times \times P_3$ and by $\overline{v}$ vectors of $\overline{V}$. For $t \in \mathbf{E}^\times$,

$$\overline{\pi}(t)\overline{v} = \omega_\pi(t)\overline{v}.$$

According to the P_n-theory of Bernstein-Zelevinskii [1], $\overline{V}$ has the following filtration of P_3-subspaces:

$$V_2 \subset V_1 \subset \overline{V}. \tag{2.3}$$

Define the character ψ_{U_3} on U_3 by

$$\psi_{U_3}(\begin{bmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{bmatrix}) = \psi(x_1 + x_3).$$

Put

$$\overline{V}_{\psi_{U_3}} = \overline{V}/\langle \psi_{U_3}(u')\overline{v} - \overline{\pi}(u')\overline{v} \mid u' \in U_3 \rangle.$$

Then, V_2 is a direct sum of $\dim_{\mathbb{C}} \overline{V}_{\psi_{U_3}}$ copies of an irreducible representation (c.f. p. 49 of loc. cite., this irreducible representation is denoted by $\tau^0_{P\cdot}$.). However,

$\dim_{\mathbf{C}} \overline{V}_{\psi_{U_3}} = 1$ by the uniqueness of Whittaker models of V (c.f. Shalika [6]). Thus, V_2 is irreducible.

Now, we are going to give a functional equation of π. Set

$$\begin{aligned} \mathbf{S} &= \{g = \begin{bmatrix} h & \\ & h' \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} \in \mathbf{G} \mid h \in \mathrm{GL}_2(\mathbf{F})\}, \\ \mathbf{T} &= \{g \in \mathbf{S} \mid h \in P_2\}, \end{aligned} \tag{2.4}$$

where

$$h' = \det(h) \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} {}^t h^{-1} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}.$$

These sets are subgroup of $\mathbf{G}$, and $\mathbf{S}$ is called Shalika subgroup. For $s \in \mathbf{C}$, define the 1-dimensional representation $\nu_s\psi$ of $\mathbf{S}$ by

$$\nu_s\psi(\begin{bmatrix} h & \\ & h' \end{bmatrix} \begin{bmatrix} 1 & X \\ & 1 \end{bmatrix}) = |\det(h)|^{2s} \psi_{\mathbf{E}}(\frac{1}{2}\mathrm{tr}(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} X)).$$

Let $\mathbf{S}$ act on $\phi \in \mathscr{S}(\mathbf{F}^2)$ by

$$\begin{bmatrix} h & x \\ & h' \end{bmatrix} \cdot \phi(z) = \phi(zh).$$

Lemma 2.1. *Except for finitely many $s \in \mathbf{C}$,*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathscr{S}_0(\mathbf{F}^2), \nu_s\psi) \leq \dim_{\mathbf{C}} \mathrm{Hom}_{\mathbf{T}}(\pi, \nu_{s-1}\psi).$$

Proof. It suffices to construct a homomorphism $J : \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathscr{S}(\mathbf{F}^2), \nu_s\psi) \to \mathrm{Hom}_{\mathbf{T}}(\pi, \nu_{s-1}\psi)$ so that J is injective except for finitely many $s \in \mathbf{C}$. Take $\sigma \in \mathrm{Hom}_{\mathbf{S}}(\pi, \nu_s\psi)$. By the definition of $\nu_s\psi$,

$$\sigma(\pi(t)v) = \omega_\pi(t)\sigma(v) = |t|^{4s}\sigma(v)$$

for $t \in \mathbf{F}^\times$ and $v \in V$. Hence, except for finitely many $s \in \mathbf{C}$,

$$\mathrm{Hom}_{\mathbf{S}}(\pi, \nu_s\psi) = \{0\}. \tag{2.5}$$

By restriction of $\mathscr{S}(\mathbf{F}^2)$ to its $\mathbf{S}$-subspace $\mathscr{S}_0(\mathbf{F}^2) = \{\phi \in \mathscr{S}(\mathbf{F}^2) \mid \phi([0,0]) = 0\}$, we have a homomorphism

$$J' : \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathscr{S}(\mathbf{F}^2), \nu_s\psi) \to \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathscr{S}_0(\mathbf{F}^2), \nu_s\psi).$$

