数理解析研究所講究録
第 1973 巻 2015 年 154-161

Quasimodular forms are p-adic modular forms

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Our aim is to indicate a rather elementary proof for the result mentioned in the title. Our main ingredients are our previous results on p-adic Siegel modular forms (joint work with S.Nagaoka), in particular our proof that certain holomorphic derivatives of modular forms are p-adic and Shimura's theory of nearly holomorphic functions. Our method differs much from that of T.Ichikawa [5], who considers similar questions in an algebro-geometric setting under the condition that the level is always bigger or equal to 3.

1 Nearly holomorphic functions and forms

1.1 Generalities

We recall here some results of Shimura in a language appropriate for us:
Let $\mathbb{H}_n$ be the Siegel upper half space of degree n with the usual action of the group $Sp(n,\mathbb{R})$, given by $(M,Z)\longmapsto M<Z>:=(AZ+B)(CZ+D)^{-1}$. For a polynomial representation $\rho: GL(n,\mathbb{C})\longrightarrow Aut(V)$ on a finite-dimensional vector space $V=V_{\rho}$ we define an action of $Sp(n,\mathbb{R})$ on V-valued functions on $\mathbb{H}_n$ by

$$(f,M)\longmapsto(f\mid_{\rho}M)(Z)=\rho(CZ+D)^{-1}f(M<Z>).$$

We choose the smallest nonnegative integer k such that $\rho=\det^{k}\otimes\rho_0$ with ρ_0 is still polynomial and we call this k the weight of ρ; if ρ itself is scalar-valued, we often write k instead of $\det^{k}$.
We denote by $\mathcal{N}_{\rho}^{\nu}$ the space of all V-valued nearly holomorphic functions on $\mathbb{H}_n$; these are the functions given as polynomials in the entries of Y^{-1} of total degree smaller or equal to ν with holomorphic V-valued functions as coefficients. The subscript ρ indicates that we equip this space with the $|_{\rho}$ - action of $Sp(n,\mathbb{R})$. For a congruence subgroup Γ we may now define the nearly holomorphic modular forms of weight ρ for Γ by

$$\mathcal{N}_{\rho}^{\nu}(\Gamma):=\{f\in\mathcal{N}_{\rho}^{\nu}\mid\forall\gamma\in\Gamma: f\mid_{\rho}\gamma=f\},$$

where for $n=1$ we have to impose the usual additional conditions in the cusps.

Note that for $\nu = 0$ we get the usual holomorphic Siegel modular forms of degree n, and automorphy factor ρ for Γ; we denote them by $M_\rho(\Gamma)$.
The "constant term" of a nearly holomorphic function f (free of entries of Y^{-1}) will always be denoted by f^0. Quasimodular forms have so far been mainly considered for degree 1 (see e.g. [10]).

Definition: *A quasimodular form (with automorphy factor ρ for Γ) is a holomorphic V_ρ-valued function g, which appears as the "constant term" in a nearly holomorphic modular form of weight ρ for Γ, i.e. $g = f^0$ for some nearly holomorphic modular form f.*

Examples of nearly holomorphic functions are obtained by applying certain differential operators to holomorphic (or nearly holomorphic) functions. These operators D are polynomials (with coefficients in $\mathbb{Q}$) in the holomorphic derivatives ∂_{ij} with coefficients depending on Y^{-1}; they act on V_ρ-valued functions and map them to $V_{\rho'}$ -valued functions and they are equivariant w.r.t. the action of $Sp(n, \mathbb{R})$,i.e.

$$D(f \mid_\rho M) = D(f) \mid_{\rho'} M \qquad (M \in Sp(,\mathbb{R}))\,.$$

We call these operators Maaß-Shimura differential operators. Sometimes we write $D = D(\rho, \rho')$ to indicate the change of the automorphy factor.
A version of Shimura's structure theorem tells us that under some condition, we can obtain all nearly holomorphic functions from holomorphic ones by applying such differential operators. For this, we need vector-valued functions even if we are just interested in the scalar-valued case:

Theorem: *For given degree ν and a polynomial representation ρ_0 (of weight 0) there exists k_0 such that for all weights $k \geq k_0$, all representations $\rho = \det^k \otimes \rho_0$ and all $f \in \mathcal{N}_\rho^\nu$ there exist polynomial representations ρ_i $(0 \leq i \leq \nu)$ and Maaß-Shimura differential operators $D_i = D_i(\rho_i, \rho)$ and holomorphic V_{ρ_i}-valued functions f_i such that*

$$f = \sum_i D_i(f_i) \tag{1}$$

The differential operators D_i are of total degree i in the entries of Y^{-1}. Shimura [8] constructs them in a rather explicit way and he denotes them by

$$D_i = \theta_V^i D^i_{\rho\otimes\sigma^i}.$$

We will call them *special Shimura differential operators* in the sequel. In particular, D_0 is the constant map and we tacitly normalize it to be the identity.
We formulated the theorem above for arbitrary nearly holomorphic functions, but the same is true for nearly holomorphic modular forms for a congruence subgroup Γ, the f_i are then elements of $M_{\rho_i}(\Gamma)$; we will use both versions in the sequel. We should point out that in the general case as in the theorem, a nearly holomorphic function can have many different decompositions (1) depending on the action of $Sp(n,\mathbb{R})$ imposed; if necessary, we call (1) the $\mathcal{N}_\rho^\nu$ - decomposition of f.

