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$\mathcal{R}$-BOUNDEDNESS OF SOLUTION OPERATOR FAMILIES FOR TWO-PHASE STOKES RESOLVENT PROBLEM AND ITS APPLICATION

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ABSTRACT. The aim of this paper is to show the $\mathcal{R}$-boundedness of solution operator families of a two-phase Stokes resolvent problem and an application of the $\mathcal{R}$-boundedness to some (time-dependent) two-phase Stokes problem in a bounded domain $\dot{\Omega}=\Omega_{+}\cup\Omega_{-}$. More precisely, let Ω be a bounded domain, with two boundaries $\Gamma_{\pm}$ ($\Gamma_{+}\cap\Gamma_{-}=\emptyset$), of N-dimensional Euclidean space $\mathbf{R}^{N}$ ($N\geq 2$), and then some closed hypersurface Γ divides Ω into two domains $\Omega_{\pm}\subset\Omega$ such that $\Omega_{+}\cap\Omega_{-}=\emptyset$ and $\Omega\setminus\Gamma=\Omega_{+}\cup\Omega_{-}$. We here suppose that $\Gamma\cap\Gamma_{+}=\emptyset$, $\Gamma\cap\Gamma_{-}=\emptyset$, and the boundaries of $\Omega_{\pm}$ consist of two parts Γ, $\Gamma_{\pm}$, respectively. The domains $\Omega_{\pm}$ are filled with viscous, incompressible, and immiscible fluids with density $\rho_{\pm}$ and viscosity $\mu_{\pm}$, respectively. In addition, on the boundaries Γ, Γ_{+}, and Γ_{-}, we consider some interface conditions, free boundary conditions, and the Dirichlet boundary condition, respectively. This is a joint work with Sri Maryani from Waseda University.

1. INTRODUCTION

This article is a brief survey of [MS], mainly.

Let Ω be a bounded domain of $\mathbf{R}^{N}$, $N\geq 2$, with two boundaries $\Gamma_{\pm}$ satisfying $\Gamma_{+}\cap\Gamma_{-}=\emptyset$. Assume that some closed hypersurface Γ divides Ω into two subdomains of Ω, that is, there are domains $\Omega_{\pm}\subset\Omega$ such that $\Omega_{+}\cap\Omega_{-}=\emptyset$ and $\Omega\setminus\Gamma=\Omega_{+}\cup\Omega_{-}$. It is also assumed that $\Gamma\cap\Gamma_{+}=\emptyset$, $\Gamma\cap\Gamma_{-}=\emptyset$, and the boundaries of $\Omega_{\pm}$ consist of two parts Γ, $\Gamma_{\pm}$, respectively. Set $\dot{\Omega}=\Omega_{+}\cup\Omega_{-}$ and $\Sigma_{\varepsilon,\lambda_{0}}=\{\lambda\in\mathbf{C}\mid|\arg\lambda|\leq\pi-\varepsilon, |\lambda|\geq\lambda_{0}\}$ for $0<\varepsilon<\pi/2$ and $\lambda_{0}>0$. In this paper, we consider the $\mathcal{R}$-boundedness of solution operator families for the following two-phase Stokes resolvent problem with resolvent parameter λ varying in $\Sigma_{\varepsilon,\lambda_{0}}$:

$$
\left\{\begin{aligned}
\lambda\mathbf{u}-\rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},\theta)&=\mathbf{f}, & \operatorname{div}\mathbf{u}&=g & &\text{in } \dot{\Omega},\\
[\![\mathbf{T}(\mathbf{u},\theta)\mathbf{n}]\!]&=[\![\mathbf{h}]\!], & [\![\mathbf{u}]\!]&=0 & &\text{on } \Gamma,\\
\mathbf{T}(\mathbf{u},\theta)\mathbf{n}_{+}&=\mathbf{k} & & & &\text{on } \Gamma_{+},\\
\mathbf{u}&=0 & & & &\text{on } \Gamma_{-}.
\end{aligned}\right. \tag{1.1}
$$

Here the unknowns $\mathbf{u}=(u_{1}(x),\ldots,u_{N}(x))^{T\dagger}$ and $\theta=\theta(x)$ are an N-component vector function and a scalar function, respectively, while the right members $\mathbf{f}=(f_{1}(x),\ldots,f_{N}(x))^{T}$, $g=g(x)$, $\mathbf{h}=(h_{1}(x),\ldots,h_{N}(x))^{T}$, and $\mathbf{k}=(k_{1}(x),\ldots,k_{N}(x))^{T}$ are give functions. Let $\rho_{\pm}$, $\mu_{\pm}$ be positive constants, and let χ_{D} be the indicator function of $D\subset\mathbf{R}^{N}$. Then $\rho=\rho_{+}\chi_{\Omega_{+}}+\rho_{-}\chi_{\Omega_{-}}$, $\mu=\mu_{+}\chi_{\Omega_{+}}+\mu_{-}\chi_{\Omega_{-}}$, and $\mathbf{T}(\mathbf{u},\theta)=\mu\mathbf{D}(\mathbf{u})-\theta\mathbf{I}$, where $\mathbf{I}$ is the $N\times N$ identity matrix and $\mathbf{D}(\mathbf{u})$ is doubled deformation tensor, that is, the (i,j) entry $D_{ij}(\mathbf{u})$ of $\mathbf{D}(\mathbf{u})$ is given by $D_{ij}(\mathbf{u})=\partial_{i}u_{j}+\partial_{j}u_{i}$ for $i,j=1,\ldots,N$ and $\partial_{i}=\partial/\partial x_{i}$. In addition, $\mathbf{n}$ denotes a unit normal vector on Γ, which points from Ω_{+} to Ω_{-}, and $\mathbf{n}_{+}$ the unit outward normal vector on Γ_{+}. For any function f defined on $\dot{\Omega}$, $[\![f]\!]$ denotes jump of f across the interface Γ as follows:

$$
[\![f]\!]=[\![f]\!](x)=\lim_{y\to x,\,y\in\Omega_{+}}f(y)-\lim_{y\to x,\,y\in\Omega_{-}}f(y)\quad(x\in\Gamma).
$$

$^{\dagger}\mathbf{M}^{T}$ denotes the transposed $\mathbf{M}$.

Here and subsequently, we use the following notation for differentiations: Let $f = f(x)$, $\mathbf{g} = \mathbf{g}(x) = (g_1(x), \dots, g_N(x))^T$, and $\mathbf{M} = (M_{ij}(x))$ $(i, j = 1, \dots, N)$ be a scalar-, a vector-, and a matrix-valued function on some domain of $\mathbf{R}^N$, respectively, and then

$$\nabla f = (\partial_1 f, \dots, \partial_N f)^T, \quad \Delta f = \sum_{j=1}^N \partial_j f, \quad \Delta \mathbf{g} = (\Delta g_1, \dots, \Delta g_N)^T,$$

$$\operatorname{div} \mathbf{g} = \sum_{j=1}^N \partial_j g_j, \quad \nabla^2 \mathbf{g} = \{\partial_j \partial_k g_l \mid j, k, l = 1, \dots, N\},$$

$$\nabla \mathbf{g} = \begin{pmatrix} \partial_1 g_1 & \dots & \partial_N g_1 \\ \vdots & \ddots & \vdots \\ \partial_1 g_N & \dots & \partial_N g_N \end{pmatrix}, \quad \operatorname{Div} \mathbf{M} = \left(\sum_{j=1}^N \partial_j M_{1j}, \dots, \sum_{j=1}^N \partial_j M_{Nj} \right)^T.$$

The two-phase Stokes resolvent problem (1.1) arises from a two-phase problem of the Navier-Stokes equations, which describes the motion of two viscous, incompressible, and immiscible fluids without taking surface tension into account. There are a lot of studies of two-phase problems for the Navier-Stokes equations. To see the history of the studies briefly, we restrict ourselves to the case where the two fluids are both viscous, incompressible, and immiscible in the following. Such a situation are treated in several function spaces as follows:

L_2-in-time and L_2-in-space setting. Denisova [Den90, Den94] treated the motion of a drop Ω_{+t}, which is the region occupied by the drop at time $t > 0$, in another liquid $\Omega_{-t} = \mathbf{R}^3 \setminus \overline{\Omega_{+t}}$. More precisely, [Den90] showed some estimates of solutions for linearized problems and [Den94] the local-in-time unique existence theorem of the two-phase problem describing the above situation with or without surface tension. In addition, Denisova [Den14] proved the unique existence of global-in-time solutions for small initial data and its exponential stability in the case where Ω_{-t} is bounded and surface tension does not work. Concerning non-homogeneous incompressible fluids, Tanaka [Tan93] showed the global-in-time unique existence theorem for small initial data under the same assumption about Ω_{-t} as in [Den14], but surface tension is taken into account.

Hölder function spaces. A series of papers Denisova-Solonnikov [DS91, DS95] and Denisova [Den93] treated the same motion as in [Den90, Den94] mentioned above. Especially, [DS91, Den93] established estimates of solutions for some linearized problems, and [DS95] proved the local-in-time unique existence theorem of the two-phase problem with surface tension. The global-in-time unique existence theorem was proved by Denisova [Den07] without surface tension and by Denisova-Solonnikov [DS11] with surface tension in the case where Ω_{-t} is bounded. Furthermore, there are other topics due to Denisova [Den05] and Denisova-Nečasová [DN08], which consider thermocapillary convection and Oberbeck-Boussinesq approximation, respectively.

L_p-in-time and L_p-in-space setting. Prüss and Simonett [PS10a, PS10b, PS11] treated the situation that two fluids occupy $\Omega_{\pm t} = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, \pm(x_N - h(x', t)) > 0\}$, respectively, where $h(x', t)$ is an unknown scalar function describing the interface $\Gamma_t = \{(x', x_N) \mid x' \in \mathbf{R}^{N-1}, x_N = h(x', t)\}$ of the fluids. [PS10b] and [PS11] proved the local solvability of the two-phase problem with surface tension and with surface tension and gravity, respectively, for small initial data. On the other hand, [PS10a] pointed out that the Rayleigh-Taylor instability happens if the gravity works and the fluid occupying Ω_{+t} is heavier than the other one . Furthermore, Hieber and Saito [HS] extended the results of the Newtonian case of [PS10b, PS11] to a generalized Newtonian one.

L_p-in-times and L_q-in-space setting. Shibata-Shimizu [SS11] showed a maximal L_p-L_q regularity theorem for a linearized system of the two-phase problem considered in [PS10a, PS11] mentioned above. In addition, [MS] extended [SS11] to general domains, which contain e.g. $\dot{\mathbf{R}}^N = \mathbf{R}^N_+ \cup \mathbf{R}^N_-$,

perturbed $\dot{\mathbf{R}}^N$, layers, perturbed layers, bounded domains, and exterior domains. Here $\mathbf{R}^N_+$, $\mathbf{R}^N_-$ are the open upper and lower half spaces, respectively.

