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Some properties of harmonic univalent functions

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Abstract

A sufficient condition on harmonic univalent functions $f_1(z)$ and $f_2(z)$ in the open unit disk $\mathbb{U}$ for the convex combination $f_3(z) = tf_1(z) + (1-t)f_2(z)$ to be also harmonic univalent in $\mathbb{U}$ and its range $f_3(\mathbb{U})$ is convex in the horizontal direction is discussed. Furthermore, several illustrative examples and the images of functions satisfying the obtained condition are enumerated.

1 Introduction and Definitions

A real-valued function $\varphi(x, y)$ is real harmonic in $D \subset \mathbb{R}^2$ if and only if it satisfies Laplace's equation

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \qquad \big((x, y) \in D\big).$$

Let $\mathbb{D}$ be a simply connected domain on the complex plane $\mathbb{C}$. A continuous complex-valued function $f(z) = u(x, y) + iv(x, y)$ is harmonic in $\mathbb{D}$ if u and v are real harmonic in $\mathbb{D}$ (not necessarily harmonic conjugates), that is, if u and v satisfy

$$\Delta u = u_{xx} + u_{yy} = 0 \quad \text{and} \quad \Delta v = v_{xx} + v_{yy} = 0 \qquad (z = x + iy \in \mathbb{D})$$

where the subscripts indicate partial derivatives.

Remark 1 A function $f(z) = u(x, y) + iv(x, y)$ is analytic in $\mathbb{D}$ if it satisfies the Cauchy-Riemann equations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

in short, if it has a derivative $f'(z)$ at each point $z \in \mathbb{D}$. These relations show that every analytic function is harmonic.

Now, we consider the following two differential operators

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$$\frac{\partial f}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)f=\frac{1}{2}\Big((u_x+v_y)+i(v_x-u_y)\Big)$$

and

$$\frac{\partial f}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)f=\frac{1}{2}\Big((u_x-v_y)+i(v_x+u_y)\Big)$$

where $f=u+iv$ and $z=x+iy$. Then, by the Cauchy-Riemann equations, we see that $f(z)$ is analytic if and only if

$$\frac{\partial f}{\partial z}=u_x+iv_x \quad \text{and} \quad \frac{\partial f}{\partial \overline{z}}=0.$$

Furthermore, noting that the Laplacian Δf is denoted by

$$\Delta f=(u_{xx}+u_{yy})+i(v_{xx}+v_{yy})=4\frac{\partial^2 f}{\partial z\partial \overline{z}}=4\frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z}\right),$$

we know that $f(z)$ is harmonic in $\mathbb{D}$ if and only if $\partial f/\partial z$ is analytic in $\mathbb{D}$. From this relation, the following theorem is obtained.

Theorem A (*cf.* [3, pp.7]) *If $f(z)$ is harmonic in $\mathbb{D}$, then it can be written as*

$$f(z)=h(z)+\overline{g(z)}$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$. This representation is unique except for an additive constant. Conversely, for two analytic functions $h(z)$ and $g(z)$ in $\mathbb{D}$, a function $f(z)=h(z)+\overline{g(z)}$ is harmonic in $\mathbb{D}$.

Remark 2 The above theorem leads us that we take care of only the form of

$$f(z)=h(z)+\overline{g(z)}$$

when we discuss various properties and problems of harmonic (univalent) functions. Moreover complex-valued harmonic functions are closely related to analytic functions.

The Jacobian of a harmonic function $f=u+iv=h+\overline{g}$ can be written as

$$\mathcal{J}_f(z)=|h'(z)|^2-|g'(z)|^2.$$

The following result about the relation between the locally univalency and the Jacobian of harmonic functions is given by Lewy [6].

Theorem B *A complex-valued harmonic function $f(z)$ is locally univalent in $\mathbb{D}$ if and only if $\mathcal{J}_f(z)\neq 0$ for all $z\in\mathbb{D}$.*

This theorem is improved by applying its proof (see, for example, [2], [3]).

Theorem C *A harmonic function $f(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if*

$$\mathcal{J}_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0 \qquad (z \in \mathbb{D}),$$

that is, that

$$|h'(z)| > |g'(z)| \qquad (z \in \mathbb{D}).$$

Similarly, $f(z)$ is locally univalent and sense-reversing in $\mathbb{D}$ if and only if

$$\mathcal{J}_f(z) = |h'(z)|^2 - |g'(z)|^2 < 0 \qquad (z \in \mathbb{D}),$$

that is, that

$$|h'(z)| < |g'(z)| \qquad (z \in \mathbb{D}).$$

We note the sense-preserving property of harmonic functions.