Remarks:
1) In degree 1, Hecke's classical Eisenstein series of weight two [4] is an example of a nearly modular form, for which the statement above does not hold. Also, for degree one, the condition on weights and degrees is explicit: we need $k > 2\nu$, see [8]. In some sense, Hecke's example is the only nearly holomorphic modular form of degree one not obtained by differential operators from holomorphic ones [11].

2) The proof of Shimura provides more than stated in the theorem:
There are linear maps

$$\psi_i : \mathcal{N}_{\rho_i}^0 \longmapsto \mathcal{N}_\rho^\nu \qquad (\text{“defined over } \mathbb{Q}\text{”})$$

such that

$$f_i = \psi_i(f).$$

We do not claim that the decomposition (1) is unique, it is sufficient for us that we may choose linear maps ψ_i and keep them fixed throughout.

3) Under the condition of the theorem above, there is no problem about rationality or bounded denominators for Fourier expansions of nearly holomorphic modular forms: They inherit such properties from corresponding statements about the holomorphic modular forms f_i.

4) Under suitable growth conditions, $f_0 = \psi_0(f)$ is a holomorphic projection of f; this is true in both the function theoretic and the modular forms context.

1.2 Constant term of a Maaß-Shimura differential operator as leading term in a Rankin-Cohen bracket

We start with an example from [2]:
Example: For $0 \leq r \leq n$ we put

$$\delta^{[r]} := det(Y)^{-k+\frac{r-1}{2}} \partial^{[r]} det(Y)^{k-\frac{r-1}{2}},$$

where for any matrix A of size n we denote by $A^{[r]}$ the matrix of size $\binom{n}{r}$ consisting of the minors of A of order r. This operator is known to map modular forms of weight k to nearly holomorphic ones with automorphy factor $det^k \otimes \rho_0$ with ρ_0 being the irreducible representation of $GL(n, \mathbb{C})$ of highest weight $(2, \ldots, 2, 0, \ldots, 0)$. Obviously,

$$\left(\delta^{[r]} f\right)^0 = \partial^{[r]}(f)$$

On the other hand we have shown in [2, prop.3] that there is a Rankin-Cohen bracket operator $[\,,\,]_{k_1,k_2,\rho^{[r]}}$ mapping modular forms of weights k_1 and k_2 to modular forms of weight $\det^{k_1+k_2} \otimes \rho^{[r]}$. This Rankin-Cohen bracket is of the form

$$[f, g]_{k_1,k_2,\rho^{[r]}} = (\partial^{[r]} f) \cdot g + \ldots,$$

where ... consists of summands involving only nontrivial derivatives of g (not g itself!). This is true at least for k_1 outside a finite set and k_2 sufficiently large.
This means that *the "constant term" of $\delta^{[r]} f$ and the "leading term" of $[f, g]_{k_1,k_2,\rho^{[r]}}$ are the same (up to g).*

A weak version of this is true more generally by applying Shimura's theorem to a nearly holomorphic function of type $D(f) \cdot g$:

Proposition: *Let D be a Maaß-Shimura differential operator of degree ν, changing an automorphy factor ρ to ρ'; furthermore, let f, g be arbitrary holomorphic functions on $\mathbb{H}_n$, V_ρ-valued and scalar-valued respectively. Then in the $\mathcal{N}^{\nu}_{\rho' \otimes \det^l}$- decomposition with l large*

$$(Df) \cdot g = \sum D_i\left(((Df) \cdot g)_i\right)$$

the holomorphic functions $((Df) \cdot g)_i = \psi_i(Df \cdot g)$ are given by Rankin-Cohen brackets $\mathcal{L}_i(f, g)$, more precisely, if D is of degree ν, then

$$D(f) \cdot g = \mathcal{L}_0(f, g) + \left(\sum_{i=1}^{\nu-1} D_i(\mathcal{L}_i(f, g))\right) + D_\nu(f \cdot g) \qquad (2)$$

Example: The simplest case of the proposition above is the degree one Maaß-Shimura differential operator:

$$\delta_k := \frac{k}{2iy} + \frac{\partial}{\partial z}.$$

In this case we can write down an identity for all weights k, l:

$$(k+l) \cdot \delta_k(f) \cdot g = [f, g]_{k,l} + k \cdot \delta_{k+l}(f \cdot g).$$

Remark: Again, there is a version of the proposition above for modular forms.