However, the kernel of J' is embedded into $\mathrm{Hom}_{\mathbf{S}}(\pi, \nu_s\psi)$ since

$$\pi \otimes_{\mathbf{C}} \mathscr{S}(\mathbf{F}^2) / \pi \otimes_{\mathbf{C}} \mathscr{S}_0(\mathbf{F}^2) \simeq \pi$$

as $\mathbf{S}$-modules via the mapping: $v \otimes \phi \to \phi([0,0])v \in \pi$. Therefore, by (2.5), except for finitely many s, the kernel of J' is $\{0\}$ and J' is injective. The quotient $P_2\backslash\mathrm{GL}_2(\mathbf{F})$ acts on $\mathbf{F}^2 \setminus \{[0,0]\}$ faithfully, and $\mathbf{T}\backslash\mathbf{S} \simeq P_2\backslash\mathrm{GL}_2(\mathbf{F})$. So, the

S-space $\mathscr{S}_0(\mathbf{F}^2)$ is isomorphic to $\mathrm{ind}_{\mathbf{T}}^{\mathbf{S}}(\mathbf{1})$, the compactly supported induced representation from the trivial representation $\mathbf{1}$ of $\mathbf{T}$, via the correspondence: $\phi \mapsto \phi([0,1]h) \in \mathrm{ind}_{\mathbf{T}}^{\mathbf{S}}(\mathbf{1})$ where we write we write an element of $\mathbf{S}$ as in (2.4). So,

$$\mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathscr{S}_0(\mathbf{F}^2), \nu_s\psi) \simeq \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathrm{ind}_{\mathbf{T}}^{\mathbf{S}}(\mathbf{1}), \nu_s\psi).$$

However, in general, for representations τ, ξ of a group G, and one-dimensional representation χ, it holds that

$$\begin{aligned} \mathrm{Hom}_G(\tau \otimes_{\mathbf{C}} \xi, \chi) &\simeq \mathrm{Hom}_G(\tau, \mathrm{Hom}_{\mathbf{C}}(\xi, \chi)) \\ &\simeq \mathrm{Hom}_G(\tau, \mathrm{Hom}_{\mathbf{C}}(\chi^{-1}\xi, \mathbf{C})), \end{aligned}$$

where $g \in G$ acts on $f \in \mathrm{Hom}_{\mathbf{C}}(\xi, \chi)$ by $g \cdot f(t) = \chi(g) f(\xi(g^{-1})t)$. Therefore,

$$\begin{aligned} \mathrm{Hom}_{\mathbf{S}}(\pi \otimes_{\mathbf{C}} \mathrm{ind}_{\mathbf{T}}^{\mathbf{S}}(\mathbf{1}), \nu_s\psi) &\simeq \mathrm{Hom}_{\mathbf{S}}\left(\pi, \mathrm{Hom}_{\mathbf{C}}\left(\mathrm{ind}_{\mathbf{T}}^{\mathbf{S}}((\nu_s\psi)^{-1}), \mathbf{C}\right)\right) \\ &\simeq \mathrm{Hom}_{\mathbf{S}}(\pi, \mathrm{Ind}_{\mathbf{T}}^{\mathbf{S}}(\nu_{s-1}\psi)) \\ &\simeq \mathrm{Hom}_{\mathbf{T}}(\pi, \nu_{s-1}\psi). \end{aligned}$$

The second isomorphism is by 2.25 c) of [1], and the last is by the Frobenius reciprocity law 2.29 of loc. cite.. This completes the proof. □

Let

$$\pi^{\vee}(g) = \pi(\mu(g)^t g^{-1}).$$

Proposition 2.2. *π is equivalent to $\pi^{\vee}$.*

Proof. Consider the action γ of G on G defined by $\gamma(g)g_1 = g^{-1}g_1g$, and the homeomorphism $\sigma : g \to \mu(g)^t g^{-1}$ in $\mathbf{G}$. Then, all the conditions of the second Theorem in p.91 of [4] are satisfied. Indeed, $\gamma(g)\sigma = \sigma\gamma({}^tg^{-1})$, σ^2 is identity, and $\sigma(g) = J^{-1}gJ$. Therefore, the character $tr(\pi)$ of π (see 2.17 of [1] for the definition of $tr(\pi)$) coincides with $tr(\pi^{\vee})$. By Corollary 2.20 of loc. cite, π is equivalent to $\pi^{\vee}$. □