Remark 3
(i) $f(z) = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$ if and only if $\overline{f(z)} = \overline{h(z)} + g(z)$ is sense-reversing in $\mathbb{D}$.

(ii) If $f(z) = h(z) + \overline{g(z)}$ is sense-preserving in $\mathbb{D}$, then $h'(z) \neq 0 \quad (z \in \mathbb{D})$.

(iii) If $f(z)$ is analytic in $\mathbb{D}$, then the Jacobian $\mathcal{J}_f(z) = |f'(z)|^2 \geqq 0$. Hence the classical result that $f(z)$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $f'(z) \neq 0$ $(z \in \mathbb{D})$, that is, that " conformal mappings are sense-preserving " holds.

The canonical representation of harmonic functions $f(z)$ in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ is

$$f(z) = h(z) + \overline{g(z)} \ \text{ with } g(0) = 0.$$

Since $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$ and the representation is unique, $f(z)$ has the following power series expansion

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}.$$

Thus we can normalize harmonic univalent functions in a way similar to the normalized analytic univalent functions. If $f(z)$ is harmonic and sense-preserving in $\mathbb{U}$, then, by Theorem B, we derive that

$$|h'(z)| > |g'(z)| \quad \text{and therefore} \quad h'(z) \neq 0 \quad (z \in \mathbb{U}).$$

This shows that, for the caronical representation of a function $f(z)$ which is sense-preserving, harmonic and univalent in $\mathbb{U}$,

$$\begin{aligned} F(z) &= \frac{f(z)-h(0)}{h'(0)} = \frac{h(z)-a_0}{a_1} + \frac{\overline{g(z)}}{a_1} \\ &= z + \sum_{n=2}^{\infty} \frac{a_n}{a_1} z^n + \overline{\sum_{n=1}^{\infty} \frac{b_n}{\overline{a_1}} z^n} \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n} \\ &= H(z) + \overline{G(z)} \end{aligned}$$

is univalent and denoted by analytic functions $H(z)$ and $G(z)$ with $H(0) = A_0 = 0$, $H'(0) = A_1 = 1$ and

$$|H'(z)| = \left|\frac{h'(z)}{a_1}\right| > \left|\frac{g'(z)}{a_1}\right| = |G'(z)| \quad (z \in \mathbb{U}).$$

Let $\mathcal{S}_{\mathcal{H}}$ be the class of all functions $f(z)$ which have the caronical representation and they are sense-preserving, harmonic and univalent in $\mathbb{U}$ with $h(0) = 0$ and $h'(0) = 1$. Namely, we consider harmonic univalent functions

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in \mathcal{S}_{\mathcal{H}}.$$

Moreover, in view of the property $|b_1| = |g'(0)| < |h'(0)| = 1$ and the fact that

$$F(z) = \frac{f(z) - \overline{b_1 f(z)}}{1-|b_1|^2} \in \mathcal{S}_{\mathcal{H}}$$

for any $f(z) \in \mathcal{S}_{\mathcal{H}}$, where

$$\begin{aligned} F(z) &= z + \sum_{n=2}^{\infty} \frac{a_n - \overline{b_1} b_n}{1-|b_1|^2} z^n + \overline{\sum_{n=2}^{\infty} \frac{b_n - b_1 a_n}{1-|b_1|^2} z^n} \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=2}^{\infty} B_n z^n} \qquad (B_1 = 0), \end{aligned}$$

we obtain the following subclass

$$\mathcal{S}_{\mathcal{H}}^0 = \{f(z) : f(z) \in \mathcal{S}_{\mathcal{H}} \text{ and } g'(0) = b_1 = 0\}$$

and inclusion relations

$$\mathcal{S} \subset \mathcal{S}_{\mathcal{H}}^0 \subset \mathcal{S}_{\mathcal{H}}$$

where $\mathcal{S}$ is the standard class of analytic univalent functions.

The following result is the well-known coefficient estimate for $\mathcal{S}_{\mathcal{H}}^{0}$.

Theorem B *For all $f(z) \in \mathcal{S}_{\mathcal{H}}^{0}$, the sharp inequality $|b_2| \leqq \frac{1}{2}$ holds.*

We also discuss the coefficient estimate of functions $f(z) \in \mathcal{S}_{\mathcal{H}}^{0}$ for the case that $f(z)$ has the finite power series expansion.