We may apply this proposition for the function $g = 1$ and obtain as a (trivial)

Observation: Under the same conditions as in the proposition,

$$D(f)^0 = \mathcal{L}_0(f, 1) + \sum_{i=1}^{\nu-1} D_i(\mathcal{L}_i(f, 1)))^0 + D_\nu(f)^0 \tag{3}$$

We would like to prove some properties of quasimodular forms by using induction over the degree ν in (2) and (3). To do so, we have to overcome the problem that summands of degree ν appear on both sides of these identities. Such a procedure is possible, if $D = D(\rho \otimes \det^k, \rho' \otimes \det^k)$ is a *special* Shimura differential operator. Then such an operator decomposes in the form

$$D = R_\partial + r(k) R_Y + \mathcal{R}$$

where R_∂ is the part of D free of Y and R_Y is free of ∂ and consists of monomials of exact degree ν in the entries of Y^{-1}. The remaining unspecified terms are collected in $\mathcal{R}$. The important property here is that R_∂ and R_Y do not depend on k at all and $r(k)$ is a *nonconstant* polynomial in k.
These properties can be read off from the reasoning on page 109 in [9]. For the examples $\delta^{[r]}$ and δ_k from above, they are obviously satisfied.

Then we can reformulate (2) (if f carries a $|_{\rho \otimes \det^k}$-action and g carries a $|_{k'}$-action with $l = k + k'$) as

$$D(\rho \otimes \det^k, \rho' \otimes \det^k)(f) \cdot g - \frac{r(k)}{r(l)} D(\rho \otimes \det^l, \rho' \otimes \det^l)(f \cdot g)$$

$$= R_\partial(f) \cdot g + \dots$$

$$= \mathcal{L}_0(f,g) + \sum_{i=1}^{\nu-1} D_i(\mathcal{L}_i(f,g))$$

where ... consists only of monomials of positive degree in the derivatives of g and the entries of Y^{-1}.

We can apply this to the constant function $g = 1$ and obtain

Corollary: *If l is sufficiently large, then the constant term $D(f)^o$ of $D(f)$ is proportional to the leading term in the Rankin-Cohen operator defined by $\mathcal{L}_0(f,g)$ modulo the sum of constant terms of the $D_i(\mathcal{L}_i(f,1))$ for $0 \le i < \nu$.*

Remark: In some cases one can show (by the same kind of argument as in [2] that the constant terms $D_i(\mathcal{L}_i(f,1)$ for $i > 0$ are proportional to $D(f)^o$.

2 Quasimodular forms as p-adic modular forms

Up to now, the congruence subgroup was arbitrary; from now on we fix a prime p and consider only congruence subgroups

$$\Gamma_0(p^t) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid C \equiv 0 \bmod p^t\}$$

For most of our considerations, the level p^t can be arbitrary, therefore we use the somewhat unusual notation Γ_p for a congruence subgroups of type $\Gamma_0(p^t)$; note however that t may possibly vary within a statement.

Main Theorem *All quasimodular forms for $\Gamma = \Gamma_p$ with coefficients in $\mathbb{Q}$ are p-adic.*

We start from a quasimodular form h^0 with $h \in \mathcal{N}_\rho^\nu(\Gamma_p)$ with Fourier coefficients in $\mathbb{Q}$. Furthermore we fix a power p^m and we have to prove that h^0 is congruent modulo p^m to a holomorphic modular form for Γ_p. Due to the results in [2] (and their -not at all straightforward- generalization to vector-valued situations) we automatically also get a congruence mod p^m to a modular form of level one.

After multiplication by a holomorphic modular form a quasimodular form is still quasimodular, therefore, by multiplying h by a modular form F for

$\Gamma_0(p)$ of sufficiently large weight and satisfying

$$F \equiv 1 \bmod p^l \qquad (l \geq m)$$

we may assume from the beginning that the degree ν of the nearly holomorphic modular form h , is small enough compared with the weight of ρ to allow the application of Shimura's theorem for h. Note that the existence of F is assured by [1].
In view of Shimura's theorem it is then enough to prove

Proposition: *For any $f \in M_{\rho_i}(\Gamma_p)$ and any special Shimura differential operator D, the "constant term" $(Df)^0$ is p-adic.*

We cannot use the corollary of the previous section directly, because (unlike the other statements of the previous section) there is no straightforward analogue for modular forms. We may however use F as above as g in the proposition of the previous section to obtain a congruence (with a suitable constant c)

$$\begin{aligned} c(D(f))^o &\equiv c(D(f)\cdot F)^o \bmod p^t \\ &\equiv \mathcal{L}_0(f,F) + \sum_{i=1}^{\nu-1} (D_i(\mathcal{L}_i(f,F)))^o \end{aligned}$$

The first summand on the right hand side is then a modular form for Γ_p and the remaining terms carry special Shimura differential operators D_i, whose degree is smaller than the degree of D; by induction on that degree we may then assume that the $(D_i(\mathcal{L}_i(f,F)))^o$ are congruent mod p^t to modular forms for Γ_p; by our results from [2], the leading term in the Rankin-Cohen operator $\mathcal{L}_0(f,F)$ is also a p-adic modular form.
Remark: Note that (after choosing F appropriately as in [2]), the term $\mathcal{L}_0(f,F)$ is congruent to $\Theta(f)$ for a suitable "theta operator", given by holomorphic derivatives of f.
This work was initiated by a discussion with A.Pantchichkine, I wish to thank him for his encouragement and interest.

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