In the present paper, we restrict ourselves to the case where Ω is bounded, and introduce the $\mathcal{R}$-boundedness of solution operator families of the two-phase Stokes resolvent problem (1.1), which is one of main objects proved in [MS]. In addition, as an application of the $\mathcal{R}$-boundedness, we prove a maximal L_p-L_q regularity theorem with exponential stability for some time-dependent problem associated with (1.1). The maximal L_p-L_q regularity theorem plays an important role to prove the global-in-time unique existence theorem for two-phase problems of the Navier-Stokes equations.

This paper consists of four sections.

Section 2 first introduces notation and definition used throughout this article. Next, our main results, that is, the $\mathcal{R}$-boundedness of solution operator families of (1.1) is stated.

Section 3 first gives us some reduced problem of (1.1), which are obtained by elimination of pressure term θ from (1.1). To elimination the pressure term θ, we use a result concerning the unique solvability of the weak Dirichlet-Neumann problem. In addition, we introduce some auxiliary problem, which is corresponding to the weak Dirichlet-Neumann problem with resolvent parameter λ. Subsection 3.1 tell us the fact that solutions to (1.1) is also solutions to the reduced problem with help of the auxiliary problem for suitable right members $\mathbf{f}$, g, $\mathbf{h}$, and $\mathbf{k}$. Subsection 3.2 shows that the opposite direction of Subsection 3.1 also holds. Namely, solutions of the reduced problem become one of (1.1). Subsection 3.3 introduce the $\mathcal{R}$-boundedness of solution operator families of the reduced problem, and then we have Theorem 2.3 in view of subsections 3.1, 3.2.

Section 4 proves a maximal L_p-L_q regularity theorem with exponential stability for some time-dependent problem associated with the two-phase Stokes resolvent problem (1.1). To show the maximal regularity theorem, we divide the time-dependent problem into two parts as follows: one is equations for non-zero initial data with homogeneous external forces and the other is equations for zero initial data with non-homogeneous external forces. In Subsection 4.1, we show an estimate with exponential stability of solutions to the case of non-zero initial data by means of analytic semigroup. In Subsection 4.2, we show an estimate with exponential stability of solution to the case of zero initial data.

2. Notation and main results

In this section, we first introduce the notation used throughout this paper. After that our main results will be stated.

2.1. **Notation.** Let D be an open set of $\mathbf{R}^N$, and let $1 \leq q \leq \infty$ and $1 \leq r < \infty$. Then $L_q(D)$, $W_q^m(D)$ with $m \in \mathbf{N}$, and $W_r^s(D)$ with $s \in (1,\infty) \setminus \mathbf{N}$ denote the usual Lebesgue spaces, Sobolev spaces, Sobolev-Slobodeckij spaces on D, while $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^m(D)}$, and $\|\cdot\|_{W_q^s(D)}$ their norms, respectively. For two Banach spaces X and Y, $\mathcal{L}(X,Y)$ is the set of all bounded linear operators from X to Y, and $\mathcal{L}(X)$ the abbreviation of $\mathcal{L}(X,X)$. Let U be a domain of $\mathbf{C}$, and then $\mathrm{Hol}\,(U,\mathcal{L}(X,Y))$ stands for the set of all $\mathcal{L}(X,Y)$-valued holomorphic functions defined on U. For $d \in \mathbf{N}$ with $d \geq 2$, X^d denotes the d-product space of a Banach space X. Let $\|\cdot\|_X$ be the norm of X, while $\|\cdot\|_X$ also denotes the norm of the product space X^d for short, that is, $\|\mathbf{f}\|_X = \sum_{j=1}^d \|f_j\|_X$ for $\mathbf{f} = (f_1,\dots,f_d)^T \in X^d$. Let $\mathbf{a} = (a_1,\dots,a_N)^T$ and $\mathbf{b} = (b_1,\dots,b_N)^T$, and then we write $\mathbf{a}\cdot\mathbf{b} =< \mathbf{a},\mathbf{b} >= \sum_{j=1}^N a_j b_j$. On the other hand, for any vector functions $\mathbf{u}$, $\mathbf{v}$ on D, we set $(\mathbf{u},\mathbf{v})_D = \int_D \mathbf{u}\cdot\mathbf{v}\,dx$ and $(\mathbf{u},\mathbf{v})_{\partial D} = \int_{\partial D} \mathbf{u}\cdot\mathbf{v}\,d\sigma$, where ∂D is the boundary of D and $d\sigma$ the surface element on ∂D.

We here introduce the definition of the $\mathcal{R}$-boundedness of operator families.

Definition 2.1. *Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for each natural number n, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and for all sequences $\{r_j(u)\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, there holds the inequality:*

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(u) T_j f_j \Big\|_Y^p \, du \le C \int_0^1 \Big\| \sum_{j=1}^n r_j(u) f_j \Big\|_X^p \, du.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$ on $\mathcal{L}(X,Y)$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}$.

Remark 2.2. It is well-known that $\mathcal{T}$ is $\mathcal{R}$-bounded for any $p \in [1,\infty)$, provided that $\mathcal{T}$ is $\mathcal{R}$-bounded for some $p \in [1,\infty)$. This fact follows from Kahane's inequality ([KW04, Theorem 2.4]).

To state our main results, we here introduce several function spaces. Given $1 < q < \infty$, we set $q' = q/(q-1)$. Let $W^1_{q,\Gamma_+}(\Omega) = \{f \in W^1_q(\Omega) \mid f = 0 \text{ on } \Gamma_+\}$, and also we define a solenoidal space $J_q(\Omega)$ by

$$J_q(\Omega) = \{\mathbf{f} \in L_q(\Omega)^N \mid (\mathbf{f}, \nabla\varphi)_\Omega = 0 \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega)\}.$$

Set $W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega) = \{\theta = \theta_1 + \theta_2 \mid \theta_1 \in W^1_q(\dot\Omega),\ \theta_2 \in W^1_{q,\Gamma_+}(\Omega)\}$. In addition, we introduce a space $\mathcal{DI}_q(\dot\Omega)$ defined by

$$\mathcal{DI}_q(\dot\Omega) = \{g \in W^1_q(\dot\Omega) \mid \exists \mathbf{G} \text{ s.t. } (g,\varphi)_{\dot\Omega} = -(\mathbf{G}, \nabla\varphi)_{\dot\Omega} \text{ for all } \varphi \in W^1_{q',\Gamma_+}(\Omega)\}. \tag{2.1}$$

In this case, we write $\mathbf{G} = \mathcal{G}(g)$. Let $\mathbf{n}_-$ be the unit outward normal vector on Γ_-. The space $\mathcal{DI}_q(\dot\Omega)$ is a date space for the divergence equation $\operatorname{div}\mathbf{u} = g$ in $\dot\Omega$ with boundary conditions: $[\![\mathbf{u}]\!] \cdot \mathbf{n} = 0$ on Γ and $\mathbf{u} \cdot \mathbf{n}_- = 0$ on Γ_-. This fact arises from the following observation: suppose that the divergence equation is solvable, and then

$$(g,\varphi)_{\dot\Omega} = (\operatorname{div}\mathbf{u}, \varphi)_{\dot\Omega} = -(\mathbf{u}, \nabla\varphi)_{\dot\Omega} \quad \text{for any } \varphi \in W^1_{q',\Gamma_+}(\Omega),$$

which implies the existence of $\mathbf{G}$ in (2.1). On the other hand, let $g \in \mathcal{DI}_q(\dot\Omega)$, and then

$$(g,\varphi)_{\dot\Omega} = -(\mathcal{G}(g), \nabla\varphi)_{\dot\Omega} = (\operatorname{div}\mathcal{G}(g), \varphi)_{\dot\Omega} - ([\![\mathcal{G}(g)]\!] \cdot \mathbf{n}, \varphi)_\Gamma - (\mathcal{G}(g) \cdot \mathbf{n}_-, \varphi)_{\Gamma_-} \tag{2.2}$$

for any $\varphi \in W^1_{q',\Gamma_+}(\Omega)$. Choosing $\varphi \in C_0^\infty(\Omega)$ with $\operatorname{supp}\varphi \subset \dot\Omega$ in (2.2) yields that $\operatorname{div}\mathcal{G}(g) = g$ in $\dot\Omega$. We also see that $[\![\mathcal{G}(g)]\!] \cdot \mathbf{n} = 0$ on Γ and $\mathcal{G}(g) \cdot \mathbf{n}_- = 0$ on Γ_- by choosing suitable φ in (2.2). Thus, $\mathbf{u} = \mathcal{G}(g)$ solves the divergence equation. If we set $\|g\|_{\mathcal{DI}_q(\dot\Omega)} = \|g\|_{W^1_q(\dot\Omega)} + \|\mathcal{G}(g)\|_{L_q(\dot\Omega)}$ for $g \in \mathcal{DI}_q(\dot\Omega)$, then $\mathcal{DI}_q(\dot\Omega)$ is a Banach space with norm $\|\cdot\|_{\mathcal{DI}_q(\dot\Omega)}$.

2.2. **Main results.** We here introduce main results of [MS].

Theorem 2.3. *Let $1 < q < \infty$, $0 < \pi < \pi/2$, $N < r < \infty$, and $\max(q,q') \le r$ with $q' = q/(q-1)$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W^{2-1/r}_r$ class. Then the following properties hold.*

(1) **Existence.** *Set*

$$\begin{aligned}
X_q &= \{(\mathbf{f}, g, \mathbf{h}, \mathbf{k}) \mid \mathbf{f} \in L_q(\dot\Omega)^N,\ g \in \mathcal{DI}_q(\dot\Omega),\ \mathbf{h} \in W^1_q(\dot\Omega)^N,\ \mathbf{k} \in W^1_q(\Omega_+)^N\},\\
\mathcal{X}_q &= \{(F_1, \dots, F_8) \mid F_1, F_2, F_4, F_6 \in L_q(\dot\Omega)^N,\\
&\qquad F_3 \in L_q(\dot\Omega),\ F_5 \in L_q(\dot\Omega)^{N^2},\ F_7 \in L_q(\Omega_+)^{N^2},\ F_8 \in L_q(\Omega_+)^N\}.
\end{aligned}$$

Then there exist a constant $\lambda_0 \ge 1$ and operator families:

$$\mathbf{A}(\lambda) \in \operatorname{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q, W^2_q(\dot\Omega)^N)), \quad \mathbf{P}(\lambda) \in \operatorname{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_q, W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)))$$

such that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ *and* $(\mathbf{f}, g, \mathbf{h}, \mathbf{k}) \in X_q$,

$$\mathbf{u} = \mathbf{A}(\lambda)F_\lambda(\mathbf{f}, g, \mathbf{h}, \mathbf{k}) \quad \text{and} \quad \theta = \mathbf{P}(\lambda)F_\lambda(\mathbf{f}, g, \mathbf{h}, \mathbf{k})$$

are solutions to the equations (1.1), *and furthermore,*

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, L_q(\dot\Omega)^{\tilde N})}\left(\left\{\left(\frac{d}{d\lambda}\right)^l (G_\lambda \mathbf{A}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\right\}\right) \leq M,$$