Theorem 1 *If $f(z) \in \mathcal{S}_{\mathcal{H}}^{0}$ is a polynomial function given by*

$$f(z) = h(z) + \overline{g(z)} = z + \sum_{j=2}^{l} a_j z^j + \overline{\sum_{j=2}^{m} b_j z^j}$$

for some l $(l = 2, 3, 4, \cdots)$ and m $(m = 2, 3, 4, \cdots)$, then

$$|a_k| + |b_k| \leqq \frac{1}{k}$$

where $k = \max\{l, m\}$, $a_j = 0$ $(j \geqq l + 1)$ and $b_j = 0$ $(j \geqq m + 1)$. The result is sharp.

Proof. Since $f(z) \in \mathcal{S}_{\mathcal{H}}^{0}$, we know that

$$|h'(z)| \neq |g'(z)| \qquad (z \in \mathbb{U}),$$

that is, that

$$h'(z) - e^{i\theta} g'(z) \neq 0 \qquad (z \in \mathbb{U})$$

for any $\theta \in \mathbb{R}$. This means that, for every $z \in \mathbb{U}$,

$$1 + \sum_{j=2}^{k} j\left(a_j - e^{i\theta} b_j\right) z^{j-1} = (1 + \alpha_1 z)(1 + \alpha_2 z)(1 + \alpha_3 z) \cdots (1 + \alpha_{k-1} z) \neq 0.$$

Therefore we obtain that

$$|\alpha_j| \leqq 1 \qquad (j = 1, 2, 3, \cdots, k - 1)$$

and

$$\left|k\left(a_k - e^{i\theta} b_k\right)\right| = |\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{k-1}| \leqq 1,$$

that is, that

$$\left|a_k - e^{i\theta} b_k\right| \leqq \frac{1}{k}$$

for any $\theta \in \mathbb{R}$. We easily know that

$$|a_k| + |b_k| \leqq \frac{1}{k}.$$

The sharpness is assured for the function

$$f(z) = z + \xi_1 z^k + \overline{\xi_2 z^k} \in \mathcal{S}_{\mathcal{H}}^0 \qquad \left(|\xi_1| + |\xi_2| = \frac{1}{k}\right).$$

□

Remark 4 The above function

$$f(z) = z + \xi_1 z^k + \overline{\xi_2 z^k} \qquad \left(|\xi_1| + |\xi_2| = \frac{1}{k}\right)$$

is harmonic starlike univalent in $\mathbb{U}$ because it satisfies the condition

$$\sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right) \leqq 1.$$

This result is guaranteed by Avci and Złotkiewicz [1] and Silverman [7]

2 Elementary transformations

The class $\mathcal{S}_{\mathcal{H}}$ is preserved under some elementary transformations. Here is a partial list.

(i) Conjugation *If* $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$*, then the function*

$$F(z) = \overline{f(\overline{z})} = \overline{h(\overline{z})} + \overline{\overline{g(\overline{z})}} \in \mathcal{S}_{\mathcal{H}}.$$

(ii) Dilatation and rotation *If* $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$*, then the function*

$$F(z) = \alpha^{-1} f(\alpha z) = \alpha^{-1} h(\alpha z) + \overline{\overline{\alpha^{-1}} g(\alpha z)} \in \mathcal{S}_{\mathcal{H}}$$

for any complex number α $(0 < |\alpha| \leqq 1)$.

(iii) Disk automorphism *If* $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$*, then the function*

$$F(z) = \frac{f\left(\dfrac{z+\xi}{1+\overline{\xi}z}\right) - f(\xi)}{\left(1-|\xi|^2\right) h'(\xi)} = \frac{h\left(\dfrac{z+\xi}{1+\overline{\xi}z}\right) - h(\xi)}{\left(1-|\xi|^2\right) h'(\xi)} + \overline{\left\{\frac{g\left(\dfrac{z+\xi}{1+\overline{\xi}z}\right) - g(\xi)}{\left(1-|\xi|^2\right) \overline{h'(\xi)}}\right\}} \in \mathcal{S}_{\mathcal{H}}$$

for any $\xi \in \mathbb{U}$.

