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STABILITY OF TRANSITION FRONT SOLUTIONS IN CAHN–HILLIARD SYSTEMS

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ABSTRACT. We consider the asymptotic stability of transition front solutions for Cahn–Hilliard systems on $\mathbb{R}$. Such equations arise naturally in the study of phase separation, and systems describe cases in which three or more phases are possible. When a Cahn–Hilliard system is linearized about a transition front solution, the linearized operator has an eigenvalue at 0 (due to shift invariance), which is not separated from essential spectrum. In many cases it is possible to show that the only non-negative eigenvalue is $\lambda = 0$, and so stability depends entirely on the nature of this neutral eigenvalue. In such cases, we identify a stability condition based on an appropriate Evans function, and we verify this condition under strong structural conditions on our equations. Due to lack of a spectral gap, nonlinear stability cannot be concluded from classical semigroup considerations and a more refined development is appropriate. One of our main results asserts that spectral stability implies nonlinear stability.

1. INTRODUCTION

We consider the spectral and asymptotic stability of transition front solutions $\bar{u}(x)$, $\bar{u}(\pm\infty) = u_\pm$, $u_- \neq u_+$, for Cahn–Hilliard systems on $\mathbb{R}$,

$$u_t = \Big(M(u)(-\Gamma u_{xx} + f(u))_x\Big)_x, \tag{1}$$

where $u, f \in \mathbb{R}^m$, m an integer greater than or equal to 2 ($m+1$ phases are possible) and $M, \Gamma \in \mathbb{R}^{m\times m}$. We will first record here, for convenient reference, a group of technical assumptions that will be made throughout the analysis.

(H0) (Assumptions on Γ) Γ denotes a constant, symmetric, positive definite matrix.

(H1) (Assumptions on f) $f \in C^3(\mathbb{R}^m)$, and f has at least two zeros on $\mathbb{R}^m$. For convenience we denote this set

$$\mathcal{M} := \{u \in \mathbb{R}^m : f(u) = 0\}. \tag{2}$$

(H2) (Transition front existence and structure) There exists a transition front solution to (1) $\bar{u}(x)$, so that

$$-\Gamma \bar{u}_{xx} + f(\bar{u}) = 0, \tag{3}$$

with $\bar{u}(\pm\infty) = u_\pm$, $u_\pm \in \mathcal{M}$. When (3) is written as a first order autonomous ODE system $\bar{u}$ arises as a transverse connection either from the m-dimensional unstable linearized subspace for u_-, denoted $\mathcal{U}^-$, to the m-dimensional stable linearized subspace for u_+, denoted $\mathcal{S}^+$, or (by isotropy) vice versa. (We recall that since our ambient manifold is $\mathbb{R}^{2m}$, the intersection of $\mathcal{U}^-$ and $\mathcal{S}^+$ is referred to as transverse if at each point of intersection the tangent spaces associated with $\mathcal{U}^-$ and $\mathcal{S}^+$ generate $\mathbb{R}^{2m}$. In particular, in this setting a transverse connection is one in which the the intersection of these two manifolds has dimension 1; i.e., our solution manifold will comprise shifts of $\bar{u}$.)

(H3) (Assumptions on M and Γ) $M \in C^2(\mathbb{R}^m)$; M is uniformly positive definite along the front; i.e., there exists $\theta > 0$ so that for all $\xi \in \mathbb{R}^m$ and all $x \in \mathbb{R}$ we have

$$\xi^{tr} M(\bar{u}(x))\xi \geq \theta|\xi|^2.$$

(H4) (Symmetry and Endstate Assumptions) We assume the $m \times m$ Jacobian matrix $f'(\bar{u}(x))$ is symmetric for all $x \in \mathbb{R}$. Setting $B_\pm := f'(u_\pm)$ and $M_\pm := M(u_\pm)$, we assume $B_\pm$ and $M_\pm$ are both symmetric and positive definite. (Of course, $M_\pm$ is already positive definite from (H3).) In addition, we assume that for each of the matrices $M_\pm B_\pm$ and $\Gamma^{-1}B_\pm$, the spectrum is distinct except possibly for repeated eigenvalues that have an associated eigenspace with dimension equal to eigenvalue multiplicity. In the case of repeated eigenvalues, we assume additionally that the solutions μ of

$$\det\Big(- \mu^4 M_\pm \Gamma + \mu^2 M_\pm B_\pm - \lambda I \Big) = 0$$

can be strictly divided into two cases: if $\mu(0) \neq 0$ then $\mu(\lambda)$ is analytic in λ for $|\lambda|$ sufficiently small, while if $\mu(0) = 0$ $\mu(\lambda)$ can be written as $\mu(\lambda) = \sqrt{\lambda}h(\lambda)$, where h is analytic in λ for $|\lambda|$ sufficiently small.

Regarding (H1) we observe that for Cahn-Hilliard systems we can often write f as the gradient of an appropriate bulk free energy density F (i.e. $f(u) = F'(u)$), where F has $m+1$ local minima on $\mathbb{R}^m$. In this way, it's natural for f to have precisely $m+1$ zeros. Since F would appear in (1) with both a u and an x derivative, we can subtract from it any affine function without changing (1). It is often convenient to subtract a supporting hyperplane from F so that F is also 0 on $\mathcal{M}$.

Regarding (H4), we first observe that the symmetry condition on $f'(\bar{u}(x))$ is natural since $F''(u)$ is a Hessian matrix. Also, we note that we can ensure that our system satisfies the determinant condition by taking arbitrarily small perturbations of the matrices M and Γ. Since we expect stability to be insensitive to such perturbations, we view this assumption as purely for technical convenience. Generally speaking, (H0)-(H4) hold for physically relevant choices of Γ, M, and f; particular examples can be found below and in [30].

When the Cahn–Hilliard system (1) is linearized about a standing wave solution $\bar{u}(x)$, as described in (H2), the resulting linear equation is

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x\Big)_x, \tag{4}$$

where (with a slight abuse of notation) $M(x) := M(\bar{u}(x))$ and $B(x) := f'(\bar{u}(x))$. In order to develop our linear theory generally, we will state conditions on Γ, $B(x)$, and $M(x)$ that are inherited from (H0)–(H4), but that could also hold for an equation of form (4) that arose in another context.

(C1) $B \in C^2(\mathbb{R})$; there exists a constant $\alpha_B > 0$ so that

$$\partial_x^j (B(x) - B_\pm) = \mathbf{O}(e^{-\alpha_B |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; $B_\pm$ are both positive definite matrices.

(C2) $M \in C^2(\mathbb{R})$; there exists a constant $\alpha_M > 0$ so that

$$\partial_x^j (M(x) - M_\pm) = \mathbf{O}(e^{-\alpha_M |x|}), \quad x \to \pm\infty,$$

for $j = 0, 1, 2$; M is uniformly positive definite on $\mathbb{R}$; Γ denotes a constant, symmetric, positive definite matrix. We will set $\alpha = \min\{\alpha_B, \alpha_M\}$.

We note, in particular, that if (3) is written as a first order system

$$\begin{pmatrix} \bar{u} \\ \bar{u}' \end{pmatrix}' = \begin{pmatrix} 0 & I \\ \Gamma^{-1} f(\bar{u}) & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{u}' \end{pmatrix},$$

then by (H1) the endstates $u_\pm$ correspond with hyperbolic equilibrium points $\begin{pmatrix} u_\pm \\ 0 \end{pmatrix}$. This guarantees that $\bar{u}(x)$ approaches its endstates at exponential rate, and $(C1)-(C2)$ follow by the continuity assumed on f and M.

(C3) We assume $B(x)$ is symmetric for all $x \in \mathbb{R}$, and the conditions described in (H4) hold for Γ, $B_\pm$. and $M_\pm$.