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_q, W_q^1(\dot\Omega)^N)}\left(\left\{\left(\frac{d}{d\lambda}\right)^l \mathbf{P}(\lambda) \mid \lambda \in \Sigma_{\varepsilon,\gamma_0}\right\}\right) \leq M \quad (l = 0, 1)$$

with some positive constant M. *Here we have set* $\widetilde N = N^3 + N^2 + N$, $G_\lambda \mathbf{u} = (\nabla^2\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \lambda\mathbf{u})$, *and*

$$F_\lambda(\mathbf{f}, g, \mathbf{h}, \mathbf{k}) = (\mathbf{f}, \nabla g, \lambda^{1/2}g, \lambda\mathcal{G}(g), \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{k}, \lambda^{1/2}\mathbf{k}).$$

(2) **Uniqueness.** *There exists a* $\lambda_0 \geq 1$ *such that if* $\mathbf{u} \in W_q^2(\dot\Omega)^N \cap J_q(\Omega)$ *and* $\theta \in W_q^1(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)$ *satisfies the homogeneous equations:*

$$\lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},\theta) = 0 \quad \text{in } \dot\Omega, \quad [\![\mathbf{T}(\mathbf{u},\theta)\mathbf{n}]\!] = 0, \quad [\![\mathbf{u}]\!] = 0 \quad \text{on } \Gamma,$$
$$\mathbf{T}(\mathbf{u},\theta)\mathbf{n}_+ = 0 \quad \text{on } \Gamma_+, \quad \mathbf{u} = 0 \quad \text{on } \Gamma_-$$

with $\lambda \in \Sigma_{\varepsilon,\lambda_0}$, *then* $\mathbf{u} = 0$ *and* $\theta = 0$.

Remark 2.4. (1) In the original paper [MS], we can treat a more general case such that the viscosity coefficients $\mu_\pm$ are functions on $\Omega_\pm$ and domains are not necessarily bounded.

(2) The symbols F_1, F_2, F_3, F_4, F_5, F_6, F_7, and F_8 are corresponding variables to $\mathbf{f}$, ∇g, $\lambda^{1/2}g$, $\lambda\mathcal{G}(g)$, $\nabla\mathbf{h}$, $\lambda^{1/2}\mathbf{h}$, $\nabla\mathbf{k}$, and $\lambda^{1/2}\mathbf{k}$, respectively. The norm of space $\mathcal{X}_q$ is given by

$$\|(F_1, \ldots, F_9)\|_{\mathcal{X}_q} = \|(F_1, \ldots, F_6)\|_{L_q(\dot\Omega)} + \|(F_7, F_8)\|_{L_q(\Omega_+)}.$$

(3) We do not give any proof of the results of [MS] in this article, but an application of Theorem 2.3 is presented in Section 4.

3. Stokes and reduced Stokes

The aim of this section is to show some equivalence between the two-phase Stokes resolvent problem (1.1) and its reduced problem. Here "reduced" means that the pressure term θ of (1.1) is eliminated. Such a reduced problem plays an important role to construct an analytic semigroup generated by the Stokes operator $\mathcal{A}$ associated with the equation (1.1).

To introduce the reduced problem, we start with the following proposition, which will be announced in [MS].

Proposition 3.1 (Unique solvability of the weak Dirichlet-Neumann problem). *Let* $1 < q < \infty$, $N < r < \infty$, *and* $\max(q, q') \leq r$ *with* $q' = q/(q-1)$. *Suppose that* Ω *is a bounded domain and* Γ, $\Gamma_\pm$ *are closed hypersurfaces of* $W_r^{2-1/r}$ *class, and set* $\rho = \rho_+\chi_{\Omega_+} + \rho_-\chi_{\Omega_-}$ *for positive constants* $\rho_\pm$. *Then, for any* $\mathbf{f} \in L_q(\Omega)^N$, *there is a unique* $\theta \in W^1_{q,\Gamma_+}(\Omega)$ *satisfying the variational equation:*

$$(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_\Omega \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega),$$

which possesses the estimate: $\|\theta\|_{W_q^1(\Omega)} \leq C\|\mathbf{f}\|_{L_q(\Omega)}$ *with a positive constant* C *independent of* θ, φ, *and* $\mathbf{f}$.

Remark 3.2. (1) Let $\mathbf{f} \in L_q(\Omega)^N$, and let $Q_q\mathbf{f} := \theta \in W^1_{q,\Gamma_+}(\Omega)$ in Proposition 3.1 with $\rho_\pm = 1$. Then, setting $P_q\mathbf{f} = \mathbf{f} - \nabla Q_q\mathbf{f}$, we have $P_q\mathbf{f} \in J_q(\Omega)$. We thus obtain a decomposition: $\mathbf{f} = P_q\mathbf{f} + \nabla Q_q\mathbf{f} \in J_q(\Omega) + G_q(\Omega)$ with $G_q(\Omega) = \{\mathbf{g} \mid \mathbf{g} = \nabla\psi,\ \psi \in W^1_{q,\Gamma_+}(\Omega)\}$. Moreover, we see that the decomposition is determined uniquely. In fact, let $\mathbf{f} \in J_q(\Omega) \cap G_q(\Omega)$ with $\mathbf{f} = \nabla\psi$ for some $\psi \in W^1_{q,\Gamma_+}(\Omega)$, and then $\mathbf{f} \in J_q(\Omega)$ implies that

$$(\nabla\psi, \nabla\varphi)_\Omega = (\mathbf{f}, \nabla\varphi)_\Omega = 0 \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega),$$

which, combined with the uniqueness of Proposition 3.1, furnishes that $\psi = 0$. Hence, it holds the so-called *Helmholtz decomposition*: $L_q(\Omega)^N = J_q(\Omega) \oplus G_q(\Omega)$.

(2) By Proposition 3.1, we see that, for any $\mathbf{f} \in L_q(\dot\Omega)^N$, $g \in W^{1-1/q}_q(\Gamma)$, and $h \in W^{1-1/q}_q(\Gamma_+)$, there exists a unique $\theta \in W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)$ satisfying the weak problem:

$$\begin{aligned} &(\rho^{-1}\nabla\theta, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \\ &[\![\theta]\!] = g \quad \text{on } \Gamma, \quad \theta = h \quad \text{on } \Gamma_+, \end{aligned}$$

which possesses the estimate:

$$\|\theta\|_{W^1_q(\dot\Omega)} \le C\left(\|\mathbf{f}\|_{L_q(\dot\Omega)} + \|g\|_{W^{1-1/q}_q(\Gamma)} + \|h\|_{W^{1-1/q}_q(\Gamma_+)}\right)$$

with some positive constant C independent of θ, φ, $\mathbf{f}$, g, and h. Thus, it is possible to define a linear operator $\mathcal{K}$ as follows:

$$\mathcal{K} : L_q(\dot\Omega)^N \times W^{1-1/q}_q(\Gamma) \times W^{1-1/q}_q(\Gamma_+) \to W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)$$

satisfying the following weak problem:

$$\begin{aligned} &(\rho^{-1}\nabla\mathcal{K}(\mathbf{f}, g, h), \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \\ &[\![\mathcal{K}(\mathbf{f}, g, h)]\!] = g \quad \text{on } \Gamma, \quad \mathcal{K}(\mathbf{f}, g, h) = h \quad \text{on } \Gamma_+ \end{aligned}$$

and the estimate:

$$\|\mathcal{K}(\mathbf{f}, g, h)\|_{W^1_q(\dot\Omega)} \le C\left(\|\mathbf{f}\|_{L_q(\dot\Omega)} + \|g\|_{W^{1-1/q}_q(\Gamma)} + \|h\|_{W^{1-1/q}_q(\Gamma_+)}\right)$$

with some positive constant C independent of φ, $\mathbf{f}$, g, and h.

By using the operator $\mathcal{K}$ mentioned above, we set, for $\mathbf{u} \in W^2_q(\dot\Omega)^N$, $K(\mathbf{u}) = \mathcal{K}(\mathbf{f}, g, h)$ with

$$\begin{aligned} &\mathbf{f} = \rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u})) - \nabla\operatorname{div}\mathbf{u}, \\ &g = [\![<\mu\mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n}> - \operatorname{div}\mathbf{u}]\!], \quad h = <\mu\mathbf{D}(\mathbf{u})\mathbf{n}_+, \mathbf{n}_+> - \operatorname{div}\mathbf{u}. \end{aligned}$$

Then the two-phase reduced Stokes resolvent problem is given by

$$\left\{\begin{aligned} \lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, K(\mathbf{u})) &= \mathbf{f} && \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}]\!] &= [\![\mathbf{h}]\!] && \text{on } \Gamma, \\ [\![\mathbf{u}]\!] &= 0 && \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}_+ &= \mathbf{k} && \text{on } \Gamma_+, \\ \mathbf{u} &= 0 && \text{on } \Gamma_-. \end{aligned}\right. \tag{3.1}$$

In the following subsections, we will show some equivalence between (1.1) and (3.1). To this end, we consider an auxiliary problem as follows:

$$(\lambda u, \varphi)_{\dot\Omega} + (\nabla u, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \tag{3.2}$$

$$[\![u]\!] = [\![g]\!] \quad \text{on } \Gamma, \quad u = h \quad \text{on } \Gamma_+. \tag{3.3}$$

Let $\Sigma_\varepsilon = \{\lambda \in \mathbf{C} \setminus \{0\} \mid |\arg\lambda| < \pi - \varepsilon\}$ for $0 < \varepsilon < \pi/2$. Then the following proposition holds.

Proposition 3.3. *Let $0 < \varepsilon < \pi/2$, $1 < q < \infty$, $N < r < \infty$, and $\max(q, q') \le r$ with $q' = q/(q-1)$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W_r^{2-1/r}$ class. Then, for any $\lambda \in \Sigma_\varepsilon \cup \{0\}$ and any $\mathbf{f} \in L_q(\dot\Omega)^N$, $g \in W_q^1(\dot\Omega)$, and $h \in W_q^1(\Omega_+)$, there is a unique solution $u \in W_q^2(\dot\Omega)$ to the equations* (3.2)-(3.3).

Remark 3.4. The symbols H_1, H_2, H_3, H_4, and H_5 are corresponding variables to $\mathbf{f}$, ∇g, $\lambda^{1/2}g$, ∇h, and $\lambda^{1/2}h$, respectively.

