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On uniformly-type Sakaguchi functions

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Abstract

Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disc $\mathbb{U}$. Furthermore, let $\mathcal{US}_s$ and $\mathcal{US}_s(\alpha,\beta)$ be the subclasses of $\mathcal{A}$ consisting of functions $f(z)$ related to uniformly convex and Sakaguchi functions. The object of the present paper is trying to guess inclusive relations between uniformly convexity, $\mathcal{S}_p$ and $\mathcal{US}_s$, and considering coefficient inequalities for $f(z)$ belonging to the class $\mathcal{US}_s(\alpha,\beta)$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

which are analytic in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)\in\mathcal{A}$ is said to be starlike with respect to symmetrical points in $\mathbb{U}$ if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>0 \qquad (z\in\mathbb{U}).$$

This class is introduced by Sakaguchi [7]. A function $f(z)\in\mathcal{A}$ is said to be in the class of uniformly convex (or starlike) functions denoted by $\mathcal{UCV}$ (or $\mathcal{UST}$) if $f(z)$ is convex (or starlike) in $\mathbb{U}$ and maps every circle or circular arc in $\mathbb{U}$ with center at ζ in $\mathbb{U}$ onto the convex arc (or the starlike arc with respect to $f(\zeta)$). These classes are introduced by Goodman [1] (see also [2]). For the class $\mathcal{UCV}$, it is defined as the one variable characterization by Rønning [5] and [6], that is, a function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{UCV}$ if it satisfies

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in\mathbb{U}).$$

It is independently studied by Ma and Minda [3]. Further, a function $f(z)\in\mathcal{A}$ is said to be the corresponding class denoted by $\mathcal{S}_p$ if it satisfies

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$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z \in \mathbb{U}).$$

This class $\mathcal{S}_p$ was introduced by Rønning [5]. We easily know that the relation $f(z) \in \mathcal{UCV}$ if and only if $zf'(z) \in \mathcal{S}_p$. By virtue of these classes, we define the subclass $\mathcal{US}_s(\alpha, \beta)$ of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > \alpha\left|\frac{zf'(z)}{f(z) - f(-z)} - \frac{1}{2}\right| + \beta \qquad (z \in \mathbb{U})$$

for some α ($\alpha \geqq 0$) and β ($0 \leqq \beta < \frac{1}{2}$). We denote $\mathcal{US}_s(1,0) \equiv \mathcal{US}_s$.

2 Some examples of relation between the class $\mathcal{S}_p$, $\mathcal{UCV}$ and $\mathcal{US}_s$

We don't have the inclusion relation between the class $\mathcal{S}_p$, $\mathcal{UCV}$ and $\mathcal{US}_s$. However, we give two examples to consider some relations between these classes.

Example 2.1. *Let us consider the function $f(z) \in \mathcal{A}$ as given by*

$$f(z) = z + \frac{1}{5}z^3.$$

Then we obtain

$$\begin{aligned}
&\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) - \left|\frac{zf'(z)}{f(z) - f(-z)} - \frac{1}{2}\right| \qquad (z = re^{i\theta})\\
&= \frac{25 + 20r^2\cos 2\theta + 3r^4}{50 + 20r^2\cos 2\theta + 2r^4} - \frac{r^2}{\sqrt{25 + 10r^2\cos 2\theta + r^4}}\\
&\geqq \frac{(1-r)(5+2r)}{2(5-2r)} > 0
\end{aligned}$$

which shows that $f(z) \in \mathcal{US}_s$. On the other hand, choosing $z = \frac{2}{3}e^{\frac{\pi}{2}i}$, we get

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \left|\frac{zf''(z)}{f'(z)}\right| = -\frac{5}{11}$$

which shows that $f(z) \notin \mathcal{UCV}$.

Example 2.2. *Let us consider the function $f(z) \in \mathcal{A}$ as given by*

$$f(z) = z + \frac{1}{7}z^4.$$

Then we obtain

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) - \left|\frac{zf'(z)}{f(z)} - 1\right| \qquad (z = re^{i\theta})$$

$$= \frac{49 + 35r^3 \cos 2\theta + 4r^6}{49 + 14r^3 \cos 2\theta + r^6} - \frac{3r^3}{\sqrt{49 + 14r^3 \cos 2\theta + r^6}}$$
$$\geqq \frac{7(1 - r^3)}{7 - r^3} > 0$$

which shows that $f(z) \in \mathcal{S}_p$. *On the other hand, choosing* $z = \frac{23}{24} e^{\frac{\pi}{3} i}$, *we get*

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z) - f(-z)} \right) - \left| \frac{zf'(z)}{f(z) - f(-z)} - \frac{1}{2} \right| = -\frac{71}{24192}$$

which shows that $f(z) \notin \mathcal{US}_s$.

3 Coefficient inequalities for the class $\mathcal{US}_s(\alpha, \beta)$

Our aim of this section is to discuss some coefficient inequalities for function $f(z)$ to be in the class $\mathcal{US}_s(\alpha, \beta)$.

Theorem 3.1. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\text{(3.1)} \qquad \sum_{n=2}^{\infty} [2(n-1)(1+\alpha)|a_{2n-2}| + \{2n - 1 - 2\beta + 2\alpha(n-1)\}|a_{2n-1}|] \leqq 1 - 2\beta$$

for some α $(\alpha \geqq 0)$ *and* β $(0 \leqq \beta < \frac{1}{2})$, *then* $f(z) \in \mathcal{US}_s(\alpha, \beta)$.

Let us consider an example for Theorem 3.1.

Example 3.1. *supposing that the function* $f(z) \in \mathcal{A}$ *as given by*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{(1-2\beta)t\delta_{2n-2}}{2n(n-1)^2(1+\alpha)} z^{2n-2} + \sum_{n=2}^{\infty} \frac{(1-2\beta)(1-t)\delta_{2n-1}}{\{2n-1-2\beta+2\alpha(n-1)\}n(n-1)} z^{2n-1}$$

for some α $(\alpha \geqq 0)$, β $(0 \leqq \beta < \frac{1}{2})$, t $(0 \leqq t \leqq 1)$ *and* $|\delta_{2n-2}| = |\delta_{2n-1}| = 1$. *Then the coefficient* (3.1) *yields*

$$\sum_{n=2}^{\infty} [2(n-1)(1+\alpha)|a_{2n-2}| + \{2n - 1 - 2\beta + 2\alpha(n-1)\}|a_{2n-1}|]$$
$$= \sum_{n=2}^{\infty} \left\{ \frac{(1-2\beta)t}{n(n-1)} + \frac{(1-2\beta)(1-t)}{n(n-1)} \right\}$$
$$\leqq 1 - 2\beta.$$

This implies that $f(z) \in \mathcal{US}_s(\alpha, \beta)$.

Theorem 3.2. *If $f(z) \in \mathcal{US}_s(\alpha,\beta)$, then*

$$|a_2| \leqq \frac{1-2\beta}{|1-\alpha|},$$

$$|a_3| \leqq \frac{1-2\beta}{|1-\alpha|},$$

$$|a_{2n}| \leqq \frac{1-2\beta}{n|1-\alpha|}\prod_{j=1}^{n-1}\left(1+\frac{1-2\beta}{j|1-\alpha|}\right) \qquad (n=2,3,4,\cdots)$$

and

$$|a_{2n+1}| \leqq \frac{1-2\beta}{n|1-\alpha|}\prod_{j=1}^{n-1}\left(1+\frac{1-2\beta}{j|1-\alpha|}\right) \qquad (n=2,3,4,\cdots).$$

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