The eigenvalue problem associated with (4) has the form

$$L\phi := \Big(M(x)(-\Gamma\phi'' + B(x)\phi)'\Big)' = \lambda\phi. \tag{5}$$

In many cases it's possible to verify that the only non-negative eigenvalue for this equation is $\lambda = 0$ (see, for example, [1, 2, 46] and our spectral paper [30]), and so stability depends entirely on the nature of this neutral eigenvalue. In [30], we identify an appropriate stability condition for this leading eigenvalue. Briefly, this condition is constructed in terms of the asymptotically growing/decaying solutions of (5). For $|\lambda| > 0$ sufficiently small, and $\mathrm{Arg}\lambda \neq \pi$ (i.e., excluding negative real numbers), there are $2m$ linearly independent solutions of (5) that decay as $x \to -\infty$ and $2m$ linearly independent solutions of (5) that decay as $x \to +\infty$. Moreover, these functions can be constructed so that they are analytic in $\rho = \sqrt{\lambda}$. If we denote these functions $\{\phi_j^\pm(x;\rho)\}_{j=1}^{2m}$ and set $\Phi_j^\pm = (\phi_j^\pm, \phi_j^{\pm\prime}, \phi_j^{\pm\prime\prime}, \phi_j^{\pm\prime\prime\prime})^{\mathrm{tr}}$, the Evans function can be expressed as

$$D_a(\rho) = \det(\Phi_1^+, \dots, \Phi_{2m}^+, \Phi_1^-, \dots, \Phi_{2m}^-)\Big|_{x=0}. \tag{6}$$

In terms of this function the stability condition of [30] can be stated as follows:

Condition 1. *The set $\sigma(L)\backslash\{0\}$ lies entirely in the negative half-plane $Re\lambda < 0$, and*

$$\frac{d^{m+1}}{d\rho^{m+1}} D_a(\rho)\Big|_{\rho=0} \neq 0.$$

Remark 1. *As discussed in Section 3 of* [30], *our assumptions (H0)-(H4) ensure that the essential spectrum of L (defined here as any value that is not in the resolvent set of L and is not an isolated eigenvalue of finite multiplicity) is confined to the negative real axis* $(-\infty, 0]$. *(This follows immediately from our assumptions that Γ, $B_\pm$, and $M_\pm$ are all symmetric and positive definite.) In addition, since L can be regarded as a perturbation of the sectorial operator $-(M(x)\Gamma\phi''')'$ (by lower order perturbation $(M(x)(B(x)\phi)')'$), we can conclude from standard perturbation theory for sectorial operators (e.g. Theorem VI.2.3.4 in* [37]*) that Condition 1 implies that aside from the leading eigenvalue $\lambda = 0$ the point spectrum of L is bounded to the left of a wedge with vertex on the negative real axis:*

$$\Gamma_\theta := \{\lambda : Re\ \lambda = -\theta_1 - \theta_2|Im\ \lambda|\} \tag{7}$$

for some positive values θ_1, θ_2 sufficiently small. If we make one additional natural assumption, that $M(\bar{u}(x))$ is symmetric for all $x \in \mathbb{R}$, we can ensure that the point spectrum of L is entirely real-valued (since our analysis does not require this assuption we have not added it to (H0)-(H4)). Finally, we verify in [30] *that*

$$D_a(0) = D_a'(0) = \cdots = D_a^{(m)}(0) = 0$$

Moreover, we establish in [31, 32] that Condition 1 is sufficient to guarantee nonlinear (phase-asymptotic) stability for the front $\bar{u}(x)$, for which we employ the pointwise Green's function approach of [17, 18, 49], along with the local tracking developed in [33].

2. Phase separation models

Our analysis is particularly motivated by the study of spinodal decomposition, a phenomenon in which the rapid cooling of a homogeneously mixed alloy with $m + 1$ components causes separation to occur, resolving the mixture into regions of different crystalline structure, separated by steep transition layers, in which one or more component concentrations rise above their high-temperature concentrations while one or more fall below their high-temperature concentrations. In this context, the vector u typically contains concentrations for m components of the alloy, and the final component concentration is obtained from conservation of mass

$$\sum_{j=1}^{m+1} u_j = 1 \tag{8}$$

(by a non-essential choice of scaling). Each component of u is a conserved quantity, so if we denote by J_j the (vector) flux associated with concentration u_j we have

$$u_{j_t} + \nabla \cdot J_j = 0; \quad j = 1, \dots, m. \tag{9}$$

The molecular transfer during spinodal decomposition corresponds with motion from configurations in which small fluctuations in concentration correspond with large fluctuations in system internal energy to configurations in which small fluctuations in concentration correspond with small fluctuations in system internal energy. In order to capture this behavior the J_j are typically chosen to have the form (see [7], p. 12)

$$J_j = -\sum_{i=1}^{m} M_{ji}(u) \nabla \frac{\delta E}{\delta u_i}, \tag{10}$$

where $M_{ji}(u)$ denotes (scalar) molecular mobility, E denotes a total free energy functional for the alloy, and $\frac{\delta E}{\delta u_i}$ denotes the kernel of the variational derivative of E with respect to u_i. The Cahn–Hilliard system arises from these considerations and a form of the free energy functional suggested by Cahn and Hilliard in 1958 for the case of binary alloys ($m = 1$) [6] and generalized by de Fontaine to multicomponent alloys in 1967 [7]. For the case of a bounded domain $U \subset \mathbb{R}^n$, de Fontaine's functional can be written as (see [7], p. 10)

$$E(u) = \int_U F(u) + \frac{1}{2} Du : (\Gamma(u) Du) dx, \tag{11}$$

where $F(u)$ denotes the bulk free energy density for the alloy with uniform composition u, $\Gamma(u)$ is a gauge of interfacial energy (so, in particular, the term involving Γ describes energy associated with a transition of composition), Du denotes the $m \times n$ Jacobian of u, and the notation $A : B$ refers to matrix inner product

$$A : B := \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}. \tag{12}$$

We note for comparison with our references that Eyre uses the same energy in [12], though he replaces $Du : (\Gamma(u)Du)$ with the equivalent expression $\mathrm{tr}((Du)^{tr}\Gamma(u)Du)$. If Γ is taken to be constant (which certainly need not be the case physically) then

$$\frac{\delta E}{\delta u_i} = F_{u_i}(u) - (\Gamma \Delta u)_i, \tag{13}$$

and we obtain the Cahn–Hilliard system on $\mathbb{R}^n$

$$u_{j_t} = \nabla \cdot \Big\{ \sum_{i=1}^{m} M_{ji}(u) \nabla \Big((-\Gamma \Delta u)_i + F_{u_i}(u) \Big) \Big\}, \quad j = 1, 2, \dots, m. \tag{14}$$

This corresponds, for example, with equation (24) in [7] and equation (4) in [40], except that the author in [7] adds an inhomogeneous term and the authors in [40] take Γ as the identity. (Also, in contrast to the case here, the authors in [40] are considering degenerate mobility, but that does not show up in the notation.) We note that for $n = 1$ (14) is a special case of our equation (1), obtained by taking f to be the Jacobian (with respect to u) of F. (For the case of a binary alloy (14) first appeared in Cahn's 1961 paper [5], and de Fontaine suggests the Cahn–Hilliard equation would more correctly be designated the Cahn equation. Hilliard, de Fontaine's advisor, apparently referred to this equation as 'the last unnumbered equation after Eq. (18) in Cahn's 1961 paper" [10].)

Equation (14) has been the subject of considerable study [7, 8, 9, 12, 15, 40, 42], though certainly the case $m = 1$ is much better understood than the case currently under investigation.

Alternatively, we can derive a form of (1) by regarding the total internal energy as a map on all m+1 component concentrations and introducing a Lagrange multiplier to impose the total mass constraint (7). In this approach we will assume the flow is governed by a functional L, equivalent with total internal energy along (1), which includes a Lagrange multiplier as a constraint term. Following the analysis of Boyer and Lapuerta in [5], we obtain the system

$$\begin{aligned} u_{j_t} &= \Big(\frac{\tilde{M}(u)}{\gamma_j} \Big(-\gamma_j u_{j_{xx}} + \gamma_0 \sum_{i=1}^{m+1} \frac{1}{\gamma_i} (F_{u_j}(u) - F_{u_i}(u)) \Big)_x \Big)_x, \quad j = 1, 2, \dots, m \\ u_{m+1} &= 1 - \sum_{j=1}^{m} u_j, \end{aligned} \tag{15}$$

where in this case $\tilde{M}$ and F are regarded as functions of $m+1$ variables, and we note that Boyer and Lapuerta focused on the case $m = 2$. We remark for clarity of comparison that system (15) is taken from equation (8) of [4], given in the case $m = 2$, with

$$\tilde{M} = \frac{3\epsilon}{4} M_0; \quad \gamma_j = \Sigma_j; \quad j = 1, 2, 3; \quad \gamma_0 = \frac{16\Sigma_T}{3\epsilon^2},$$
$$\Sigma_T = \frac{3}{\frac{1}{\Sigma_1} + \frac{1}{\Sigma_2} + \frac{1}{\Sigma_3}},$$

where the expressions on the right hand sides are in the notation of [4]. This is the special case of our (1) obtained by taking $M(u)$ diagonal, with entries $\tilde{M}/\gamma_j$, Γ diagonal with entries $\{\gamma_j\}_{j=1}^m$, and

$$f_j(u) = \Gamma_0 \sum_{i=1}^{m+1} \frac{1}{\gamma_i} \Big(F_{u_j}(u, 1 - \sum_{k=1}^{m} u_k) - F_{u_i}(u, 1 - \sum_{k=1}^{m} u_k) \Big).$$

Remark 2. *As the analysis of systems of form (15) differs somewhat from the analysis of systems of form (14), we will find it convenient to have a terminological difference between the cases. While we certainly regard both cases as Cahn-Hilliard systems, we will, for brevity, sub-categorize equations (15) as Boyer-Lapuerta systems and equations*

$$u_t = \Big(M(u)(-\Gamma u_{xx} + F'(u))_x \Big)_x, \tag{16}$$

(slightly more general than (14)) as gradient systems.