3.1. **Stokes implies reduced Stokes.** We shall solve (3.1) by means of solutions to (1.1).

Given $\mathbf{f} \in L_q(\dot\Omega)^N$, $\mathbf{h} \in W_q^1(\dot\Omega)^N$, and $\mathbf{k} \in W_q^1(\Omega_+)^N$, we choose by Proposition 3.3 some g in such a way that g solves the weak problem:

$$(\lambda g, \varphi)_{\dot\Omega} + (\nabla g, \nabla\varphi)_{\dot\Omega} = -(\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \tag{3.4}$$

$$[\![g]\!] =< [\![\mathbf{h}]\!], \mathbf{n} > \quad \text{on } \Gamma, \quad g =< \mathbf{k}, \mathbf{n}_+ > \quad \text{on } \Gamma_+. \tag{3.5}$$

Let $\mathbf{u} \in W_q^2(\dot\Omega)^N$ and $\theta \in W_q^1(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)$ be solutions to (1.1) with $\mathbf{f}$, g, $\mathbf{h}$, and $\mathbf{k}$ mentioned above. Let $\varphi \in W^1_{q',\Gamma_+}(\Omega)$. Then, by the definition of $K(\mathbf{u})$, $[\![\mathbf{u}]\!] = 0$ on Γ, and $\mathbf{u} = 0$ on Γ_-,

$$\begin{aligned}(\mathbf{f}, \nabla\varphi)_{\dot\Omega} =& (\lambda\mathbf{u} - \nabla \operatorname{div}\mathbf{u} - \rho^{-1}\nabla K(\mathbf{u}) + \rho^{-1}\nabla\theta, \nabla\varphi)_{\dot\Omega}\\ =& -(\lambda g, \varphi)_{\dot\Omega} - (\nabla g, \nabla\varphi)_{\dot\Omega} + (\rho^{-1}\nabla(\theta - K(\mathbf{u})), \nabla\varphi)_{\dot\Omega},\end{aligned}$$

which, combined with (3.4), furnishes that

$$(\rho^{-1}\nabla(\theta - K(\mathbf{u})), \nabla\varphi)_{\dot\Omega} = 0 \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega).$$

In addition, we have $[\![K(\mathbf{u}) - \theta]\!] = 0$ on Γ and $K(\mathbf{u}) - \theta = 0$ on Γ_+, since g satisfies (3.5) and

$$\begin{aligned}< [\![\mathbf{h}]\!], \mathbf{n} > =& [\![< \mu\mathbf{D}(\mathbf{u})\mathbf{n}, \mathbf{n} >]\!] - [\![\theta]\!] = [\![K(\mathbf{u}) - \theta]\!] + [\![\operatorname{div}\mathbf{u}]\!]\\ =& [\![K(\mathbf{u}) - \theta]\!] + [\![g]\!] \quad \text{on } \Gamma,\\ < \mathbf{k}, \mathbf{n}_+ > =& < \mu\mathbf{D}(\mathbf{u})\mathbf{n}_+, \mathbf{n}_+ > -\theta = K(\mathbf{u}) - \theta + \operatorname{div}\mathbf{u}\\ =& K(\mathbf{u}) - \theta + g \quad \text{on } \Gamma_+.\end{aligned}$$

Thus the uniqueness of Proposition 3.1 implies $K(\mathbf{u}) = \theta$, which means that the solution $\mathbf{u} \in W_q^2(\dot\Omega)^N$ of (1.1) solves (3.1) for $\mathbf{f} \in L_q(\dot\Omega)^N$, $\mathbf{h} \in W_q^1(\dot\Omega)$, $\mathbf{k} \in W_q^1(\Omega_+)$, and g of (3.4)-(3.5).

3.2. **Reduced Stokes implies Stokes.** We shall solve (1.1) by means of solutions to (3.1).

Given $\mathbf{f} \in L_q(\dot\Omega)^N$, $\mathbf{h} \in W_q^1(\dot\Omega)^N$, and $\mathbf{k} \in W_q^1(\Omega_+)^N$, let $\kappa \in W_q^1(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)$ be the solution to the weak problem:

$$\begin{aligned}&(\rho^{-1}\nabla\kappa, \nabla\varphi)_{\dot\Omega} = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega),\\ &[\![\kappa]\!] = - < [\![\mathbf{h}]\!], \mathbf{n} > \quad \text{on } \Gamma, \quad \kappa = - < \mathbf{k}, \mathbf{n}_+ > \quad \text{on } \Gamma_+.\end{aligned}$$

Then the problem (1.1) is reduced to

$$\begin{cases}\lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, \theta - \kappa) = \mathbf{f} - \rho^{-1}\nabla\kappa, & \operatorname{div}\mathbf{u} = g \quad \text{in } \dot\Omega,\\ [\![\mathbf{T}(\mathbf{u}, \theta - \kappa)\mathbf{n}]\!] = [\![\mathbf{h}]\!] - < [\![\mathbf{h}]\!], \mathbf{n} > \mathbf{n}, & [\![\mathbf{u}]\!] = 0 \quad \text{on } \Gamma,\\ \mathbf{T}(\mathbf{u}, \theta - \kappa)\mathbf{n}_+ = \mathbf{k} - < \mathbf{k}, \mathbf{n}_+ > \mathbf{n}_+ & \text{on } \Gamma_+,\\ \mathbf{u} = 0 & \text{on } \Gamma_-.\end{cases}$$

It thus suffices to consider the problem (1.1) under the condition that

$$(\mathbf{f}, \nabla\varphi)_{\dot\Omega} = 0 \text{ for all } \varphi \in W^1_{q',\Gamma_+}(\Omega), \quad < [\![\mathbf{h}]\!], \mathbf{n} >= 0 \text{ on } \Gamma, \quad < \mathbf{k}, \mathbf{n}_+ >= 0 \text{ on } \Gamma_+. \tag{3.6}$$

Given $g \in \mathcal{DI}_q(\dot\Omega)$, let $K_\lambda(g) = \mathcal{K}(\lambda\mathcal{G}(g) - \nabla g, -g, -g)$ by the operator $\mathcal{K}$ of Remark 3.2 (2), that is, $K_\lambda(g)$ satisfies the weak problem:

$$(\rho^{-1}\nabla K_\lambda(g), \nabla\varphi)_{\dot\Omega} = (\lambda\mathcal{G}(g) - \nabla g, \nabla\varphi)_{\dot\Omega} \quad \text{for all } \varphi \in W^1_{q',\Gamma_+}(\Omega),$$
$$[\![K_\lambda(g)]\!] = -[\![g]\!] \quad \text{on } \Gamma, \quad K_\lambda(g) = -g \quad \text{on } \Gamma_+.$$

Let $\mathbf{u} \in W^2_q(\dot\Omega)^N$ be a solution to the two-phase reduced Stokes resolvent problem as follows:

$$\begin{cases} \lambda\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u}, K(\mathbf{u})) = \mathbf{f} + \rho^{-1}\nabla K_\lambda(g) & \text{on } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}]\!] = [\![\mathbf{h}]\!] + [\![g]\!]\mathbf{n} & \text{on } \Gamma, \\ [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}_+ = \mathbf{k} + g\mathbf{n}_+ & \text{on } \Gamma_+, \\ \mathbf{u} = 0 & \text{on } \Gamma_-. \end{cases}$$

Let $\varphi \in W^1_{q',\Gamma_+}(\Omega)$. Then, by (3.6) and the definitions of $K(\mathbf{u})$, $K_\lambda(g)$,

$$\begin{aligned} 0 = (\mathbf{f}, \nabla\varphi)_{\dot\Omega} &= (\lambda\mathbf{u} - \rho^{-1}\operatorname{Div}(\mu\mathbf{D}(\mathbf{u})) + \rho^{-1}\nabla K(\mathbf{u}) - \rho^{-1}\nabla K_\lambda(g), \nabla\varphi)_{\dot\Omega} \\ &= -(\lambda\operatorname{div}\mathbf{u}, \varphi)_{\dot\Omega} - (\nabla\operatorname{div}\mathbf{u}, \nabla\varphi)_{\dot\Omega} - (\lambda\mathcal{G}(g) - \nabla g, \nabla\varphi)_{\dot\Omega} \\ &= -(\lambda(\operatorname{div}\mathbf{u} - g), \varphi)_{\dot\Omega} - (\nabla(\operatorname{div}\mathbf{u} - g), \nabla\varphi)_{\dot\Omega}. \end{aligned}$$

In addition, by (3.6) and the definition of $K(\mathbf{u})$,

$$[\![g]\!] = < [\![\mu\mathbf{D}(\mathbf{u})\mathbf{n}]\!], \mathbf{n} > -[\![K(\mathbf{u})]\!] = [\![\operatorname{div}\mathbf{u}]\!] \quad \text{on } \Gamma,$$
$$g = < \mu\mathbf{D}(\mathbf{u})\mathbf{n}_+, \mathbf{n}_+ > -K(\mathbf{u}) = \operatorname{div}\mathbf{u} \quad \text{on } \Gamma_+,$$

which implies that

$$[\![\operatorname{div}\mathbf{u} - g]\!] = 0 \quad \text{on } \Gamma, \quad \operatorname{div}\mathbf{u} - g = 0 \quad \text{on } \Gamma_+.$$

Thus, the uniqueness of Proposition 3.3 furnishes that $\operatorname{div}\mathbf{u} = g$ in $\dot\Omega$, which means that $\mathbf{u}$ and $\theta = K(\mathbf{u}) - K_\lambda(g)$ solves (1.1).

3.3. $\mathcal{R}$-boundedness for two-phase reduced Stokes resolvent problem. According to what was pointed out in Subsection 3.2, we obtain Theorem 2.3 by the $\mathcal{R}$-boundedness of solution operator families for the two-phased reduced Stokes resolvent problem (3.1), which is also one of main objects of [MS], as follows.