(iv) Affine transformation *If* $f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$*, then the function*

$$F(z) = \varphi_\varepsilon \circ f(z) \in \mathcal{S}_{\mathcal{H}}$$

for an affine mapping

$$\varphi_\varepsilon(z) = (1 - \varepsilon b_1)z + \varepsilon\overline{z}$$

where ε *satisfies*

$$\left|\varepsilon + \frac{\overline{b_1}}{1 - |b_1|^2}\right| < \frac{1}{1 - |b_1|^2}.$$

The proofs of (i) – (iv) are fairly straight forward, and hence we omit the details involved.

We now note that even if $f_1(z)$ and $f_2(z)$ are univalent in $\mathbb{U}$, the convex combination of $f_1(z)$ and $f_2(z)$ is not necessarily univalent in $\mathbb{U}$. For example, although

$$f_1(z) = \frac{2z - z^2}{2(1-z)^2} + \overline{\frac{z^2}{2(1-z)^2}} \quad \text{and} \quad f_2(z) = -if_1(iz) = \frac{2z - iz^2}{2(1-iz)^2} + \overline{\frac{-iz^2}{2(1-iz)^2}}$$

are in the class $\mathcal{S}_\mathcal{H}$ (for details, see [4]), the convex combination $f_3(z)$ of these functions defined as

$$f_3(z) = tf_1(z) + (1-t)f_2(z) \qquad (0 \leqq t \leqq 1)$$

is ***not*** a member of $\mathcal{S}_\mathcal{H}$.

The present investigation is motivated by the above. It is important and interesting to discuss the condition for $f_3(z)$ to belong to $\mathcal{S}_\mathcal{H}$.

3 Preliminary Results

For some θ $\left(-\frac{\pi}{2} \leqq \theta < \frac{\pi}{2}\right)$, a domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex in the direction of $e^{i\theta}$ if, for each $a \in \mathbb{C}$, the intersection

$$\mathbb{D} \cap \{a + te^{i\theta} : t \in \mathbb{R}\}$$

is either connected or empty. In particular, if $\theta = 0$ then $\mathbb{D}$ is said to be convex in the direction of the real axis or convex in the horizontal direction (CHD). Clunie and Sheil-Small [2] have shown the next results concerned with CHD.

Lemma 1 *Let* $\mathbb{D} \subset \mathbb{C}$ *be CHD, and let* $p(w)$ *be a real-valued continuous function on* $\mathbb{D}$. *Then, the mapping*

$$w \longmapsto w + p(w)$$

is univalent in $\mathbb{D}$ *if and only if it is locally univalent in* $\mathbb{D}$. *If it is univalent, then its range is CHD.*

Theorem C *Let $f(z) = h(z) + \overline{g(z)}$ be harmonic and locally univalent in $\mathbb{U}$. Then, $f(z)$ is univalent and its range is CHD if and only if the analytic function*

$$\psi(z) = h(z) - g(z)$$

has the same properties.

This theorem leads us the following shearing technique.

Lemma 2 *Let $\psi(z)$ be analytic and univalent in $\mathbb{U}$ such that its range $\psi(\mathbb{U})$ is CHD and let $w(z)$ be an analytic function in $\mathbb{U}$ which satisfies $|w(z)| < 1$ $(z \in \mathbb{U})$. Then, by solving the simultaneous differential equations*

$$\begin{cases} h'(z) - g'(z) = \psi'(z) \\ w(z)h'(z) - g'(z) = 0, \end{cases}$$

we can find a function

$$f(z) = h(z) + \overline{g(z)} \in \mathcal{S}_{\mathcal{H}}$$

whose range $f(\mathbb{U})$ is CHD.

For example, let $\psi(z) = \dfrac{1}{3}\log\left(\dfrac{1+2z+z^2}{1-z+z^2}\right)$ and $w(z) = z$. Then, since

$$s(z) = \frac{1+2z+z^2}{1-z+z^2}$$

is univalent in $\mathbb{U}$ and it maps $\mathbb{U}$ onto the domain

$$\mathbb{C} \setminus \{t : t \leqq 0 \text{ or } t \geqq 4\},$$

we see that $\psi(z) = \dfrac{1}{3}\log(s(z))$ is univalent and it maps $\mathbb{U}$ onto the domain

$$\psi(\mathbb{U}) = \left\{w : -\frac{\pi}{3} < \mathrm{Im}(w) < \frac{\pi}{3}\right\} \setminus \left\{t : t \geqq \frac{2}{3}\log 2\right\}$$

which is CHD. Thus, solving the simultaneous differential equations

$$\begin{cases} h'(z) - g'(z) = \dfrac{1-z}{1+z^3} = \psi'(z) \\ zh'(z) - g'(z) = 0, \end{cases}$$