Qualitatively, we expect that at high temperatures the bulk free energy density F will decrease as entropy increases (according to the Helmholtz free energy relation $\mathcal{F} = U - TS$, where $\mathcal{F}$ denotes free energy, U denotes internal energy, T denotes system temperature, and S denotes system entropy), and so F will have a global minimum in the configuration that maximizes entropy. For example, if our system has $m+1$ components, present in equal amounts, we expect (at high temperature) F to have a global minimum at a concentration vector

$$u = (\frac{1}{m+1}, \frac{1}{m+1}, \dots, \frac{1}{m+1}),$$

and to have global maxima at the $m+1$ low-entropy single-component configurations corresponding with concentration vectors $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$ etc., with also $(0, 0, \dots, 0)$. (To be clear, this discussion is only intuitive, and we are not adding any hypotheses on F.) As temperature decreases (and assuming internal energy remains constant) we have the thermodynamic relation

$$\frac{\partial \mathcal{F}}{\partial T} = -S,$$

and so $\mathcal{F}$ increases (again, as temperature decreases) at a rate proportional to entropy. In this way the free energy increases most rapidly where it was previously minimized and increases most slowly where it was previously maximized. Heuristically, then, we expect that at low temperatures F will have a local maximum where it was previously minimized and that it will have $m+1$ local minima associated (possibly by equivalence) with the $m+1$ global single-component maxima. More precisely, in [7, 8] de Fontaine attributes the following form of the bulk free energy density for a ternary alloy to Prigogine [43]:

$$F(u_1, u_2, u_3) = \sum_{i \neq j} \omega_{ij} u_i u_j + \kappa T \sum_{i=1}^{3} u_i \ln u_i,$$

where κ denotes Boltzmann's constant, T denotes system temperature, and we haven't yet employed mass conservation to reduce the number of variables. For a system with $m+1$ components it is natural to consider the generalized form

$$F(u) = \frac{1}{2} u \cdot Au + \kappa T \sum_{i=1}^{m+1} l_i u_i \ln u_i, \tag{17}$$

where A is an $(m+1) \times (m+1)$ matrix and the values $\{l_i\}_{i=1}^{m+1}$ are constants associated with the alloy.

A form commonly examined due to its simplicity is

$$F(u) = \sum_{i<j} \alpha_{ij} u_i^2 u_j^2. \tag{18}$$

In the case $m = 2$ Alikakos et al. have carefully examined bulk free energy functions of the form

$$F(u_1, u_2) = |h(u_1 + iu_2)|^2, \tag{19}$$

where h is analytic on $\mathbb{C}$, and the third component has been eliminated by conservation of mass [1].

Finally, we note that there are alternative derivations, such as the one employed in [4], for which f does not have divergence form. We refer the interested reader to [4] or [30].

3. Existence and Structure of Transition Waves

We look for stationary solutions $\bar{u}(x)$ for (1) that satisfy $\bar{u}(\pm\infty) = u_\pm \in \mathcal{M}$. (We recall that our notation $\mathcal{M}$ is defined in (H1).) Upon substitution of $\bar{u}(x)$ into (1), and after integrating twice and using $f(u_\pm) = 0$, we find

$$-\Gamma \bar{u}_{xx} + f(\bar{u}) = 0. \tag{20}$$

We set $U = \bar{u}$ and $V = \bar{u}_x$, and write this as a first order system

$$\begin{aligned} U' &= V \\ V' &= \Gamma^{-1} f(U). \end{aligned} \tag{21}$$

Upon linearization about the endstates $(u_\pm, 0)$ we obtain

$$\begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}' = \begin{pmatrix} 0 & I \\ \Gamma^{-1} f'(u_\pm) & 0 \end{pmatrix} \begin{pmatrix} \tilde{U} \\ \tilde{V} \end{pmatrix}.$$

The associated eigenvalues are $\{-\sqrt{\nu_j^\pm}\}_{j=1}^m$ and $\{+\sqrt{\nu_j^\pm}\}_{j=1}^m$, where the values $\{\nu_j^\pm\}_{j=1}^m$ are the (necessarily positive) eigenvalues of $\Gamma^{-1} f'(u_\pm)$. Clearly, the points $(u_\pm, 0)$ both have an m-dimensional unstable manifold and an m-dimensional stable manifold. In this way we see that $\bar{u}(x)$ must correspond with a connection either between the m-dimensional unstable manifold of $(u_-, 0)$ and the m-dimensional stable manifold of $(u_+, 0)$ or vice versa.

The existence theories for our two cases (gradient systems and Boyer-Lapuerta systems) are quite different, and for space considerations we will focus here on the theory for gradient systems and note that interested readers can find the analysis for Boyer-Lapuerta systems in Section 2.2 of [30].

3.1. Gradient Systems.

In the standard case that f can be written as the gradient of some function, $f = F'$, system (21) has been the subject of considerable study. In particular, existence of transition front solutions in this case has been established by Alikakos and Fusco in [2] (see also the related analysis of Stefanopoulos [46]). The following theorem is a slight Modification of Theorem 3.6 of [2].

Theorem 1 (Existence for gradient systems). *Let (H0) hold and suppose $F \in C^4(\mathbb{R}^m)$ has precisely $m+1$ local minima $\{\xi_j\}_{j=1}^{m+1}$ such that $F''(\xi_j)$ is positive definite for each $j = 1, 2, ..., m+1$. In addition, suppose u_- and u_+ are elements of the set $\{\xi_j\}_{j=1}^{m+1}$, $u_- \neq u_+$, and*

$$F(tu_- + (1-t)u_+) > 0,$$

for all $t \in (0,1)$. Then there exists a transition front $\bar{u} \in C^5(\mathbb{R})$ so that (3) holds and $\bar{u}(\pm\infty) = u_\pm$.

Moreover, $\bar{u}(x)$ minimizes the energy functional

$$E(u) = \int_{\mathbb{R}} F(u) + \frac{1}{2}(\Gamma u_x, u_x) dx,$$

where $(\cdot, \cdot)$ denotes inner product on $\mathbb{R}^n$ (i.e., dot product).

Notes on the proof of Theorem 1. Aside from the brief observations we make here, this theorem was established in [2] Theorem 3.6.

While the analysis of [2] is carried out with Γ taken as the identity matrix, we can reduce our equation to their case by setting $\bar{v} := \Gamma^{1/2}\bar{u}$ and

$$F_0(\bar{v}) := F((\Gamma^{-1/2})\bar{v}).$$

That is, we now have

$$-\bar{v}_{xx} + F_0'(\bar{v}) = 0.$$

Clearly, F_0 has precisely $m+1$ local minima at $\{\Gamma^{1/2}\xi_j\}_{j=1}^{m+1}$, and

$$F_0(t\Gamma^{1/2}u_- + (1-t)\Gamma^{1/2}u_+) = F(u_-t + u_+(1-t)) > 0.$$

Under these conditions, Theorem 3.6 of [2], along with Extension Theorem 3.8 from the same paper, assert the existence of a weak $W^{1,2}_{\text{loc}}(\mathbb{R})$ solution to (20). According to Theorem 4.2 in [14] (also Theorem 4.4 on p. 277 of [13]), this solution must have Holder continuous derivatives, and so consequently it must agree with the Picard solution for system (21). Our claimed regularity is immediate. □

Here, we note for later reference,

$$E'(\bar{u})(\varphi) = \int_{\mathbb{R}} (F'(\bar{u}) - \Gamma\bar{u}_{xx}, \varphi)dx, \tag{22}$$

and

$$E''(\bar{u})(\psi, \phi) = \int_{\mathbb{R}} \Big((-\Gamma\partial_x^2 + F''(\bar{u}))\psi, \varphi\Big)dx, \tag{23}$$

where $(\cdot,\cdot)$ denotes Euclidean inner product. In particular, the assertion that $\bar{u}$ is a minimizer of E ensures that the operator

$$H := -\Gamma\partial_x^2 + F''(\bar{u})$$

is non-negative.