Theorem 3.5. *Let $1 < q < \infty$, $0 < \varepsilon < \pi/2$, $N < r < \infty$, and $\max(q, q') \le r$ with $q' = q/(q-1)$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W^{2-1/r}_r$ class. Let $X_{\mathcal{R},q}$ and $\mathcal{X}_{\mathcal{R},q}$ be given by*

$$X_{\mathcal{R},q} = \{(\mathbf{f}, \mathbf{h}, \mathbf{k}) \mid \mathbf{f} \in L_q(\dot\Omega)^N, \mathbf{h} \in W^1_q(\dot\Omega)^N, \mathbf{k} \in W^1_q(\Omega_+)^N\},$$
$$\mathcal{X}_{\mathcal{R},q} = \{(F_1, F_2, F_3, F_4, F_5) \mid F_1, F_3 \in L_q(\dot\Omega)^N, F_2 \in L_q(\dot\Omega)^{N^2}, F_4 \in L_q(\Omega_+)^{N^2}, F_5 \in L_q(\Omega_+)^N\}.$$

Then there exist a positive number $\lambda_0 \ge 1$ and an operator family $\mathbf{B}(\lambda)$ with

$$\mathbf{B}(\lambda) \in \operatorname{Hol}(\Sigma_{\varepsilon,\lambda_0}, \mathcal{L}(\mathcal{X}_{\mathcal{R},q}, W^2_q(\dot\Omega)^N))$$

such that, for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $(\mathbf{f}, \mathbf{h}, \mathbf{k}) \in X_{\mathcal{R},q}$, $\mathbf{u} = \mathbf{B}(\lambda)\mathbf{F}_\lambda(\mathbf{f}, \mathbf{h}, \mathbf{k})$ is a unique solution to the equations (3.1), and furthermore,

$$\mathcal{R}_{\mathcal{L}(\mathcal{X}_{R,q}, L_q(\dot\Omega)^{\tilde N})}\Big(\Big\{\Big(\lambda\frac{d}{d\lambda}\Big)^l (G_\lambda\mathbf{B}(\lambda)) \mid \lambda \in \Sigma_{\varepsilon,\lambda_0}\Big\}\Big) \le M \quad (l = 0, 1)$$

with some positive constant M, where $\mathbf{F}_\lambda(\mathbf{f}, \mathbf{h}, \mathbf{k}) = (\mathbf{f}, \nabla\mathbf{h}, \lambda^{1/2}\mathbf{h}, \nabla\mathbf{k}, \lambda^{1/2}\mathbf{k})$.

Remark 3.6. (1) As mentioned above, Theorem 2.3 follows from Theorem 3.5.
(2) If $\mathbf{u}$ satisfies (3.1) with $\mathbf{f} \in J_q(\Omega)$, $< [\![\mathbf{h}]\!], \mathbf{n} >= 0$ on Γ, and $< \mathbf{k}, \mathbf{n}_+ >= 0$ on Γ_+, then $\mathbf{u}$ belongs to $J_q(\Omega)$. This fact can be obtained in the same manner as in Subsection 3.2 with $g = 0$.

4. An application of $\mathcal{R}$-boundedness

In this section, we apply Theorem 2.3 to a two-phase problem of time-dependent Stokes equations as follows:

$$
(4.1)\qquad \begin{cases} \partial_t \mathbf{u} - \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{u}, \theta) = \mathbf{f} & \text{in } \dot\Omega \times (0, \infty), \\ \operatorname{div} \mathbf{u} = g = \operatorname{div} \mathbf{g} & \text{in } \dot\Omega \times (0, \infty), \\ [\![\mathbf{T}(\mathbf{u}, \theta)\mathbf{n}]\!] = [\![\mathbf{h}]\!], \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma \times (0, \infty), \\ \mathbf{T}(\mathbf{u}, \theta)\mathbf{n}_+ = \mathbf{k} & \text{on } \Gamma_+ \times (0, \infty), \\ \mathbf{u} = 0 & \text{on } \Gamma_- \times (0, \infty), \\ \mathbf{u}|_{t=0} = \mathbf{u}_0 & \text{in } \dot\Omega, \end{cases}
$$

and prove some maximal regularity property of (4.1). To state the maximal regularity theorem, we introduce some function spaces and symbols. For a Banach space X, we denote the usual Lebesgue and Sobolev spaces of X-valued functions defined on time interval I by $L_p(I, X)$ and $W_p^m(I, X)$ ($m \in \mathbf{N}$), and their associated norms by $\|\cdot\|_{L_p(I,X)}$ and $\|\cdot\|_{W_p^m(I,X)}$, respectively. We set

$$L_{p,0}(\mathbf{R}, X) = \{f \in L_p(\mathbf{R}, X) \mid f(t) = 0 \text{ for } t < 0\}.$$

Let $\mathcal{L}$, $\mathcal{L}_\lambda^{-1}$, $\mathcal{F}$, and $\mathcal{F}_\tau^{-1}$ denote the Laplace transform, the Laplace inverse transform, the Fourier transform, and the Fourier inverse transform, which are denoted by

$$
\begin{aligned}
\mathcal{L}[f](\lambda) &= \int_{-\infty}^{\infty} e^{-\lambda t} f(t)\, dt, \quad & \mathcal{L}_\lambda^{-1}[g(\lambda)](t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\lambda)\, d\tau \quad (\lambda = \gamma + i\tau), \\
\mathcal{F}[f](\tau) &= \int_{-\infty}^{\infty} e^{-i\tau t} f(t)\, dt, & \mathcal{F}_\tau^{-1}[g](t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\tau t} g(\tau)\, d\tau.
\end{aligned}
$$

Note that we have the following relations:

$$(4.2)\qquad \mathcal{L}[f](\lambda) = \mathcal{F}[e^{-\gamma t} f(t)](\tau), \quad \mathcal{L}_\lambda^{-1}[g(\lambda)](t) = e^{\gamma t} \mathcal{F}_\tau^{-1}[g(\gamma + i\tau)](t) \quad (\lambda = \gamma + i\tau).$$

For any real number $s \geq 0$, let $H_p^s(\mathbf{R}, X)$ be the Bessel potential space of order s defined by

$$H_p^s(\mathbf{R}, X) = \{f \in L_p(\mathbf{R}, X) \mid \mathcal{F}_\tau^{-1}[(1 + |\tau|^2)^{s/2} \mathcal{F}[f](\tau)](t) \in L_p(\mathbf{R}, X)\}.$$

In addition, we set, for an open set D of $\mathbf{R}^N$,

$$H_{q,p,0}^{1,1/2}(D \times \mathbf{R}) = H_p^{1/2}(\mathbf{R}, L_q(D)) \cap L_{p,0}(\mathbf{R}, W_q^1(D)).$$

Here, we introduce the Stokes operator $\mathcal{A}$ with domain $\mathcal{D}_q(\mathcal{A})$ defined by

$$
(4.3)\qquad
\begin{aligned}
\mathcal{D}_q(\mathcal{A}) = &\{\mathbf{u} \in W_q^2(\dot\Omega)^N \cap J_q(\Omega) \mid [\![\mathcal{T}_\mathbf{n} \mathbf{D}(\mathbf{u})\mathbf{n}]\!] = 0 \text{ on } \Gamma, \\
&[\![\mathbf{u}]\!] = 0 \text{ on } \Gamma, \quad \mathcal{T}_{\mathbf{n}_+} \mathbf{D}(\mathbf{u})\mathbf{n}_+ = 0 \text{ on } \Gamma_+, \quad \mathbf{u} = 0 \text{ on } \Gamma_-\}, \\
\mathcal{A}\mathbf{u} = &- \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{u}, K(\mathbf{u})) \quad \text{for } \mathbf{u} \in \mathcal{D}_q(\mathcal{A}),
\end{aligned}
$$

where $K(\mathbf{u})$ is defined as in Section 3 and we have set

$$\mathcal{T}_\mathbf{n} \mathbf{f} = \mathbf{f} - < \mathbf{f}, \mathbf{n} > \mathbf{n}, \quad \mathcal{T}_{\mathbf{n}_+} \mathbf{f} = \mathbf{f} - < \mathbf{f}, \mathbf{n}_+ > \mathbf{n}_+$$

that are the tangential components of N-vector $\mathbf{f}$ with respect to $\mathbf{n}$, $\mathbf{n}_+$, respectively. Then we set

$$\mathcal{D}_{q,p}(\dot\Omega) = (J_q(\Omega), \mathcal{D}_q(\mathcal{A}))_{1-1/p,p},$$

where $(\cdot,\cdot)_{1-1/p,p}$ denotes the real interpolation functor.

The aim of this section is to prove the following maximal L_p-L_q regularity property with exponential stability for (4.1).

Theorem 4.1. *Let $1 < p < \infty$, $N < q < \infty$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W_q^{2-1/q}$ class. Then there exists a positive constant ε_0 such that if the right members $\mathbf{f}$, g, $\mathbf{g}$, $\mathbf{h}$, $\mathbf{k}$, and $\mathbf{u}_0$ of (4.1) satisfy the conditions:*

$$e^{\varepsilon_0 t}\mathbf{f} \in L_{p,0}(\mathbf{R}, L_q(\dot\Omega))^N, \quad e^{\varepsilon_0 t}g \in H^{1,1/2}_{q,p,0}(\dot\Omega \times \mathbf{R}), \quad e^{\varepsilon_0 t}\mathbf{h} \in H^{1,1/2}_{q,p,0}(\dot\Omega \times \mathbf{R})^N,$$
$$e^{\varepsilon_0 t}\mathbf{k} \in H^{1,1/2}_{q,p,0}(\Omega_+ \times \mathbf{R})^N, \quad \mathbf{u}_0 \in \mathcal{D}_{q,p}(\dot\Omega),$$
$$e^{\varepsilon_0 t}(\partial_t \mathbf{g}, \mathbf{g}) \in L_{p,0}(\mathbf{R}, L_q(\dot\Omega))^{N\times N} \text{ with } [\![\mathbf{g}(t)]\!]\cdot\mathbf{n} = 0 \text{ on } \Gamma,\ \mathbf{g}(t)\cdot\mathbf{n}_+ = 0 \text{ on } \Gamma_- \ (t>0),$$

then the equations (4.1) admits a unique solution $(\mathbf{u},\theta)$ with

$$\mathbf{u} \in (W_p^1((0,\infty), L_q(\dot\Omega)) \cap L_p((0,\infty), W_q^2(\dot\Omega)))^N, \quad \theta \in L_p((0,\infty), W_q^1(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)),$$

which possesses the estimate:

$$\|e^{\varepsilon_0 t}(\partial_t\mathbf{u}, \mathbf{u}, \nabla\mathbf{u}, \nabla^2\mathbf{u})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\theta\|_{L_p((0,\infty),W_q^1(\dot\Omega))} \leq C\Big(\|\mathbf{u}_0\|_{\mathcal{D}^{2(1-1/p)}_{q,p}(\dot\Omega)} \tag{4.4}$$
$$+ \|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g,\mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big)$$

with some positive constant C.

Remark 4.2. The uniqueness follows from the solvability of some dual problem (cf. e.g. [Sai15, Section 7]), so that we only prove the estimate (4.4) in the following subsections.