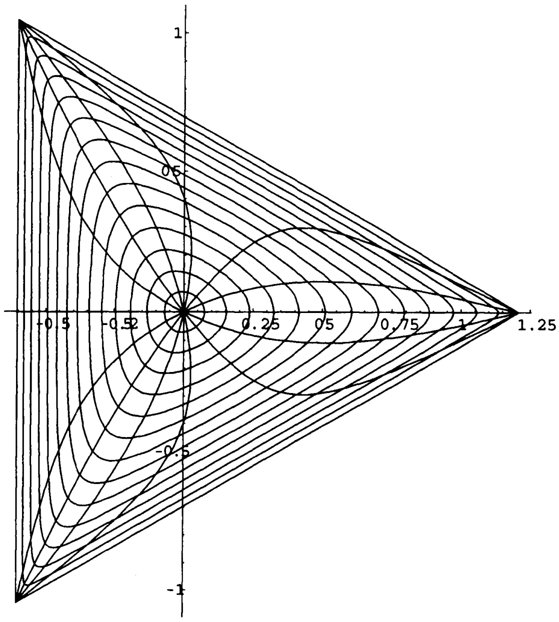
we obtain that

$$h'(z) = \frac{1}{1+z^3} \qquad \text{and} \qquad g'(z) = \frac{z}{1+z^3}.$$

It follows from Theorem C that

$$f(z) = z\,{}_2F_1\left(\frac{1}{3},1;\frac{4}{3};-z^3\right) + \overline{\frac{z^2}{2}\,{}_2F_1\left(\frac{2}{3},1;\frac{5}{3};-z^3\right)} \in \mathcal{S}_{\mathcal{H}}$$

where ${}_2F_1(a,b;c;z)$ represent the Gaussian hypergeometric function and $f(\mathbb{U})$ is a triangle, or CHD (see, for example, [5]) as follows:



The aim of this article is to find a sufficient condition on $f_1(z) = h_1(z) + \overline{g_1(z)} \in \mathcal{S}_{\mathcal{H}}$ and $f_2(z) = h_2(z) + \overline{g_2(z)} \in \mathcal{S}_{\mathcal{H}}$ for the convex combination

$$f_3(z) = tf_1(z) + (1-t)f_2(z) = h_3(z) + \overline{g_3(z)} \qquad (0 \leqq t \leqq 1)$$

to be also a member of $\mathcal{S}_{\mathcal{H}}$ and its range $f_3(\mathbb{U})$ is CHD.

4 Main Result

Our result is contained in

Theorem 2 *Let $f_j(z) = h_j(z) + \overline{g_j(z)} \in \mathcal{S}_{\mathcal{H}}$ and its range $f_j(\mathbb{U})$ be CHD $(j = 1,2)$. If*

$$\operatorname{Re}\left(h_1'(z)\overline{h_2'(z)} - g_1'(z)\overline{g_2'(z)}\right) > 0 \qquad (z \in \mathbb{U})$$

and there exists an analytic function $\psi(z)$ such that

$$\psi(z) = h_j(z) - g_j(z) \qquad (j = 1,2),$$

then $f_3(z) = tf_1(z) + (1-t)f_2(z) \in \mathcal{S}_{\mathcal{H}}$ and its range $f_3(\mathbb{U})$ is CHD.

Proof. We verify the locally univalency of $f_3(z)$. It follows from

$$h_3(z) = th_1(z) + (1-t)h_2(z) \qquad \text{and} \qquad g_3(z) = tg_1(z) + (1-t)g_2(z)$$

that

$$\left|\frac{g_3'(z)}{h_3'(z)}\right| = \left|\frac{tg_1'(z) + (1-t)g_2'(z)}{th_1'(z) + (1-t)h_2'(z)}\right|.$$