We observe that the transition front guaranteed by Theorem 1 may not be a unique minimizer of E, and so this does not guarantee that $\bar{u}(x)$ correponds with a transverse connection in system (21). On the other hand, the derivative statement in our Condition 1 can be regarded as a transversality condition. For the case $m = 2$, and for bulk free energy densities of form

$$F(u_1, u_2) = |h(u_1 + iu_2)|^2, \tag{24}$$

where h is analytic on $\mathbb{C}$, Alikakos, Betelu, and Chen have shown that the transition fronts guaranteed by Theorem 1 are unique: i.e., given a valid pair of endstates u_- and u_+ there is precisely one transition wave $\bar{u}(x)$ that solves (20) and satisfies $\bar{u}(\pm\infty) = u_\pm$. (See [1].) Generically, these can be reversed, so that there will also be a solution so that $\bar{u}(\pm\infty) = u_\mp$. For example, if we would like to work with the case of three minima located at the standard points $(0,0)$, $(1,0)$, and $(0,1)$, we can take

$$h(z) = z(z-1)(z-i),$$

and the theorem of Alikakos, Betelu, and Chen guarantees we have a unique solution. (Of course, this corresponds with a sixth order bulk free energy density polynomial rather than the more standard quartic.) This uniqueness guarantees transversality, and also that the waves we investigate numerically by the methods of [31] are the waves guaranteed by Theorem 1.

4. ODE Estimates

The general eigenvalue problem is

$$\Big(M(x)(-\Gamma\phi_{xx} + B(x)\phi)_x\Big)_x = \lambda\phi, \tag{25}$$

where $B(x) := f'(\bar{u}(x))$ and (with a slight abuse of notation) $M(x) := M(\bar{u}(x))$. We set $W_j = \partial_x^{j-1}\phi$, $j = 1,2,3,4$, and regard this equation as the first order system $W' = \mathbb{A}(x;\lambda)W$. As $x \to \pm\infty$, we can write this system as

$$W' = \mathbb{A}_\pm(\lambda)W + Q_\pm(x;\lambda)W, \tag{26}$$

where

$$\mathbb{A}_{\pm}(\lambda) = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -\lambda\Gamma^{-1}M_{\pm}^{-1} & 0 & \Gamma^{-1}B_{\pm} & 0 \end{pmatrix}, \tag{27}$$

and there exists $\eta > 0$ so that

$$Q_{-}(x;\lambda) = \mathbf{O}(e^{-\eta|x|}), x \to -\infty; \quad Q_{+}(x;\lambda) = \mathbf{O}(e^{-\eta|x|}), x \to +\infty,$$

uniformly for λ sufficiently small. We note that the $4m \times 4m$ matrices $Q_{\pm}$ only have non-zero entries in their last m rows.

While the eigenvalues of $\mathbb{A}_{\pm}(\lambda)$ can be computed directly using, for example, the determinant identity

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B),$$

valid whenever A is a non-singular matrix, it is more straightforward (and is equivalent) to simply look for solutions of the form $\phi = e^{\mu x}r$ of the asymptotic form of (25),

$$-M_{\pm}\Gamma\phi'''' + M_{\pm}B_{\pm}\phi'' = \lambda\phi.$$

We find

$$\Big(- \mu^4 M_{\pm}\Gamma + \mu^2 M_{\pm}B_{\pm} - \lambda I\Big)r = 0.$$

We will divide our discussion of the growth and decay modes $\mu(\lambda)$ into two cases: (1) *fast rates*, for which $\mu(0) \neq 0$; and (2) *slow rates*, for which $\mu(0) = 0$.

Fast rates. For the fast rates, $\mu(0) = \mu_0 \neq 0$, and for $\lambda = 0$ we have

$$\det(-\mu_0^4 M_{\pm}\Gamma + \mu_0^2 M_{\pm}B_{\pm}) = \mu_0^{2m}\det(M_{\pm}\Gamma)\det(\Gamma^{-1}B_{\pm} - \mu_0^2 I) = 0.$$

Since Γ and $M_{\pm}$ are positive definite, we find that the values for μ_0^2 are the eigenvalues of $\Gamma^{-1}B_{\pm}$. If the eigenvalues of this matrix are distinct, it follows from Theorem XII.1 of [45] that the associated fast $\mu(\lambda)^2$ are analytic. If the eigenvalues of $\Gamma^{-1}B_{\pm}$ are not distinct, we have from (H4) that the fast eigenvalues are nonetheless analytic in λ. In either case, it's clear that since the matrices $B_{\pm}$ are additionally symmetric and positive definite the eigenvalues of $\Gamma^{-1}B_{\pm}$ are positive. I.e.,

$$\Gamma^{-1}B_{\pm}v = \mu_0^2 v \Rightarrow B_{\pm}v = \mu_0^2\Gamma v \Rightarrow \langle v, B_{\pm}v\rangle = \mu_0^2\langle v, \Gamma v\rangle \Rightarrow \mu_0^2 = \frac{\langle v, B_{\pm}v\rangle}{\langle v, \Gamma v\rangle} > 0.$$

Our notation will be

$$\sigma(\Gamma^{-1}B_{\pm}) = \{\nu_j^{\pm}\}_{j=1}^{m}, \tag{28}$$

ordered so that $j < k \Rightarrow \nu_j^{\pm} \le \nu_k^{\pm}$. We conclude that for $j = 1, \dots, m$ the fast rates $\{\mu_j^{\pm}\}_{j=1}^{m}$ and $\{\mu_j^{\pm}\}_{j=3m+1}^{4m}$ are given by

$$\begin{aligned} \mu_j^{\pm}(\lambda) &= -\sqrt{\nu_{m+1-j}^{\pm}} + \mathbf{O}(|\lambda|) \\ \mu_{3m+j}^{\pm}(\lambda) &= \sqrt{\nu_j^{\pm}} + \mathbf{O}(|\lambda|), \end{aligned} \tag{29}$$

where for consistency the indices are chosen so that $j < k \Rightarrow \mu_j^\pm \leq \mu_k^\pm$. The eigenvectors $\{V_j^\pm(\lambda)\}_{j=1}^m$ and $\{V_j^\pm(\lambda)\}_{j=3m+1}^{4m}$ associated with these eigenvalues have the form

$$V_j^\pm = \begin{pmatrix} r_j^\pm \\ \mu_j^\pm r_j^\pm \\ (\mu_j^\pm)^2 r_j^\pm \\ (\mu_j^\pm)^3 r_j^\pm \end{pmatrix}, \tag{30}$$

where $r_j^\pm(\lambda)$ satisfies

$$\Big(- (\mu_j^\pm)^4 M_\pm \Gamma + (\mu_j^\pm)^2 M_\pm B_\pm - \lambda I \Big) r_j^\pm = 0. \tag{31}$$

Since $\mu_j^\pm = -\mu_{4m+1-j}^\pm$ we clearly have

$$r_j^\pm(\lambda) = r_{4m+1-j}^\pm(\lambda),$$

for $j = 1, \dots, m$. Finally, the leading term $r_j^\pm(0)$ is an eigenvector of $\Gamma^{-1}B_\pm$ associated with the eigenvalue $(\mu_j^\pm(0))^2$.