To show Theorem 4.1, we divide the equations (4.1) into the following two systems:

$$\begin{cases} \partial_t\mathbf{v} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{v},\pi) = 0, & \operatorname{div}\mathbf{v} = 0 \quad \text{in } \dot\Omega\times(0,\infty) \\ [\![\mathbf{T}(\mathbf{v},\pi)\mathbf{n}]\!] = 0, & [\![\mathbf{v}]\!] = 0 \quad \text{on } \Gamma\times(0,\infty), \\ \mathbf{T}(\mathbf{v},\pi)\mathbf{n}_+ = 0 & \text{on } \Gamma_+\times(0,\infty), \\ \mathbf{v} = 0 & \text{on } \Gamma_-\times(0,\infty), \\ \mathbf{v}|_{t=0} = \mathbf{u}_0 & \text{in } \dot\Omega; \end{cases} \tag{4.5}$$

$$\begin{cases} \partial_t\mathbf{w} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{w},\kappa) = \mathbf{f}, & \operatorname{div}\mathbf{w} = g \quad \text{in } \dot\Omega\times(0,\infty), \\ [\![\mathbf{T}(\mathbf{w},\kappa)\mathbf{n}]\!] = [\![\mathbf{h}]\!], & [\![\mathbf{w}]\!] = 0 \quad \text{on } \Gamma\times(0,\infty), \\ \mathbf{T}(\mathbf{w},\kappa)\mathbf{n}_+ = \mathbf{k} & \text{on } \Gamma_+\times(0,\infty), \\ \mathbf{w} = 0 & \text{on } \Gamma_-\times(0,\infty), \\ \mathbf{w}|_{t=0} = 0 & \text{in } \dot\Omega, \end{cases} \tag{4.6}$$

where we note that solutions of the equations (4.1) are given by $\mathbf{u} = \mathbf{v} + \mathbf{w}$ and $\theta = \pi + \kappa$. In the following subsections, we will discuss the equations (4.5)-(4.6).

4.1. **Analysis of the equations** (4.5). In this subsection, we shall solve the equations (4.5) by means of analytic semigroup. We start with the following equations:

$$\partial_t\mathbf{v} + \mathcal{A}\mathbf{v} = 0 \quad \text{in } \dot\Omega\times(0,\infty), \quad \mathbf{v}|_{t=0} = \mathbf{u}_0 \quad \text{in } \dot\Omega$$

with $\mathbf{u}_0 \in J_q(\Omega)$, which are equivalent to

$$
(4.7) \qquad \begin{cases} \partial_t \mathbf{v} - \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{v}, K(\mathbf{v})) = 0 & \text{in } \dot\Omega \times (0,\infty) \\ [\![\mathbf{T}(\mathbf{v}, K(\mathbf{v}))\mathbf{n}]\!] = 0, \quad [\![\mathbf{v}]\!] = 0 & \text{on } \Gamma \times (0,\infty), \\ \mathbf{T}(\mathbf{v}, K(\mathbf{v}))\mathbf{n}_+ = 0 & \text{on } \Gamma_+ \times (0,\infty), \\ \mathbf{v} = 0 & \text{on } \Gamma_- \times (0,\infty), \\ \mathbf{v}|_{t=0} = \mathbf{u}_0 & \text{in } \dot\Omega. \end{cases}
$$

Then, by Theorem 3.5, the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains $\Sigma_{\varepsilon,\lambda_0}$. Denoting the resolvent operator of $\mathcal{A}$ by $(\lambda+\mathcal{A})^{-1}$, we have $(\lambda+\mathcal{A})^{-1}\mathbf{f} = \mathbf{B}(\lambda)(\mathbf{f},0,0,0,0)$ for any $\lambda \in \Sigma_{\varepsilon,\lambda_0}$ and $\mathbf{f} \in J_q(\Omega)$. Note that $(\lambda+\mathcal{A})^{-1}\mathbf{f}$ belongs to $J_q(\Omega)$ by Remark 3.6 (2). Since the $\mathcal{R}$-boundednee of $\mathbf{B}(\lambda)$ implies the usual boundedness, we obtain

$$
(4.8) \qquad \|(\lambda\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u})\|_{L_q(\dot\Omega)} \leq C\|\mathbf{f}\|_{J_q(\Omega)} \quad (\lambda \in \Sigma_{\varepsilon,\lambda_0}),
$$

where we have set $\mathbf{u} = (\lambda+\mathcal{A})^{-1}\mathbf{f}$. The resolvent estimate (4.8) furnishes that the following proposition holds.

Proposition 4.3. *Let $1 < q < \infty$, $N < r < \infty$, and $\max(q,q') \leq r$ with $q' = q/(q-1)$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W_r^{2-1/r}$ class. Then the Stokes operator $\mathcal{A}$, defined as* (4.3), *generates a C_0-semigroup $\{e^{-\mathcal{A}t}\}_{t\geq 0}$ on $J_q(\Omega)$, which is analytic.*

Let $\mathbf{v} = e^{-\mathcal{A}t}\mathbf{u}_0$ and $\pi = K(e^{-\mathcal{A}t}\mathbf{u}_0)$ for $\mathbf{u}_0 \in J_q(\Omega)$. Then $(\mathbf{v},\pi)$ satisfies (4.5). In fact, since $\mathbf{v}$ satisfies (4.7) and belongs to $J_q(\Omega)$, we see for any $\varphi \in W^1_{q',\Gamma_+}(\Omega)$ that

$$
0 = (\mathbf{v}, \nabla\varphi)_{\dot\Omega} = -(\operatorname{div}\mathbf{v}, \varphi)_{\dot\Omega} + ([\![\mathbf{v}]\!]\cdot\mathbf{n}, \varphi)_\Gamma + (\mathbf{v}\cdot\mathbf{n}_-, \varphi)_{\Gamma_-} = -(\operatorname{div}\mathbf{v}, \varphi)_{\dot\Omega},
$$

which implies that $\operatorname{div}\mathbf{v} = 0$ in $\dot\Omega$.

The aim of this subsection is to prove the following theorem concerning (4.5).

Theorem 4.4. *Let $1 < p < \infty$ and $N < q < \infty$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W_q^{2-1/q}$ class. Then, for any initial data $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\dot\Omega)$, $(\mathbf{v},\pi) = (e^{-\mathcal{A}t}\mathbf{u}_0, K(e^{-\mathcal{A}t}\mathbf{u}_0))$ solves the equations* (4.5) *uniquely and*

$$
\begin{aligned} &\mathbf{v} \in (W^1_p((0,\infty), L_q(\dot\Omega)) \cap L_p((0,\infty), W^2_q(\dot\Omega)))^N, \\ &\pi \in L_p((0,\infty), W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)) \end{aligned}
$$

with the estimate:

$$
(4.9) \qquad \|e^{\varepsilon_0 t}(\partial_t\mathbf{v}, \mathbf{v}, \nabla\mathbf{v}, \nabla^2\mathbf{v})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\pi\|_{L_p((0,\infty),W^1_q(\dot\Omega))} \leq C\|\mathbf{u}_0\|_{\mathcal{D}_{q,p}(\dot\Omega)}
$$

for some positive constants ε_0 and C.

Proof of Theorem 4.4. It suffices to show an exponential stability of $\{e^{-\mathcal{A}t}\}_{t\geq 0}$, that is, there exits a positive constant ε_0 sufficiently small such that

$$
(4.10) \qquad \|e^{-\mathcal{A}t}\mathbf{u}_0\|_{J_q(\Omega)} \leq Ce^{-2\varepsilon_0 t}\|\mathbf{u}_0\|_{J_q(\Omega)} \quad (t > 0)
$$

for some positive constant C. In fact, if the estimate (4.10) holds, then we can obtain the estimate (4.9) in the same manner as [SS08, Theorem 3.9] since $\{e^{-\mathcal{A}t}\}_{t\geq 0}$ is analytic.

To show the exponential stability, we consider the resolvent problem with resolvent parameter $\lambda \in \mathbf{C}$ as follows:

$$
\begin{cases}
\lambda \mathbf{u} - \rho^{-1} \operatorname{Div} \mathbf{T}(\mathbf{u}, K(\mathbf{u})) = \mathbf{f} & \text{in } \dot{\Omega}, \\
[\![\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}]\!] = 0, \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\
\mathbf{T}(\mathbf{u}, K(\mathbf{u}))\mathbf{n}_+ = 0 & \text{on } \Gamma_+, \\
\mathbf{u} = 0 & \text{on } \Gamma_-,
\end{cases} \tag{4.11}
$$

and prove that the resolvent set $\rho(\mathcal{A})$ contains $\Sigma_\varepsilon \cup \{0\}$ for $0 < \varepsilon < \pi/2$. By Theorem 3.5, let $\mathbf{u} = R\mathbf{f}$ be the solution to (4.11) with $\lambda = 2\lambda_0$ and $\mathbf{f} \in L_q(\dot{\Omega})^N$, where R is the solution operator satisfying $R : L_q(\dot{\Omega})^N \to W_q^2(\dot{\Omega})^N$. Then we have

$$
\begin{cases}
\lambda(R\mathbf{f}) - \rho^{-1} \operatorname{Div} \mathbf{T}(R\mathbf{f}, K(R\mathbf{f})) = [I + (\lambda - 2\lambda_0)R]\mathbf{f} & \text{in } \dot{\Omega}, \\
[\![\mathbf{T}(R\mathbf{f}, K(R\mathbf{f}))\mathbf{n}]\!] = 0, \quad [\![R\mathbf{f}]\!] = 0 & \text{on } \Gamma, \\
\mathbf{T}(R\mathbf{f}, K(R\mathbf{f}))\mathbf{n}_+ = 0 & \text{on } \Gamma_+, \\
R\mathbf{f} = 0 & \text{on } \Gamma_-,
\end{cases}
$$

which means that if there exists the inverse mapping of $[I + (\lambda - 2\lambda_0)R] : L_q(\dot{\Omega})^N \to L_q(\dot{\Omega})^N$, then $\mathbf{u} = R[I + (\lambda - 2\lambda_0)R]^{-1}\mathbf{f}$ is a solution to the equations (4.11). On the other hand, the invertibility of $[I + (\lambda - 2\lambda_0)R]$ $(\lambda \neq 2\lambda_0)$ on $L_q(\dot{\Omega})^N$ follows from the uniqueness of (4.11) by the following observation: By Rellich's theorem, $W_q^2(\dot{\Omega})$ is compactly embedded into $L_q(\dot{\Omega})$, so that $R : L_q(\dot{\Omega})^N \to L_q(\dot{\Omega})^N$ becomes a compact operator. This combined with the Riesz-Schauder theorem furnishes that the existence of inverse mapping of $[I + (\lambda - 2\lambda_0)R]$ is equivalent to the injectivity of $[I + (\lambda - 2\lambda_0)R]$. We shall prove the injectivity under the assumption that the uniqueness holds for (4.11). Suppose that $[I + (\lambda - 2\lambda_0)R]\mathbf{f} = 0$. Then $\mathbf{u} = R\mathbf{f}$ satisfies (4.11) with $\lambda \in \mathbf{C} \setminus \{2\lambda_0\}$ and $\mathbf{f} = 0$, which, combined with the uniqueness of (4.11), furnishes that $R\mathbf{f} = 0$. Hence, we have $\mathbf{f} = 0$ since $\mathbf{f} = 2\lambda_0(R\mathbf{f}) - \rho^{-1} \operatorname{Div} \mathbf{T}(R\mathbf{f}, K(R\mathbf{f})) = 0$ by the definition of R and $R\mathbf{f} = 0$. This implies the injectivity.