By the assumption of the theorem, we know that, for all $z \in \mathbb{U}$,

$$|th_1'(z) + (1-t)h_2'(z)|^2 - |tg_1'(z) + (1-t)g_2'(z)|^2$$

$$= t^2 |h_1'(z)|^2 + t(1-t)\left(h_1'(z)\overline{h_2'(z)} + \overline{h_1'(z)}h_2'(z)\right) + (1-t)^2 |h_2'(z)|^2$$
$$-t^2 |g_1'(z)|^2 - t(1-t)\left(g_1'(z)\overline{g_2'(z)} - \overline{g_1'(z)}g_2'(z)\right) + (1-t)^2 |g_2'(z)|^2$$

$$= t^2\left(|h_1'(z)|^2 - |g_1'(z)|^2\right) + (1-t)^2\left(|h_2'(z)|^2 - |g_2'(z)|^2\right)$$
$$+2t(1-t)\mathrm{Re}\left(h_1'(z)\overline{h_2'(z)} - g_1'(z)\overline{g_2'(z)}\right) > 0.$$

This implies that $|h_3'(z)| > |g_3'(z)|$, that is, that $f_3(z)$ is locally univalent and sense-preserving in $\mathbb{U}$. By Theorem C, $w = \psi(z) = h_j(z) - g_j(z)$ is univalent in $\mathbb{U}$ and its range $\psi(\mathbb{U})$ is CHD because $f_j(z) \in \mathcal{S}_{\mathcal{H}}$ and $f_j(\mathbb{U})$ is CHD. Then,

$$f_j(z) = h_j(z) - g_j(z) + \left(g_j(z) + \overline{g_j(z)}\right) = \psi(z) + 2\mathrm{Re}\,(g_j(z))$$

and the composition $f_j \circ \psi^{-1}(w)$ can be written as

$$f_j\left(\psi^{-1}(w)\right) = \psi\left(\psi^{-1}(w)\right) + 2\mathrm{Re}\left\{g_j\left(\psi^{-1}(w)\right)\right\} = w + p_j(w) \qquad (j = 1, 2)$$

for some real-valued continuous function $p_j(w)$ in $\psi(\mathbb{U})$. We derive that

$$\begin{aligned} f_3\left(\psi^{-1}(w)\right) &= tf_1\left(\psi^{-1}(w)\right) + (1-t)f_2\left(\psi^{-1}(w)\right) \\ &= t(w + p_1(w)) + (1-t)(w + p_2(w)) \\ &= w + (tp_1(w) + (1-t)p_2(w)) \\ &= w + p_3(w) \end{aligned}$$

is locally univalent in $\psi(\mathbb{U})$, and hence it is univalent in $\psi(\mathbb{U})$ and its range is CHD by Lemma 1. The proof is completed. □

Setting $\psi(z) = h_j(z) - g_j(z) = z$ $(j = 1, 2)$ in Theorem 2, we obtain the following examples.

Example 1 *Let*

$$f_1(z) = h_1(z) + \overline{g_1(z)} = z + \frac{1}{4}z^2 + \frac{1}{4}\overline{z}^2$$

and

$$f_2(z) = h_2(z) + \overline{g_2(z)} = z + \frac{1}{6}z^3 + \frac{1}{6}\overline{z}^3.$$

Then $f_1(z)$, $f_2(z) \in \mathcal{S}_{\mathcal{H}}$, $\psi(z) = h_j(z) - g_j(z) = z$. *Furthermore, we know that* $f_1(\mathbb{U})$, $f_2(\mathbb{U})$ *and* $\psi(\mathbb{U}) = \mathbb{U}$ *are CHD. For all* $z = x + iy \in \mathbb{U}$ $(x^2 + y^2 < 1)$,

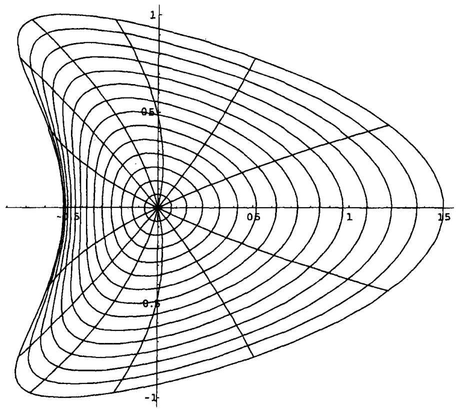
$$\begin{aligned}
\mathrm{Re}\left(h_1'(z)\overline{h_2'(z)} - g_1'(z)\overline{g_2'(z)}\right) &= \mathrm{Re}\left\{\left(1+\frac{1}{2}z\right)\left(1+\frac{1}{2}\overline{z}^2\right) - \frac{1}{2}z \cdot \frac{1}{2}\overline{z}^2\right\} \\
&= \mathrm{Re}\left(1 + \frac{1}{2}z + \frac{1}{2}\overline{z}^2\right) \\
&= 1 + \frac{1}{2}x + \frac{1}{2}(x^2 - y^2) \\
&= \frac{1}{2}(1+x) + \frac{1}{2}(1 + x^2 - y^2) \\
&> \frac{1}{2}(1+x) + \frac{1}{2}\left((x^2+y^2) + x^2 - y^2\right) \\
&= \frac{1}{2} + \frac{1}{2}x + x^2 \\
&= \left(x + \frac{1}{4}\right)^2 + \frac{7}{16} \\
&\geqq \frac{7}{16} \;=\; 0.4375 \;>\; 0.
\end{aligned}$$