Slow rates. For the slow rates, for which $\mu(0) = 0$, we set $\omega = \mu^2$ so that our characteristic equation becomes

$$\det \Big(- \omega^2 M_\pm \Gamma + \omega M_\pm B_\pm - \lambda I \Big) = 0. \tag{32}$$

In this case, when $\lambda = 0$ we have that $\omega_0 = 0$ is repeated m times, and so Theorem XII.1 of [45] does not apply directly. Instead, we work with the scaled variable ζ, defined so that $\omega = \lambda\zeta$. In this way, (32) becomes

$$\det \Big(- \lambda^2 \zeta^2 M_\pm \Gamma + \lambda \zeta M_\pm B_\pm - \lambda I \Big) = 0.$$

Upon dividing by λ^m, we have

$$\det \Big(- \lambda \zeta^2 M_\pm \Gamma + \zeta M_\pm B_\pm - I \Big) = 0,$$

where now setting $\lambda = 0$ we find the values of $\zeta(0)$ are precisely the eigenvalues of $B_\pm^{-1} M_\pm^{-1}$. If the eigenvalues of $B_\pm^{-1} M_\pm^{-1}$ are distinct, we can conclude, again from Theorem XII.1 of [45], that the $\zeta(\lambda)$ are analytic in λ. We have, then,

$$\omega(\lambda) = \sum_{j=1}^\infty a_j \lambda^j,$$

and so the slow modes $\mu^\pm(\lambda)$ have the form $\sqrt{\lambda} h(\lambda)$, where h is an analytic function in λ (for $|\lambda|$ sufficiently small) with $h(0) \neq 0$. Calculating almost precisely as for the case of $\Gamma^{-1}B_\pm$ we find that the eigenvalues of $B_\pm^{-1} M_\pm^{-1}$ are all real and positive. (We recall from (H1) and (H3) that $B_\pm^{-1}$ and $M_\pm^{-1}$ are both symmetric positive definite matrices.) Our notation will be

$$\sigma(M_\pm B_\pm) = \{\beta_j^\pm\}_{j=1}^m,$$

where again our choice of ordering is $j < k \Rightarrow \beta_j \leq \beta_k$. We conclude that for each $j = 1, \ldots, m$ we have $\omega_j(\lambda) = \frac{\lambda}{\beta_j^\pm} + \mathbf{O}(|\lambda|^2)$, and so the slow rates are $\{\mu_j^\pm\}_{j=m+1}^{3m}$

$$
\begin{aligned}
\mu_{m+j}^\pm(\lambda) &= -\sqrt{\frac{\lambda}{\beta_j^\pm}} + \mathbf{O}(|\lambda|^{3/2}), \quad j = 1, \ldots, m \\
\mu_{2m+j}^\pm(\lambda) &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^\pm}} + \mathbf{O}(|\lambda|^{3/2}), \quad j = 1, \ldots, m.
\end{aligned}
\tag{33}
$$

Similarly as with the case of fast modes, if the eigenvalues of $B_\pm^{-1}M_\pm^{-1}$ are not distinct, we have directly from (H4) that for $|\lambda|$ sufficiently small we can write $\mu(\lambda) = \sqrt{\lambda}h(\lambda)$ for some function $h(\lambda)$ that is analytic in λ.

The eigenvectors $\{V_j^\pm(\lambda)\}_{j=m+1}^{3m}$ associated with these eigenvalues have the form (30) where $r_j^\pm(\lambda)$ satisfies (31). Since $\mu_j^\pm = -\mu_{4m+1-j}^\pm$ for $j = 1, \ldots, 2m$ we clearly have

$$
r_j^\pm(\lambda) = r_{4m+1-j}^\pm(\lambda),
$$

for $j = m+1, \ldots, 2m$. Finally, the values $\{\beta_j\}_{j=1}^m$ and $\{r_j^-(0)\}_{j=2m+1}^{3m}$, along with $\{r_j^+(0)\}_{j=m+1}^{2m}$ are sufficiently important to our later calculations that we summarize their roles in the following remark.

We are now prepared to state our basic ODE lemma.

Lemma 1. *Under Conditions (C1)-(C3), there exist values $\eta > 0$ and $r > 0$ so that for a choice of linearly independent solutions of the eigenvalue problem (25), we have the following estimates, uniformly in the set $\{\lambda : \lambda \in B(0,r), Arg\lambda \neq \pi\}$:*

(I) For $x \leq 0$ and $k = 0, 1, 2, 3$ we have:

(i) For $j = 1, \ldots, 2m$

$$
\partial_x^k \phi_j^-(x;\lambda) = e^{\mu_{2m+j}^-(\lambda)x}\Big(\mu_{2m+j}^-(\lambda)^k r_{2m+j}^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);
$$

(ii) For $j = 1, \ldots, m$

$$
\partial_x^k \psi_j^-(x;\lambda) = e^{\mu_j^-(\lambda)x}\Big(\mu_j^-(\lambda)^k r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);
$$

(iii) For $j = m+1, \ldots, 2m$

$$
\partial_x^k \psi_j^-(x;\lambda) = \frac{1}{\mu_j^-(\lambda)}\Big(\mu_j^-(\lambda)^k e^{\mu_j^-(\lambda)x} - (-\mu_j^-(\lambda))^k e^{-\mu_j^-(\lambda)x}\Big) r_j^-(\lambda) + \mathbf{O}(e^{-\eta|x|});
$$

(II) For $x \geq 0$ and $k = 0, 1, 2, 3$ we have:

(i) For $j = 1, \ldots, 2m$

$$
\partial_x^k \phi_j^+(x;\lambda) = e^{\mu_j^+(\lambda)x}\Big(\mu_j^+(\lambda)^k r_j^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big);
$$

(ii) For $j = 1, \ldots, m$

$$
\partial_x^k \psi_j^+(x;\lambda) = \frac{1}{\mu_{2m+j}^+(\lambda)}\Big(\mu_{2m+j}^+(\lambda)^k e^{\mu_{2m+j}^+(\lambda)x} - (-\mu_{2m+j}^+(\lambda))^k e^{-\mu_{2m+j}^+(\lambda)x}\Big) r_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|});
$$

(iii) For $j = m+1, \ldots, 2m$

$$
\partial_x^k \psi_j^+(x;\lambda) = e^{\mu_{2m+j}^+(\lambda)x}\Big(\mu_{2m+j}^+(\lambda)^k r_{2m+j}^+(\lambda) + \mathbf{O}(e^{-\eta|x|})\Big).
$$

Remark 3. *The proof of Lemma 1 can be found in* [30]*; here, we simply record some useful remarks.*

1. The fast decay modes are $\{\phi_j^-\}_{j=m+1}^{2m}$ *and* $\{\phi_j^+\}_{j=1}^{m}$. *Likewise, the slow decay modes are* $\{\phi_j^-\}_{j=1}^{m}$ *and* $\{\phi_j^+\}_{j=m+1}^{2m}$.

2. The rates of growth and decay can be characterized for convenient reference as follows: for $j = 1, \dots, m$,

$$\begin{aligned} \mu_j^{\pm}(\lambda) &= -\sqrt{\nu_{m+1-j}^{\pm}} + \mathbf{O}(|\lambda|) \\ \mu_{m+j}^{\pm}(\lambda) &= -\sqrt{\frac{\lambda}{\beta_j^{\pm}}} + \mathbf{O}(|\lambda|^{3/2}), \\ \mu_{2m+j}^{\pm}(\lambda) &= \sqrt{\frac{\lambda}{\beta_{m+1-j}^{\pm}}} + \mathbf{O}(|\lambda|^{3/2}), \\ \mu_{3m+j}^{\pm}(\lambda) &= \sqrt{\nu_j^{\pm}} + \mathbf{O}(|\lambda|). \end{aligned} \tag{34}$$

3. We recall that for $j = 1, \dots, 2m$

$$\begin{aligned} \mu_j^{\pm}(\lambda) &= -\mu_{4m+1-j}^{\pm}(\lambda) \\ r_j^{\pm}(\lambda) &= r_{4m+1-j}^{\pm}(\lambda). \end{aligned} \tag{35}$$

4. For $j = 1, \dots, m$ *and* $j = 3m+1, \dots, 4m$ *the leading terms* $r_j^{\pm}(0)$ *are eigenvectors of* $\Gamma^{-1}B_{\pm}$ *associated with the eigenvalue* $(\mu_j^{\pm}(0))^2$. *Likewise, for* $j = m+1, \dots, 3m$ *the leading terms* $r_j^{\pm}(0)$ *are eigenvectors of* $B_{\pm}^{-1}M_{\pm}^{-1}$.