For a local Whittaker function $W \in \mathscr{W}_{\psi}$, and $\phi \in \mathscr{S}(\mathbf{F}^2)$, we define the zeta integral by

$$Z(s, W, \phi) = \int_{N_2\backslash \mathrm{GL}_2(\mathbf{F})} \int_{\mathbf{F}} W(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & x & 1 & \\ & & & 1 \end{bmatrix} i(h))\phi([0,1]h)|\det(h)|^s dx dh.$$

A common argument shows this integral is absolutely convergent when $\Re(s)$ is sufficiently large. Define $L(s, \pi)$ in the same way as in the GSp(4) case. Let $\mathbf{S}'$ (resp. $\mathbf{T}'$) be the conjugation of $\mathbf{S}$ (resp. $\mathbf{T}$) by

$$\begin{bmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}.$$

Theorem 2.3. *There exists a function* $\varepsilon(s,\pi,\psi)$ *such that*

$$\frac{Z(1-s,W',\phi^\sharp)}{L(1-s,\pi)} = \varepsilon(s,\pi,\psi)\frac{Z(s,W,\phi)}{L(s,\pi)} \tag{2.6}$$

for arbitrary $W \in V$ *and* $\phi \in \mathscr{S}(\mathbf{F}^2)$. $\varepsilon(s,\pi,\psi)$ *is in a form of* $\varepsilon q^{-N_\pi(s-1/2)}$ *with* $\varepsilon \in \{\pm 1\}, N_\pi \in \mathbb{Z}$.

Proof. Consider the functionals on $\pi \otimes_{\mathbb{C}} \mathscr{S}(\mathbf{F}^2)$ sending $W \otimes \phi$ to $\frac{Z(s,W,\phi)}{L(s,\pi)}$ and to $\frac{Z(1-s,W',\phi^\sharp)}{L(1-s,\pi)}$. Both these functionals belong to $\mathrm{Hom}_{\mathbf{S}'}(\pi \otimes_{\mathbb{C}} \mathscr{S}(\mathbf{F}^2), \nu_s\psi)$. First, we will show that, except for finitely many s,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{S}'}(\pi \otimes_{\mathbb{C}} \mathscr{S}(\mathbf{F}^2), \nu_s\psi) \leq 1. \tag{2.7}$$

By Lemma 2.1, it suffices to show $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbf{T}'}(\pi,\nu_s\psi) \leq 1$, except for finitely many s. Take a $\Lambda \in \mathrm{Hom}_{\mathbf{T}'}(\pi,\nu_s\psi)$. For $t \in \mathbf{F}^\times, x \in \mathbf{E}, z \in \mathbf{F}$,

$$\Lambda(\pi(\mathbf{n}(x,y,z))v) = \psi_E(x)\Lambda(v),$$

$$\Lambda(\pi(\begin{bmatrix} t & & & \\ & t & & \\ & * & 1 & \\ & & & 1 \end{bmatrix})v) = \nu_s(t)\Lambda(v).$$

Via (2.2), Λ corresponds to the functional λ on the P_3-space $\overline{V}$, such that

$$\lambda(\begin{bmatrix} 1 & & \\ & 1 & z \\ & & 1 \end{bmatrix}v) = \psi(z)\lambda(v),$$

$$\lambda(\begin{bmatrix} t & & \\ * & 1 & \\ & & 1 \end{bmatrix}v) = \nu_s(t)\lambda(v).$$

The proof for Proposition 2.5.7.of [5] says that, except for finitely many s, the space $\{\lambda\}$ is at most one-dimensional. (see also Lemma 2.5.4., 2.5.5., and 2.5.6.of loc.cite.) Thus, (2.7) is showed. Therefore, there is a function $\varepsilon(s,\pi,\psi)$ depending only on s,π,ψ such that (2.6) holds except for finitely many s. By definition, there is a finite set of pairs $\{(W_i,\phi_i)\}$ such that $L(s,\pi) = \sum_i Z(s,W_i,\phi_i)$. By (2.6),

$$\varepsilon(s,\pi,\psi) = \frac{\sum_i Z(1-s,W_i',\phi_i^\sharp)}{L(1-s,\pi)} \in \mathbb{C}[q^{-s},q^s].$$

(Each W_i' belongs to V.) Therefore, we may write $\varepsilon(s,\pi,\psi) = R(q^{-s})q^{-Ms}$ by some $R[X] \in \mathbb{C}[X]$ and $M \in \mathbb{Z}$. From (2.6) and the fact that $(W')' = W$, $(\phi^\sharp)^\sharp(z) = \phi(-z)$, it follows that

$$\varepsilon(s,\pi,\psi)\varepsilon(1-s,\pi,\psi) = 1.$$

Therefore, R has no zeros, and is a monomial. Now the assertion follows immediately. □