Therefore,

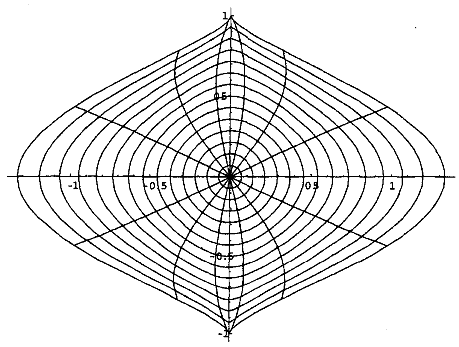
$$\begin{aligned}
f_3(z) &= tf_1(z) + (1-t)f_2(z) \\
&= z + \frac{t}{4}z^2 + \frac{1-t}{6}z^3 + \overline{\frac{t}{4}z^2 + \frac{1-t}{6}z^3} \qquad (0 \leqq t \leqq 1)
\end{aligned}$$

is also in the class $\mathcal{S}_{\mathcal{H}}$ *and its range* $f_3(\mathbb{U})$ *is CHD.*

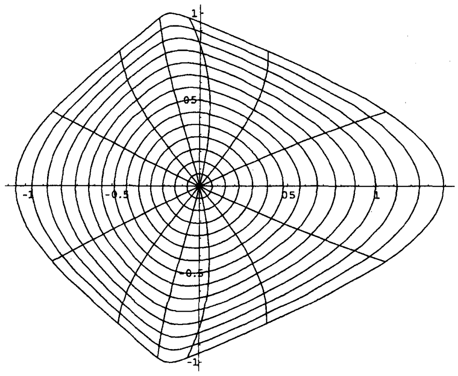
We actually check images of $f_1(z)$, $f_2(z)$ and $f_3(z)$ with $t = \dfrac{1}{3}$.



$$f_1(z) = z + \frac{1}{4}z^2 + \frac{1}{4}\overline{z^2} \in \mathcal{S}_{\mathcal{H}}$$



$$f_2(z) = z + \frac{1}{6}z^3 + \frac{1}{6}\overline{z^3} \in \mathcal{S}_{\mathcal{H}}$$



$$f_3(z) = \frac{1}{3}f_1(z) + \frac{2}{3}f_2(z) \in \mathcal{S}_{\mathcal{H}}$$

Example 2 *Let* $\psi(z) = h_j(z) - g_j(z) = z$ *and* $\dfrac{g_j'(z)}{h_j'(z)} = z^j$ $(j = 1, 2)$. *Then solving the following simultaneous differential equations*

$$\begin{cases} h_1'(z) - g_1'(z) = 1 \\ zh_1'(z) - g_1'(z) = 0 \end{cases} \quad \text{and} \quad \begin{cases} h_2'(z) - g_2'(z) = 1 \\ z^2 h_2'(z) - g_2'(z) = 0, \end{cases}$$

we obtain

$$f_1(z) = -\log(1-z) + \overline{(-z - \log(1-z))} \in \mathcal{S}_{\mathcal{H}}$$

and

$$f_2(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + \overline{\left(-z + \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right)} \in \mathcal{S}_{\mathcal{H}}.$$

Moreover, we see that their ranges $f_1(\mathbb{U})$ *and* $f_2(\mathbb{U})$ *are CHD. In view of* $|z|^2 < 1$ *and* $\mathrm{Re}\left(\dfrac{1}{1+z}\right) > \dfrac{1}{2} > 0$, *we know that*