5. Asymptotic stability

Generally, if the initial value for (1) is taken as a small perturbation of $\bar{u}(x)$, the solution $u(t,x)$ will approach a shift of $\bar{u}(x)$ rather than the front itself (orbital stability). Following [33], we proceed by tracking this shift locally in time, our location denoted by $\delta(t)$, which is standard notation in the literature and should not be confused with a Dirac delta function. More precisely, we include this shift in our analysis by defining our perturbation $v(t,x)$ as

$$v(t,x) := u(t, x+\delta(t)) - \bar{u}(x). \tag{36}$$

At this point, $\delta(t)$ is yet undetermined, and indeed one of the most important aspects of our approach to this problem is that it allows us to make an effective choice of $\delta(t)$. Upon substitution of $u(t, x+\delta(t))$ into (1) we obtain the perturbation equation

$$v_t = \Big(M(x)(-\Gamma v_{xx} + B(x)v)_x\Big)_x + \bar{u}_x(x)\dot{\delta}(t) + v_x\dot{\delta}(t) + Q_x, \tag{37}$$

where $Q = Q(x, v, v_x, v_{xxx})$ is at least C^2 in all its variables, and if

$$|v| + |v_x| + |v_{xxx}| \le \tilde{C}$$

for some constant $\tilde{C}$, then there exists a constant C so that

$$|Q| \le C\Big(|v||v_x| + e^{-\alpha|x|}|v|^2 + |v||v_{xxx}|\Big), \tag{38}$$

where α is described in (C1)-(C2) above. On one hand, this is a beneficial nonlinearity, because $|v_x|$ and $|v_{xxx}|$ will generally decay faster than $|v|$ as $|x|$ or t tends to ∞, and so each of these bounds is better than the standard nonlinearity $|v|^2$ encountered in the analysis of viscous conservation laws. On the other hand, for small values of t, derivatives of v generally blow up, and v_{xxx} is problematic in this regard. Our short time analysis is designed primarily to address this difficulty.

Let $G(t, x; y)$ denote the Green's function associated with the linear equation $v_t = Lv$, where L is as in (5), so that, in the standard distributional sense,

$$\begin{aligned} G_t &= LG \\ G(0, x; y) &= \delta_y(x) I, \end{aligned} \tag{39}$$

where I denotes an $m \times m$ identity matrix, and of course $\delta_y(x)$ is a standard Dirac delta function. Integrating (37), we find

$$\begin{aligned} v(t, x) &= \int_{-\infty}^{+\infty} G(t, x; y) v_0(y) dy + \delta(t) \bar{u}_x(x) \\ &- \int_0^t \int_{-\infty}^{+\infty} G_y(t - s, x; y) \Big[\dot{\delta}(s) v(s, y) + Q(s, y)\Big] dy ds, \end{aligned} \tag{40}$$

where in deriving this equation we have (1) observed that since $\bar{u}_x(x)$ is a stationary solution for $v_t = Lv$ we must have $e^{Lt}\bar{u}_x(x) = \bar{u}_x(x)$; (2) assumed our eventual choice of $\delta(t)$ has the natural property $\delta(0) = 0$; and (3) integrated the standard nonlinear integral by parts. To be clear, we do not assume at this stage that solutions of (40) are necessarily solutions of (37). Rather, our approach will be to work directly with (40) and use our estimates on G and v to establish the correspondence. We consider the condition $\delta(0) = 0$ to be natural, because $\delta(t)$ should capture the shift obtained as perturbation mass accumulates near the transition layer, and generally this accumulation will take some time.

We remark that (40) can be expressed in terms of the semigroup e^{Lt} as

$$v(t, x) = e^{Lt} v_0 + \delta(t) \bar{u}_x(x) + \int_0^t e^{L(t-s)} \Big[\dot{\delta}(s) v(s, \cdot) + Q(s, \cdot)\Big]_y ds. \tag{41}$$

Using the resolvent kernel estimates we derive in Section 3.2 of [31], we can verify that $-L$ is sectorial on $L^p(\mathbb{R})$, $1 \le p < \infty$, and so generates an analytic semigroup. (See, for example, [36] or [44].) If we assume additional regularity on f and M we can ensure $-L$ is sectorial on $W^{k,p}$ spaces: precisely, if $f \in C^{3+k}$ and $M \in C^{1+k}$ then $-L$ will be sectorial on $W^{k,p}$. This serves to establish the spectral representation (inverse Laplace transform)

$$e^{Lt} = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} (\lambda I - L)^{-1} d\lambda, \tag{42}$$

where Γ is a contour in the resolvent set of L, entirely to the right of $\sigma(L)$, so that $\arg \lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$. (In fact, we can relax this last condition to the extent allowed by analyticity and Cauchy's Theorem.) If ϕ is in an appropriate Banach space, such as those listed above, then

$$(\lambda I - L)^{-1} \phi = \int_{-\infty}^{+\infty} G_\lambda(x, y) \phi(y) dy, \tag{43}$$

where G_λ denotes the resolvent kernel associated with L. We have, then,

$$e^{Lt} \phi = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \int_{-\infty}^{+\infty} G_\lambda(x, y) \phi(y) dy d\lambda. \tag{44}$$

Our estimates on G_λ, derived in Section 3 of [31], will verify that we can exchange the order of integration, and so we have

$$e^{Lt} \phi = \int_{-\infty}^{+\infty} G(t, x; y) \phi(y) dy,$$

where

$$G(t,x;y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\lambda(x,y)d\lambda. \tag{45}$$

More directly, we can employ our estimates on $G_\lambda(x,y)$ to show that if G is defined as in (45) then (39) can be verified directly.

In our detailed analysis of G we proceed by decomposing G into two parts, an *excited* term E that does not decay as $t \to \infty$ (and is associated with the leading eigenvalue $\lambda = 0$ and mass accumulation at the origin), and a higher order term $\tilde{G}(t,x;y)$ that does decay as $t \to \infty$. This approach, following [22, 33, 47, 49] and others, will allow us to choose our shift $\delta(t)$. We will find that E can be written as $E(t,x;y) = \bar{u}_x(x)e(t;y)$, for a function $e(t;y)$ that will be specified below, and so we can express G as

$$G(t,x;y) = \bar{u}_x(x)e(t;y) + \tilde{G}(t,x;y), \tag{46}$$

so that (40) becomes

$$\begin{aligned} v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds \\ &+ \bar{u}_x(x)\Big\{\delta(t) + \int_{-\infty}^{+\infty} e(t;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds\Big\}. \end{aligned} \tag{47}$$

Our goal will be to choose $\delta(t)$ in such a way that the entire expression multiplying $\bar{u}_x(x)$ in (47) is annihilated. That is, we would like $\delta(t)$ to solve the integral equation

$$\delta(t) = -\int_{-\infty}^{+\infty} e(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_y(t-s;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds, \tag{48}$$

leaving v to solve

$$v(t,x) = \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y) + Q(s,y)\Big]dyds. \tag{49}$$

In principle now, we would like to establish existence of v, along with a bound on asymmmptotic behavior, by closing an iteration on (48)–(49). For such an argument we must be clear about which functions must be carried through the iteration and which can be analyzed after the iteration, using the obtained bounds. Of particular importance in this regard, $\delta(t)$ does not appear directly in (49), and so it suffices to couple (49) with an equation for $\dot{\delta}(t)$, rather than for $\delta(t)$ itself. (Of course, v depends on δ, and we will accommodate that dependence with our short-time theory; see Section 6.2 of [31].) Afterward, estimates on $\delta(t)$ can be obtained directly from (48). Also, the nonlinearity Q depends on v_x and v_{xxx} (in addition, of course, to dependence on x and v), and so we must either couple (49) with integral equations for these functions or obtain estimates on them in terms of the functions we do iterate. It's straightforward to show that v_{xxx} can be bounded in terms of x, v_x, and $\delta(t)$ for t bounded away from 0, and can easily be estimated for t near 0, and so our approach will be to iterate with the variables v, v_x, and $\dot{\delta}(t)$, and to obtain estimates on v_{xxx} and $\delta(t)$ after the iteration. (Though the connection between v_x and v_{xxx} will be used during the course of the iteration.) In this way, we will carry out an iteration on the $2m+1$ integral equations,

$$\begin{aligned}
v(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_y(t-s,x;y)\Big[\dot{\delta}(s)v(s,y)+Q(s,y)\Big]dyds \\
v_x(t,x) &= \int_{-\infty}^{+\infty} \tilde{G}_x(t,x;y)v_0(y)dy - \int_0^t \int_{-\infty}^{+\infty} \tilde{G}_{xy}(t-s,x;y)\Big[\dot{\delta}(s)v(s,y)+Q(s,y)\Big]dyds \quad (50) \\
\dot{\delta}(t) &= -\int_{-\infty}^{+\infty} e_t(t;y)v_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_{ty}(t-s;y)\Big[\dot{\delta}(s)v(s,y)+Q(s,y)\Big]dyds.
\end{aligned}$$

In deriving this third equation we have anticipated the choice of $e(t,y)$ taken below, which satisfies $e(0,y)\equiv 0$.