3 D-paramodular vectors

For a nonnegative integer m, set

$$\mathfrak{O}_m = \mathfrak{o} + \mathfrak{p}^m \sqrt{d}$$

and

$$K_d(2m) = \{g \in \mathbf{G} \cap \begin{bmatrix} \mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m & \mathfrak{p}^{-2m}\mathfrak{O}_m \\ \mathfrak{p}^{m}\mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m \\ \mathfrak{p}^{m}\mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m \\ \mathfrak{p}^{2m}\mathfrak{O}_m & \mathfrak{p}^{m}\mathfrak{O}_m & \mathfrak{p}^{m}\mathfrak{O}_m & \mathfrak{O}_m \end{bmatrix} \mid \mu(g) \in \mathfrak{o}^\times\},$$

$$K_d(2m+1) = \{g \in \mathbf{G} \cap \begin{bmatrix} \mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m & \mathfrak{p}^{-2m-1}\mathfrak{O}_m \\ \mathfrak{p}^{m+1}\mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m \\ \mathfrak{p}^{m+1}\mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{O}_m & \mathfrak{p}^{-m}\mathfrak{O}_m \\ \mathfrak{p}^{2m+1}\mathfrak{O}_m & \mathfrak{p}^{m+1}\mathfrak{O}_m & \mathfrak{p}^{m+1}\mathfrak{O}_m & \mathfrak{O}_m \end{bmatrix} \mid \mu(g) \in \mathfrak{o}^\times\}.$$

$\mathfrak{O}_m$ is a subring of the ring of integers of $\mathbf{E}$, and $K_d(n)$ is a compact subgroup of $\mathbf{G}$. We call $K_d(n)$ the D-paramodular group of level n. Let ϖ be a generator of $\mathfrak{p}$. The followings are elements of $\mathbf{G}$:

$$t_n = \begin{bmatrix} & & & -\varpi^{-n} \\ & 1 & & \\ & & 1 & \\ \varpi^n & & & \end{bmatrix},$$

$$u_{2m} = \begin{bmatrix} & -\varpi^{-m} & & \\ \varpi^m & & & \\ & & & \varpi^{-m} \\ & & -\varpi^m & \end{bmatrix}, v_{2m} = \begin{bmatrix} & & -\varpi^{-m} & \\ & & & -\varpi^{-m} \\ \varpi^m & & & \\ & \varpi^m & & \end{bmatrix},$$

$$u_{2m+1} = \begin{bmatrix} & -\varpi^{-m} & & \\ \varpi^{m+1} & & & \\ & & & \varpi^{-m} \\ & & -\varpi^{m+1} & \end{bmatrix}, v_{2m+1} = \begin{bmatrix} & & -\varpi^{-m} & \\ & & & -\varpi^{-m} \\ \varpi^{m+1} & & & \\ & \varpi^{m+1} & & \end{bmatrix}.$$

Note that $t_{2m}, u_{2m}, v_{2m} \in K_d(n)$. Note that $t_{2m+1} \in K_d(2m+1), u_{2m+1}, v_{2m+1} \notin K_d(2m+1)$, however, u_{2m+1}, v_{2m+1} are normalizers of $K_d(2m+1)$. We call $v \in V$ a D-paramodular vector if v is invariant under some $K_d(n)$, and denote by $V(n)$ the subspace consisting of $K_d(n)$-invariant vectors. Similar to the argument of [5], we may find a $W \in V$ fixed by a *'Klingen-type'* compact subgroup such that $W(1) \neq 0$, and construct a D-paramodular form $W' = \int_{K_d(n)} \pi(k)W dk$ for some n, such that $W'(1) \neq 0$. So,

$$V(n) \neq \{0\}$$

for some n. When $n = 2m+1$, the subspace $V(2m+1)$ is decomposed to the direct sum $V(2m+1)_+ + V(2m+1)_-$ with

$$V(2m+1)_\varepsilon = \{v \in V(2m+1) \mid \pi(u_{2m+1})v = \varepsilon v\}.$$

Note that $\pi(v_{2m+1})v = \pi(u_{2m+1})v$ since $t_{2m+1}^{-1}u_{2m+1}t_{2m+1} = v_{2m+1}$. Set

$$\eta = \begin{bmatrix} \varpi^{-1} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{bmatrix}.$$