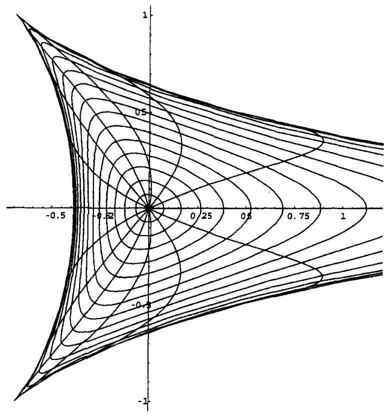
$$\begin{aligned} \mathrm{Re}\left(h_1'(z)\overline{h_2'(z)} - g_1'(z)\overline{g_2'(z)}\right) &= \mathrm{Re}\left(\frac{1}{1-z}\cdot\overline{\frac{1}{1-z^2}} - \frac{z}{1-z}\cdot\overline{\frac{z^2}{1-z^2}}\right) \\ &= \mathrm{Re}\left((1-|z|^2\overline{z})\left|\frac{1}{1-z}\right|^2\cdot\overline{\frac{1}{1+z}}\right) \\ &= \left|\frac{1}{1-z}\right|^2 \mathrm{Re}\left(\frac{1}{1+z} - \frac{|z|^2 z}{1+z}\right) \\ &> \left|\frac{1}{1-z}\right|^2 \mathrm{Re}\left(\frac{|z|^2(1-z)}{1+z}\right) \\ &= \left|\frac{z}{1-z}\right|^2 \mathrm{Re}\left(\frac{1-z}{1+z}\right) \geqq 0 \qquad (z \in \mathbb{U}). \end{aligned}$$

Therefore, for any t $(0 \leqq t \leqq 1)$,

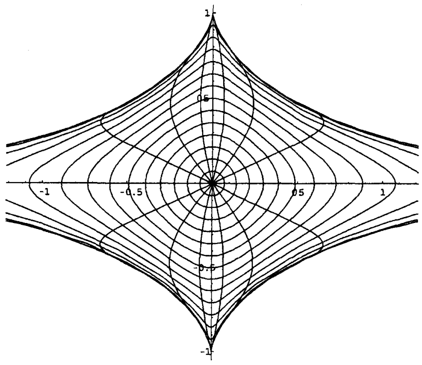
$$\begin{aligned} f_3(z) &= tf_1(z) + (1-t)f_2(z) \\ &= -tz\log(1-z) + \frac{1-t}{2}\log\left(\frac{1+z}{1-z}\right) + \overline{\left(-z - t\log(1-z) + \frac{1-t}{2}\log\left(\frac{1+z}{1-z}\right)\right)} \end{aligned}$$

is also a member of $\mathcal{S}_{\mathcal{H}}$ *and its range* $f_3(\mathbb{U})$ *is CHD.*

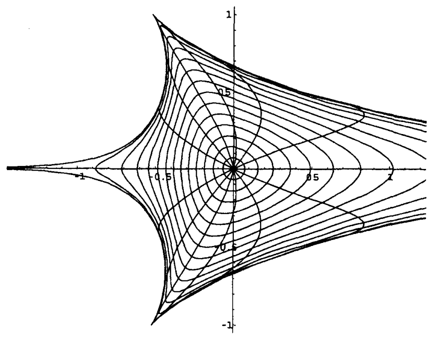
Indeed, the images of $f_1(z)$, $f_2(z)$ and $f_3(z)$ with $t = \dfrac{3}{4}$ are below.



$$f_1(z) = -\log(1-z) + \overline{-z - \log(1-z)} \in \mathcal{S}_{\mathcal{H}}$$



$$f_2(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) + \overline{-z + \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)} \in \mathcal{S}_{\mathcal{H}}$$



$$f_3(z) = \frac{3}{4}f_1(z) + \frac{1}{4}f_2(z) \in \mathcal{S}_{\mathcal{H}}$$

References

[1] Y. Avci and E. Złotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Marie Curie-Sklodowska Sect. A **44**(1991), 1–7.

[2] J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I **9**(1984), 3–25.

[3] P. L. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge, 2004.

[4] T. Hayami, *Coefficient conditions for harmonic close-to-convex functions*, Abstr. Appl. Anal. Vol.2012, Article ID 413965, 1–12.

[5] T. Hayami, *A sufficient condition for p-valently harmonic functions*, Complex Var. Elliptic Equ. **59**(2014), 1214–1222.

[6] H. Lewy, *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc. **42**(1936), 689–692.

[7] H. Silverman, *Harmonic univalnet functions with negative coefficients*, J. Math. Anal. Appl. **220**(1998), 283–289.

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