Our first result regards estimates on $G(t,x;y)$ and its derivatives. In expressing this theorem, it's convenient to separate short and long time behavior by taking a function $\varrho(t)$ so that $\varrho \in C^\infty[0,\infty)$, with $\varrho(t)=0$ for $0\le t<T_1$ and $\varrho(t)=1$ for $t\ge T_2$, where T_1 and T_2 are any constants so that $0<T_1<T_2$. Simply to be specific, we take

$$\varrho(t) := \varphi_\epsilon * \chi(t),$$

where φ is the standard mollifier

$$\varphi(t) := \begin{cases} Ce^{\frac{1}{t^2-1}} & |t|<1 \\ 0 & |t|\ge 1, \end{cases}$$

(C chosen so that $\int_{\mathbb{R}} \varphi(t)dt = 1$), $\varphi_\epsilon(t) := \frac{1}{\epsilon}\varphi(\frac{t}{\epsilon})$, and $\chi(t)$ denotes a characteristic function on $[\frac{1}{2},\infty)$. Taking $\epsilon=\frac{1}{4}$, we obtain $T_1=\frac{1}{4}$ and $T_2=\frac{3}{4}$.

Theorem 2. *Suppose Conditions (C1)-(C3) hold, and also that spectral Condition 1 holds. Then given any time thresholds $T_1>0$ and $T_2>0$ there exist constants $\eta>0$ (sufficiently small), and $C>0$, $K>0$, $M>0$ (sufficiently large) so that the Green's function described in (39) can be bounded as follows: there exists a splitting*

$$G(t,x;y) = \bar{u}_x(x)e(t;y) + \tilde{G}(t,x;y),$$

so that for $y<0$:

(I) (Excited terms)

(i) Main estimates:

$$\begin{aligned}
e(t;y) &= \Big(\frac{2}{\sqrt{\pi}}\sum_{j=m+1}^{2m} c_j^- \tilde{r}_j^-(0)\int_{-\infty}^{\frac{y}{\sqrt{4\beta_{j-m}^- t}}} e^{-z^2}dz + R_e(t;y)\Big)\varrho(t) \\
e_y(t;y) &= \Big(\sum_{j=m+1}^{2m} \frac{c_j^-\tilde{r}_j^-(0)}{\sqrt{\beta_{j-m}^-\pi t}} e^{-\frac{y^2}{4\beta_{j-m}^- t}} + \partial_y R_e(t;y)\Big)\varrho(t)
\end{aligned}$$

where

$$\begin{aligned}
|R_e(t,y)| &\le Ct^{-1/2}e^{-y^2/Mt} \\
|\partial_y R_e(t,y)| &\le C\Big(t^{-1}e^{-y^2/Mt} + t^{-1/2}e^{-y^2/Mt}e^{-\eta|y|}\Big).
\end{aligned}$$

For brevity the (constant) values $\{\beta_j^-\}_{j=1}^m$ and $\{c_j^-\}_{j=m+1}^{2m}$, and the vectors $\{\tilde{r}_j^-(0)\}_{j=m+1}^{2m}$ are specified in a remark following the theorem statement.

(ii) Time derivatives:

$$\Big|e_t(t;y)\Big| \le C(1+t)^{-1}e^{-\frac{y^2}{Mt}}$$
$$\Big|e_{yt}(t;y)\Big| \le C(1+t)^{-3/2}e^{-\frac{y^2}{Mt}}.$$

(II) For $|x-y| \le Kt$ and $t \ge T_1$

$$\Big|\tilde{G}(t,x;y)\Big| \le Ct^{-1/2}e^{-\frac{(x-y)^2}{Mt}}$$
$$\Big|\tilde{G}_y(t,x;y)\Big| \le Ct^{-1}e^{-\frac{(x-y)^2}{Mt}}$$
$$\Big|\tilde{G}_x(t,x;y)\Big| \le C\Big[t^{-1/2}e^{-\eta|x|} + t^{-1}\Big]e^{-\frac{(x-y)^2}{Mt}}$$
$$\Big|\tilde{G}_{xy}(t,x;y)\Big| \le C\Big[t^{-1}e^{-\eta|x|}e^{-\frac{y^2}{Mt}} + t^{-1}e^{-\eta|x-y|} + t^{-3/2}e^{-\frac{(x-y)^2}{Mt}}\Big].$$

(III) For $|x-y| \ge Kt$ or $0 < t < T_2$

$$\Big|\partial^\alpha \tilde{G}(t,x;y)\Big| \le C\Big[t^{-\frac{1+|\alpha|}{4}}e^{-\frac{|x-y|^{4/3}}{Mt^{1/3}}} + e^{-\eta(|x|+t)}e^{-\frac{y^2}{Mt}}\Big]$$

where α is a standard multiindex in x and y with $|\alpha| \le 3$. In all cases symmetric estimates hold for $y > 0$.

Remark 4. *Using the notation of (C1)-(C3) we can, up to a choice of scaling, specify the values $\{\beta_j^\pm\}_{j=1}^m$ and $\{\tilde{r}_{m+j}^-(0)\}_{j=1}^m$ by the relation*

$$\tilde{r}_{m+j}^\pm(0)M_\pm B_\pm = \beta_j^\pm \tilde{r}_{m+j}^\pm(0).$$

Here, the $\tilde{r}_{m+j}^\pm(0)$ are row vectors, and since $\bar{u}_x(x)$ is a column vector, the product $\bar{u}_x(x)\tilde{r}_{m+j}^\pm$ is a matrix, as expected. The values $\{c_j\}_{j=m+1}^{2m}$ can be specified as

$$c_j = h_{(2m)j}^- \tilde{c}_j^-(0),$$

where the $\{\tilde{c}_j^-\}_{j=m+1}^{2m}$ are described in Lemma 3.5 of [31]*, while the values $\{h_{(2m)j}^-\}_{j=m+1}^{2m}$ are described in Lemma 3.9 of* [31] *Part (iv). Although we give these precise specifications to be complete, our analysis only requires the existence of such constants.*

The estimates on $\tilde{G}$ could be expressed in a more detailed form, similar to the expressions for $e(t;y)$, but our analysis won't require that much precision, and we have chosen to omit it. See [3, 22] *for more precise statements in the scalar case.*

Remark 5. *We will use the observation that by taking $T_2 > T_1$ we can ensure there is a region in the case $|x-y| \le Kt$ for which estimates (II) and (III) both hold.*

In Section 6 of [31], we show that the estimates of Theorem 2 are sufficient to close an iteration on the system (50). In this way, we establish the following theorem, which is the main result of our analysis.

Theorem 3. *Suppose $\bar{u}(x)$ is a transition front solution to (1) as described in (H2), and suppose (H0)-(H4) hold, as well as Spectral Condition 1. Then for Hölder continuous initial conditions $u(0,x) \in C^\gamma(\mathbb{R})$, any $0 < \gamma < 1$, with*

$$|u(0,x) - \bar{u}(x)| \le \epsilon(1+|x|)^{-3/2},$$

for some $\epsilon > 0$ sufficiently small, there exists a solution $u(t,x)$ of (1) so that for any $\sigma > 0$

$$u \in C^{1+\frac{\gamma}{4},4+\gamma}([\sigma,\infty)\times\mathbb{R}) \cap C^{\frac{\gamma}{4},\gamma}([0,\infty)\times\mathbb{R})$$

and a shift function $\delta \in C^{1+\frac{\gamma}{4}}[0,\infty)$ so that

$$\lim_{t\to 0^+}\delta(t) = 0; \quad \lim_{t\to\infty}\delta(t) = \delta_\infty \in \mathbb{R},$$

for which the following estimates hold: there exist constants $C > 0$ and $L > 0$ (sufficiently large) and a constant $\tilde{\eta} > 0$ (sufficiently small) so that

$$\begin{aligned}
|u(t,x+\delta(t)) - \bar{u}(x)| &\le C\epsilon\Big[(1+t)^{-1/2}e^{-\frac{x^2}{Lt}} + (1+|x|+\sqrt{t})^{-3/2}\Big] \\
|u_x(t,x+\delta(t)) - \bar{u}'(x)| &\le C\epsilon t^{-1/4}\Big[(1+t)^{-3/4}e^{-\frac{x^2}{Lt}} \\
&\quad + (1+t)^{-1/4}(1+|x|+\sqrt{t})^{-3/2} + (1+t)^{-1/4}e^{-\tilde{\eta}|x|}e^{-\frac{x^2}{Lt}}\Big] \\
|\delta(t) - \delta_\infty| &\le C\epsilon(1+t)^{-1/4} \\
|\dot{\delta}(t)| &\le C\epsilon(1+t)^{-1}.
\end{aligned}$$