From now on, we shall show the uniqueness of (4.11). Let $(\mathbf{f}, \mathbf{g}) = \int_{\dot{\Omega}} \mathbf{f}(x) \cdot \overline{\mathbf{g}(x)}\, dx$ and $\|\mathbf{f}\|^2 = (\mathbf{f}, \mathbf{f})$. Since $2 \le N < q < \infty$, $L_q(\dot{\Omega})$ is continuously embedded into $L_2(\dot{\Omega})$, which means that it is sufficient to consider the case $L_2(\dot{\Omega})$. We multiply (4.11) with $\mathbf{f} = 0$ by $\overline{\mathbf{u}}$, integrate the resultant formula over $\dot{\Omega}$, and use integration by parts to obtain

$$
0 = \lambda \|\sqrt{\rho}\mathbf{u}\|^2 + \|\sqrt{\mu}\mathbf{D}(\mathbf{u})\|^2 = (\Re\lambda)\|\sqrt{\rho}\mathbf{u}\|^2 + \|\sqrt{\mu}\mathbf{D}(\mathbf{u})\|^2 + i(\Im\lambda)\|\sqrt{\rho}\mathbf{u}\|^2. \tag{4.12}
$$

By (4.12), we have $\mathbf{u} = 0$ when $\Re\lambda > 0$ or $\Im\lambda \neq 0$. In addition, when $\lambda = 0$, we obtain $\mathbf{D}(\mathbf{u}) = 0$, which furnishes that $\mathbf{u} = 0$ since $\mathbf{u} = 0$ on Γ_-. Hence, we have the uniqueness for $\lambda \in \mathbf{C} \setminus (-\infty, 0)$.

Summing up the above argumentation, we see that $\rho(\mathcal{A})$ contains $\Sigma_\varepsilon \cup \{0\}$ for $0 < \varepsilon < \pi/2$, and also we can show that the unique solution to (4.11) satisfies the following resolvent estimate:

$$
(1 + |\lambda|)\|\mathbf{u}\|_{L_q(\dot{\Omega})} + (1 + |\lambda|^{1/2})\|\nabla \mathbf{u}\|_{L_q(\dot{\Omega})} + \|\nabla^2 \mathbf{u}\|_{L_q(\dot{\Omega})} \le C\|\mathbf{f}\|_{L_q(\dot{\Omega})} \tag{4.13}
$$

for any $\lambda \in \Sigma_\varepsilon \cup \{0\}$. By (4.13) and noting Remark 3.6 (2), we have (4.10). □

4.2. **Analysis of** (4.6). In this subsection, we show the following theorem.

Theorem 4.5. *Let $1 < p < \infty$, $N < q < \infty$. Suppose that Ω is a bounded domain and Γ, $\Gamma_\pm$ are closed hypersurfaces of $W_q^{2-1/q}$ class. Let ε_0 be the same positive number as in* (4.10). *If the right* *members* $\mathbf{f}$, g, $\mathbf{g}$, $\mathbf{h}$, *and* $\mathbf{k}$ *of* (4.6) *satisfy the conditions:*

$$e^{\varepsilon_0 t}\mathbf{f} \in L_{p,0}(\mathbf{R}, L_q(\dot\Omega))^N, \quad e^{\varepsilon_0 t} g \in H^{1,1/2}_{q,p,0}(\dot\Omega \times \mathbf{R}),$$
$$e^{\varepsilon_0 t}\mathbf{h} \in H^{1,1/2}_{q,p,0}(\dot\Omega \times \mathbf{R})^N, \quad e^{\varepsilon_0 t}\mathbf{k} \in H^{1,1/2}_{q,p,0}(\Omega_+ \times \mathbf{R})^N,$$
$$e^{\varepsilon_0 t}(\partial_t \mathbf{g}, \mathbf{g}) \in L_{p,0}(\mathbf{R}, L_q(\dot\Omega))^N \quad \textit{with } [\![\mathbf{g}(t)]\!]\cdot\mathbf{n} = 0 \textit{ on } \Gamma,\ \mathbf{g}(t)\cdot\mathbf{n}_+ = 0 \textit{ on } \Gamma_- \ (t>0),$$

then the equations (4.6) *admits a unique solution* $(\mathbf{w}, \kappa)$ *with*

$$\mathbf{w} \in (W^1_p((0,\infty), L_q(\dot\Omega)) \cap L_p((0,\infty), W^2_q(\dot\Omega)))^N, \quad \kappa \in L_p((0,\infty), W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)),$$

which possesses the estimate:

$$\begin{aligned}&\|e^{\varepsilon_0 t}(\partial_t\mathbf{w}, \mathbf{w}, \nabla\mathbf{w}, \nabla^2\mathbf{w})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\theta\|_{L_p((0,\infty),W^1_q(\dot\Omega))}\\ &\quad \le C\Big(\|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g, \mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big)\end{aligned}$$

with some positive constant C.

Proof of Theorem 4.5. We divide the equations (4.6) into the following three systems:

$$\text{(4.14)}\quad \begin{cases} \partial_t\mathbf{w}^1 + 2\lambda_0\mathbf{w}^1 - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{w}^1,\kappa^1) = \mathbf{f}, \quad \operatorname{div}\mathbf{w}^1 = g & \text{in } \dot\Omega\times(0,\infty),\\ [\![\mathbf{T}(\mathbf{w}^1,\kappa^1)\mathbf{n}]\!] = [\![\mathbf{h}]\!], \quad [\![\mathbf{w}^1]\!] = 0 & \text{on } \Gamma\times(0,\infty),\\ \mathbf{T}(\mathbf{w}^1,\kappa^1)\mathbf{n}_+ = \mathbf{k} & \text{on } \Gamma_+\times(0,\infty),\\ \mathbf{w}^1 = 0 & \text{on } \Gamma_-\times(0,\infty),\\ \mathbf{w}^1|_{t=0} = 0 & \text{in } \dot\Omega; \end{cases}$$

$$\text{(4.15)}\quad \begin{cases} \partial_t\mathbf{w}^2 + 2\lambda_0\mathbf{w}^2 - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{w}^2,\kappa^2) = \nabla Q_q(2\lambda_0\mathbf{w}^1), \quad \operatorname{div}\mathbf{w}^2 = 0 & \text{in } \dot\Omega\times(0,\infty),\\ [\![\mathbf{T}(\mathbf{w}^2,\kappa^2)\mathbf{n}]\!] = 0, \quad [\![\mathbf{w}^2]\!] = 0 & \text{on } \Gamma\times(0,\infty),\\ \mathbf{T}(\mathbf{w}^2,\kappa^2)\mathbf{n}_+ = 0 & \text{on } \Gamma_+\times(0,\infty),\\ \mathbf{w}^2 = 0 & \text{on } \Gamma_-\times(0,\infty),\\ \mathbf{w}^2|_{t=0} = 0 & \text{in } \dot\Omega; \end{cases}$$

$$\text{(4.16)}\quad \begin{cases} \partial_t\mathbf{w}^3 - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{w}^3,\kappa^3) = P_q(2\lambda_0\mathbf{w}^1) + 2\lambda_0\mathbf{w}^2, \quad \operatorname{div}\mathbf{w}^3 = 0 & \text{in } \dot\Omega\times(0,\infty),\\ [\![\mathbf{T}(\mathbf{w}^3,\kappa^3)\mathbf{n}]\!] = 0, \quad [\![\mathbf{w}^3]\!] = 0 & \text{on } \Gamma\times(0,\infty),\\ \mathbf{T}(\mathbf{w}^3,\kappa^3)\mathbf{n}_+ = 0 & \text{on } \Gamma_+\times(0,\infty),\\ \mathbf{w}^3 = 0 & \text{on } \Gamma_-\times(0,\infty),\\ \mathbf{w}^3|_{t=0} = 0 & \text{in } \dot\Omega. \end{cases}$$

As the first step, we show the following lemma.

Lemma 4.6. *Let* $1 < p < \infty$ *and* $N < q < \infty$. *Suppose that* Ω *is a bounded domain and* Γ, $\Gamma_\pm$ *are closed hypersurfaces of* $W^{2-1/q}_q$ *class. Let* ε_0 *be the same positive number as in* (4.10). *If the right members* $\mathbf{f}$, g, $\mathbf{g}$, $\mathbf{h}$, *and* $\mathbf{k}$ *of* (4.14) *satisfy the conditions:*

$$e^{\varepsilon_0 t}\mathbf{f} \in L_{p,0}(\mathbf{R}, L_q(\dot\Omega))^N, \quad e^{\varepsilon_0 t} g \in H^{1,1/2}_{q,p,0}(\dot\Omega \times \mathbf{R}),$$
$$e^{\varepsilon_0 t}\mathbf{h} \in H^{1,1/2}_{q,p,0}(\dot\Omega \times \mathbf{R})^N, \quad e^{\varepsilon_0 t}\mathbf{k} \in H^{1,1/2}_{q,p,0}(\Omega_+ \times \mathbf{R})^N,$$
$$e^{\varepsilon_0 t}(\partial_t \mathbf{g}, \mathbf{g}) \in L_{p,0}(\mathbf{R}, L_q(\dot\Omega))^N \quad \textit{with } [\![\mathbf{g}(t)]\!]\cdot\mathbf{n} = 0 \textit{ on } \Gamma,\ \mathbf{g}(t)\cdot\mathbf{n}_+ = 0 \textit{ on } \Gamma_- \ (t>0),$$

then the equations (4.14) *admits a unique solution* $(\mathbf{w}^1, \kappa^1)$ *with*

$$\mathbf{w}^1 \in (W^1_p((0,\infty), L_q(\dot\Omega)) \cap L_p((0,\infty), W^2_q(\dot\Omega)))^N, \quad \kappa^1 \in L_p((0,\infty), W^1_q(\dot\Omega) + W^1_{q,\Gamma_+}(\Omega)),$$

which possesses the estimate:

$$\begin{aligned}(4.17)\quad &\|e^{\varepsilon_0 t}(\partial_t \mathbf{w}^1, \mathbf{w}^1, \nabla\mathbf{w}^1, \nabla^2\mathbf{w}^1)\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\kappa^1\|_{L_p((0,\infty),W^1_q(\dot\Omega))} \\ &\le C\Big(\|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g,\mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big)\end{aligned}$$

with some positive constant C.