It is easy to see that

$$\pi(\eta) : V(n) \to V(n+2), V(2m+1)_\varepsilon \to V(2m+3)_\varepsilon.$$

Define

$$Z_d(s, W) = \int_{\mathbf{F}^\times} W(i(\begin{bmatrix} a & \\ & 1 \end{bmatrix}))|a|^{s-2}d^\times a.$$

Similar to the GSp(4)-case, we have

Proposition 3.1 (η-principle). *Let $m \geq 1$. Suppose that $W \in V(2m)$ (resp. $W \in V(2m+1)_\varepsilon$). If $Z_d(s, W) = 0$, then there exists a $W' \in V(2m-2)$ (resp. $W \in V(2m-1)_\varepsilon$) such that $\pi(\eta)W' = W$.*

4 New forms

Suppose that (π, V) is supercuspidal. The argument of [2] for L-function of a representation of GL(2) × GL(2) works for the GU(2,2) case, and we conclude $L(s, \pi)$ is one of following forms.

i) $L(s, \pi) = (1 - q^{-s})^{-1}$. We say π is distinguished.

ii) $L(s, \pi) = (1 + q^{-s})^{-1}$. We say π is quasi-distinguished.

iii) $L(s, \pi) = (1 - q^{-2s})^{-1}$. We say π is dual-distinguished.

iv) $L(s, \pi) = 1$. We say π is nondistinguished.

Define two Hecke operators for $V(n)$ by

$$T_{1,0} = \pi(K_d(n)\eta^{-1}K_d(n)),\ T_{0,1} = \pi(K_d(n)i(\begin{bmatrix} \varpi & \\ & 1 \end{bmatrix})K_d(n)),$$

which are self-adjoint. But, note that $T_{1,0}$ doesnot act on $V(2m+1)_\varepsilon$. If $V(M_\pi) \neq \{0\}$ and $V(n) = \{0\}$ for all $n < M_\pi$, then we say M_π is the minimal level of π. For $W \in V(M_\pi)$, we obtain a recursion formula of $W(i(\begin{bmatrix} \varpi^n & \\ & 1 \end{bmatrix}))$ by using the Hecke operators. The η-principle, and this recursion formula, give rough one-dimensionality of $V(M_\pi)$ (in case that M_π is odd, the one-dimensionality may lost as below). Let

$$\phi_m = \mathrm{Ch}(\mathfrak{p}^m \oplus \mathfrak{o}) \in \mathscr{S}(\mathbf{F}^2).$$

Observing the form of $L(s, \pi)$ and the functional equation, we obtain

Theorem 4.1. *Suppose that* (π, V) *is supercuspidal. Then* M_π *is odd and* $N_\pi = M_\pi + 1$.

i) *When* π *is distinguished, the subspace* $V(M_\pi)_+$ *is one dimensional, and* $V(M_\pi)_- = \{0\}$. *For the unique* $W \in V(M_\pi)_+$ *such that* $W(1) = 1$,

$$Z(s, W, \phi_{\frac{N_\pi}{2}}) = \frac{q-1}{q+1} L(s, \pi).$$

ii) *When* π *is quasi-distinguished, the subspace* $V(M_\pi)_-$ *is one dimensional, and* $V(M_\pi)_+ = \{0\}$. *For the unique* $W \in V(M_\pi)_-$ *such that* $W(1) = 1$,

$$Z(s, W, \phi_{\frac{N_\pi}{2}}) = \frac{q-1}{q+1} L(s, \pi).$$

iii) *When* π *is dual-distinguished, both of the subspaces* $V(M_\pi)_+, V(M_\pi)_-$ *are one dimensional. For each unique* $W_\pm \in V(M_\pi)_\pm$ *such that* $W_\pm(1) = 1$,

$$Z(s, W_+ + W_-, \phi_{\frac{N_\pi}{2}}) = 2\frac{q-1}{q+1} L(s, \pi).$$

iv) *When* π *is nondistinguished, both of the subspaces* $V(M_\pi)_+, V(M_\pi)_-$ *are one dimensional. For each unique* $W_\pm \in V(M_\pi)_\pm$ *such that* $W_\pm(1) = 1$,

$$Z(s, W_\pm, \phi_{\frac{N_\pi}{2}}) = \frac{q-1}{q+1} L(s, \pi).$$

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