Remark 6. *We've chosen to state our theorem with initial decay $(1+|x|)^{-3/2}$, but a similar statement can be obtained for $(1+|x|)^{-r}$, any $r > 1$. Along these lines, we verify in [32] that under the same assumptions of Theorem 3 excepting the initial rate condition, stability in $W^{1,p}(\mathbb{R})$ follows so long as $\|v_0\|_{L^1(\mathbb{R})} + \|v_0\|_{L^\infty(\mathbb{R})} \le \epsilon$.*

Our Hölder space notation $C^{\frac{\gamma}{4},\gamma}([0,\infty)\times\mathbb{R})$ designates functions bounded in the norm

$$\|u\|_{C^{\frac{\gamma}{4},\gamma}} := \sup_{\substack{x\in\mathbb{R}\\ t\in[0,\infty)}} |u(t,x)| + \sup_{\substack{x_1,x_2\in\mathbb{R},x_1\neq x_2\\ t\in[0,\infty)}} \frac{|u(t,x_1)-u(t,x_2)|}{|x_1-x_2|^\gamma} + \sup_{\substack{x\in\mathbb{R}^n\\ t_1,t_2\in[0,\infty),t_1\neq t_2}} \frac{|u(t_1,x)-u(t_2,x)|}{|t_1-t_2|^{\frac{\gamma}{4}}}, \tag{51}$$

and similarly for $C^{1+\frac{\gamma}{4},4+\gamma}([\sigma,\infty)\times\mathbb{R})$.

5.1. **Asymptotic L^p stability.** We shall also present L^p estimates on $G(t,x;y)$ and its derivatives. For the proof, we shall refer the readers to [32].

Theorem 4. *Suppose Conditions (C1)-(C3) hold, and also that spectral Condition 1 holds. Then given any time thresholds $T_1 > 0$ and $T_2 > 0$ there exists a constant $C > 0$ (depending on T_1 and T_2) so that the Green's function described in (39) can be bounded as follows: there exists a splitting*

$$G(t,x;y) = \bar{u}'(x)e(t;y) + \tilde{G}(t,x;y),$$

so that:

(I) For all $t \ge 0$

$$\|e_t(t;\cdot)\|_{L^p} \le C(1+t)^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})}; \qquad \|e_{ty}(t;\cdot)\|_{L^p} \le C(1+t)^{-1-\frac{1}{2}(1-\frac{1}{p})},$$

and $e(t;y) \equiv 0$ for all $t \le 1/4$.

(II) For $t \ge T_1$

$$\sup_{y\in\mathbb{R}}\|\tilde{G}(t,\cdot;y)\|_{L^p_x} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})}; \qquad \sup_{x\in\mathbb{R}}\|\tilde{G}(t,x;\cdot)\|_{L^p_y} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})};$$

$$\sup_{y\in\mathbb{R}}\|\tilde{G}_y(t,\cdot;y)\|_{L^p_x} \le Ct^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})}; \qquad \sup_{x\in\mathbb{R}}\|\tilde{G}_y(t,x;\cdot)\|_{L^p_y} \le Ct^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{p})};$$

$$\sup_{y\in\mathbb{R}} \|\tilde{G}_x(t,\cdot;y)\|_{L^p_x} \le Ct^{-\frac{1}{2}}; \qquad \sup_{x\in\mathbb{R}} \|\tilde{G}_x(t,x;\cdot)\|_{L^p_y} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})};$$

$$\sup_{y\in\mathbb{R}} \|\tilde{G}_{xy}(t,\cdot;y)\|_{L^p_x} \le Ct^{-1}; \qquad \sup_{x\in\mathbb{R}} \|\tilde{G}_{xy}(t,x;\cdot)\|_{L^p_y} \le Ct^{-\frac{1}{2}(1-\frac{1}{p})}.$$

(III) For $0 < t < T_2$

$$\sup_{y\in\mathbb{R}} \|\partial^\alpha \tilde{G}(t,\cdot;y)\|_{L^p_x} \le Ct^{-\frac{|\alpha|}{4}-\frac{1}{4}(1-\frac{1}{p})}$$

$$\sup_{x\in\mathbb{R}} \|\partial^\alpha \tilde{G}(t,x;\cdot)\|_{L^p_y} \le Ct^{-\frac{|\alpha|}{4}-\frac{1}{4}(1-\frac{1}{p})}$$

where α *is a standard multiindex in* x *and* y *and* $|\alpha| \le 3$*;*

Theorem 5. *Suppose* $\bar{u}(x)$ *is a transition front solution to (1) as described in (H2), and suppose (H0)-(H4) hold, as well as Condition 1. Then for Hölder continuous initial conditions* $u(0,x) \in C^\gamma(\mathbb{R})$, $0 < \gamma < 1$, *with*

$$\|u(0,x) - \bar{u}(x)\|_{L^\infty} + \|u(0,x) - \bar{u}(x)\|_{L^1} \le \epsilon,$$

for some $\epsilon > 0$ *sufficiently small, there exists a solution* $u(t,x)$ *of (1)*

$$u \in C^{1+\frac{\gamma}{4},4+\gamma}((0,\infty)\times\mathbb{R}) \cap C^{\frac{\gamma}{4},\gamma}([0,\infty)\times\mathbb{R})$$

and a shift function $\delta \in C^{1+\frac{\gamma}{4}}[0,\infty)$ *so that*

$$\lim_{t\to 0^+} \delta(t) = 0; \quad \lim_{t\to\infty} \delta(t) = \delta_\infty \in \mathbb{R},$$

for which the following estimates hold: there exists a constant $C > 0$ *so that for each* $1 \le p \le \infty$

$$\begin{aligned} \|u(t,x+\delta(t)) - \bar{u}(x)\|_{L^p} &\le C\epsilon(1+t)^{-\frac{1}{2}(1-\frac{1}{p})} \\ \|u_x(t,x+\delta(t)) - \bar{u}'(x)\|_{L^p} &\le C\epsilon t^{-1/4}(1+t)^{-1/4} \\ |\delta(t) - \delta_\infty| &\le C\epsilon(1+t)^{-1/4} \\ |\dot{\delta}(t)| &\le C\epsilon(1+t)^{-1}. \end{aligned}$$

Remark 7. *We gather several remarks associated with placing the current analysis in the broader context of related studies.*

1. This is the $L^1 \cap L^\infty \to L^p$ *analog to the pointwise theorem of* [31] *for which the authors assume*

$$|u(0,x) - \bar{u}(x)| \le \epsilon(1+|x|)^{-3/2},$$

which (with a slightly different value for ϵ*) is a special case of the assumption made in Theorem 3.*

2. The pointwise semigroup approach taken here has its origins in the study of viscous conservation laws [17, 18, 33, 49]*, and in this context Cahn-Hilliard transition fronts can be viewed as a case in which all characteristic speeds are zero, neither Lax nor undercompressive (in some sense akin to degenerate waves (see* [19, 20, 21, 24, 34]*), though also with properties of overcompressive waves (see* [49] *for further discussion of these classifications)). In particular, we contrast our current analysis with the case of phase-transitional fronts in equations of viscoelasticity, which can be undercompressive* [35]*. One salient technical point that arises in the analysis of undercompressive viscous shocks is that* y*-derivatives do not increase the decay rate in time of the Green's residual* $\tilde{G}(x,t;y)$ [35, 49]*. For such cases the nonlinear* L^p *argument employed here is insufficient, and a refined pointwise nonlinear argument is necessary (as carried out in* [35]*).*

In this way, one observation we make in the current analysis is that transition fronts for Cahn-Hilliard equations are in some sense closer to Lax-type viscous profiles than to undercompressive profiles.

3. For single Cahn-Hilliard equations stability of transition fronts was first established by Bricmont, Kupiainen, and Taskinen in [3] *using renormalization group (RG) methods, and later by Howard using the pointwise semigroup framework employed here* [22]. *Extension of the RG approach to systems appears problematic, and one of the points we would like to emphasize is how naturally the pointwise semigroup framework extends to systems (one of the key observations of* [49]*).*

4. Stability of planar transition fronts arising in single Cahn-Hilliard equations in multiple space dimensions has been established by Korvola, Kupiainen, and Taskinen for dimensions 3 *and higher (see* [38, 39]*) and by Howard for dimensions* 2 *and higher* [23, 25]. *More recently, by employing methods similar to those in the current analysis, Howard has analyzed the case of planar transition fronts arising in Cahn-Hilliard systems in multiple space dimensions. In particular, spectral behavior is analyzed in* [27], *while linear and nonlinear behavior are analyzed respectively in* [28] *and* [29].

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