Proof. Smooth functions having compact supports with respect to time variable are dense in the spaces for $\mathbf{f}$, g, $\mathbf{h}$, and $\mathbf{k}$, so that we may assume that $\mathbf{f}$, g, $\mathbf{h}$, and $\mathbf{k}$ are smooth and supported compactly with respect to time variable. Applying the Laplace transform with respect to time $t \in \mathbf{R}$ to (4.14), we have

$$(4.18)\quad \begin{cases} (\lambda + 2\lambda_0)\mathbf{u} - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{u},\theta) = \mathcal{L}[\mathbf{f}](\lambda), \quad \operatorname{div}\mathbf{u} = \mathcal{L}[g](\lambda) & \text{in } \dot\Omega, \\ [\![\mathbf{T}(\mathbf{u},\theta)\mathbf{n}]\!] = \mathcal{L}[\mathbf{h}](\lambda) \quad [\![\mathbf{u}]\!] = 0 & \text{on } \Gamma, \\ \mathbf{T}(\mathbf{u},\theta)\mathbf{n}_+ = \mathcal{L}[\mathbf{k}](\lambda) & \text{on } \Gamma_+, \\ \mathbf{u} = 0 & \text{on } \Gamma_-. \end{cases}$$

In view of Theorem 2.3, we define $\mathbf{w}^1$ and κ^1 by

$$\begin{aligned}\mathbf{w}^1 &= \mathcal{L}^{-1}_\lambda\left[\mathbf{A}(\lambda+2\lambda_0)F_{\lambda+2\lambda_0}(\mathcal{L}[\mathbf{f}](\lambda), \mathcal{L}[g](\lambda), \mathcal{L}[\mathbf{h}](\lambda), \mathcal{L}[\mathbf{k}](\lambda))\right], \\ \kappa^1 &= \mathcal{L}^{-1}_\lambda\left[\mathbf{P}(\lambda+2\lambda_0)F_{\lambda+2\lambda_0}(\mathcal{L}[\mathbf{f}](\lambda), \mathcal{L}[g](\lambda), \mathcal{L}[\mathbf{h}](\lambda), \mathcal{L}[\mathbf{k}](\lambda))\right].\end{aligned}$$

Let $\lambda = -\varepsilon_0 + i\tau$, and we set

$$\begin{aligned}\mathbf{F} =& F_{\lambda+2\lambda_0}(\mathcal{L}[\mathbf{f}](\lambda), \mathcal{L}[g](\lambda), \mathcal{L}[\mathbf{h}](\lambda), \mathcal{L}[\mathbf{k}](\lambda)) \\ =& \Big(\mathcal{F}[e^{\varepsilon_0 t}\mathbf{f}], \mathcal{F}[e^{\varepsilon_0 t}\nabla g], \mathcal{F}[\Lambda^{1/2}_{\lambda+2\lambda_0}(e^{\varepsilon_0 t}g)], \mathcal{F}[e^{\varepsilon_0 t}\partial_t\mathcal{G}(g)] + 2\lambda_0\mathcal{F}[e^{\varepsilon_0 t}\mathcal{G}(g)], \\ & \mathcal{F}[e^{\varepsilon_0 t}\nabla\mathbf{h}], \mathcal{F}[\Lambda^{1/2}_{\lambda+2\lambda_0}(e^{\varepsilon_0 t}\mathbf{h})], \mathcal{F}[e^{\varepsilon_0 t}\nabla\mathbf{k}], \mathcal{F}[\Lambda^{1/2}_{\lambda+2\lambda_0}(e^{\varepsilon_0 t}\mathbf{k})]\Big).\end{aligned}$$

Thus we obtain

$$\begin{aligned}\partial_t\mathbf{w}^1 &= \mathcal{L}^{-1}_\lambda\left[(\lambda+2\lambda_0)\mathbf{A}(\lambda+2\lambda_0)\mathbf{F}\right] - \mathcal{L}^{-1}_\lambda\left[\frac{2\lambda_0}{(\lambda+2\lambda_0)}(\lambda+2\lambda_0)\mathbf{A}(\lambda+2\lambda_0)\mathbf{F}\right], \\ \nabla\mathbf{w}^1 &= \mathcal{L}^{-1}_\lambda\left[\frac{1}{(\lambda+2\lambda_0)^{1/2}}(\lambda+2\lambda_0)^{1/2}\nabla\mathbf{A}(\lambda+2\lambda_0)\mathbf{F}\right], \quad \nabla^2\mathbf{w}^1 = \mathcal{L}^{-1}_\lambda\left[\nabla^2\mathbf{A}(\lambda+2\lambda_0)\mathbf{F}\right], \\ \mathbf{w}^1 &= \mathcal{L}^{-1}_\lambda\left[\frac{1}{(\lambda+2\lambda_0)}(\lambda+2\lambda_0)\mathbf{A}(\lambda+2\lambda_0)\mathbf{F}\right], \quad \kappa^1 = \mathcal{L}^{-1}_\lambda\left[\mathbf{P}(\lambda+2\lambda_0)\mathbf{F}\right],\end{aligned}$$

which, combined with (4.2), Theorem 2.3, and the Weis's operator valued Fourier multiplier theorem (cf. [Wei01, Theorem 3.4]), allows us to conclude that the estimate (4.17) holds. Here we note that $\mathcal{G}(g) = \mathbf{g}$ and have used the following proposition.

Proposition 4.7. *Let* $m(\lambda)$ *be a bounded function defined on a subset* Λ *in the complex plane* $\mathbf{C}$, *and let* $M_m(\lambda)$ *be a multiplication operator with* $m(\lambda)$ *defined by* $M_m(\lambda)f = m(\lambda)f$ *for any* $f \in L_q(D)$ *with an open set* $D \subset \mathbf{R}^N$. *Then,*

$$\mathcal{R}_{\mathcal{L}(L_q(D))}(\{M_m(\lambda) \mid \lambda \in \Lambda\}) \le C\|m\|_{L_\infty(\Lambda)}.$$

Finally, the same argumentation as in [Sai15, Section 7] furnishes that $\mathbf{u}(t) = 0$, $\theta(t) = 0$ for $t < 0$ and the uniqueness holds. $\square$

We now apply Lemma 4.6 to (4.15) with $\mathbf{f} = \nabla Q_q(2\lambda_0\mathbf{w}^1)$ in order to obtain

$$(4.19)\quad \begin{aligned}&\|e^{\varepsilon_0 t}(\partial_t\mathbf{w}^2, \mathbf{w}^2, \nabla\mathbf{w}^2, \nabla^2\mathbf{w}^2)\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\kappa^2\|_{L_p((0,\infty),W_q^1(\dot\Omega))}\\ &\quad\le C\Big(\|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g, \mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big)\end{aligned}$$

with some positive constant C, since $\|e^{\varepsilon_0 t}\nabla Q_q(2\lambda_0\mathbf{w}^1)\|_{L_p((0,\infty),L_q(\dot\Omega))} \le C\|e^{\varepsilon_0 t}\mathbf{w}^1\|_{L_p((0,\infty),L_q(\dot\Omega))}$.

Finally, we consider $(\mathbf{w}^3, \kappa^3)$. Let $\mathbf{W}(t) = P_q(2\lambda_0\mathbf{w}^1) + 2\lambda_0\mathbf{w}^2 \in J_q(\Omega)$, and then we have by (4.17), (4.19)

$$(4.20)\quad \begin{aligned}&\|e^{\varepsilon_0 t}\mathbf{W}\|_{L_p((0,\infty),L_q(\dot\Omega))}\\ &\quad\le C\Big(\|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g, \mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big).\end{aligned}$$

Since it holds that

$$\mathbf{w}^3(t) = \int_0^t e^{-\mathcal{A}(t-s)}\mathbf{W}(s)\,ds \quad\text{and}\quad \mathbf{W}(t) = 0 \quad \text{for } t < 0,$$

setting $\chi_+(t)$ as $\chi_+(t) = 1$ when $t > 0$ and $\chi_+(t) = 0$ when $t < 0$ yields that

$$\begin{aligned}e^{\varepsilon_0 t}\|\mathbf{w}^3(t)\|_{L_q(\dot\Omega)} &\le C\int_0^t e^{\varepsilon_0 t}e^{-2\varepsilon_0(t-s)}\|\mathbf{W}(s)\|_{L_q(\dot\Omega)}\,ds = C\int_0^t e^{-\varepsilon_0(t-s)}\left(e^{\varepsilon_0 s}\|\mathbf{W}(s)\|_{L_q(\dot\Omega)}\right)ds\\ &= C\int_{\mathbf{R}} \chi_+(t-s)e^{-\varepsilon_0(t-s)}\left(e^{\varepsilon_0 s}\|\mathbf{W}(s)\|_{L_q(\dot\Omega)}\right)ds = C\left[\left(\chi_+(\cdot)e^{-\varepsilon_0\cdot}\right) * \left(e^{\varepsilon_0\cdot}\|\mathbf{W}(\cdot)\|_{L_q(\dot\Omega)}\right)\right](t).\end{aligned}$$

Thus, by Young's inequality and (4.20), we have

$$\begin{aligned}&\|e^{\varepsilon_0 t}\mathbf{w}^3\|_{L_p(\mathbf{R},L_q(\dot\Omega))} \le C\|\chi_+(\cdot)e^{-\varepsilon_0\cdot}\|_{L_1(0,\infty)}\|e^{\varepsilon_0 t}\mathbf{W}\|_{L_p(\mathbf{R},L_q(\dot\Omega))}\\ &\le C\Big(\|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g, \mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big).\end{aligned}$$

In addition, we rewrite the equations (4.16) as follows:

$$\begin{cases}\partial_t\mathbf{w}^3 + 2\lambda_0\mathbf{w}^3 - \rho^{-1}\operatorname{Div}\mathbf{T}(\mathbf{w}^3, \kappa^3) = P_q(2\lambda_0\mathbf{w}^1) + 2\lambda_0\mathbf{w}^2 + 2\lambda_0\mathbf{w}^3 & \text{in } \dot\Omega\times(0,\infty),\\ \operatorname{div}\mathbf{w}^3 = 0 & \text{in } \dot\Omega\times(0,\infty),\\ [\![\mathbf{T}(\mathbf{w}^3, \kappa^3)\mathbf{n}]\!] = 0, \quad [\![\mathbf{w}^3]\!] = 0 & \text{on } \Gamma\times(0,\infty),\\ \mathbf{T}(\mathbf{w}^3, \kappa^3)\mathbf{n}_+ = 0 & \text{on } \Gamma_+\times(0,\infty),\\ \mathbf{w}^3 = 0 & \text{on } \Gamma_-\times(0,\infty),\\ \mathbf{w}^3|_{t=0} = 0 & \text{in } \dot\Omega,\end{cases}$$

which, combined with Lemma 4.6 and the last estimate, furnishes that

$$(4.21)\quad \begin{aligned}&\|e^{\varepsilon_0 t}(\partial_t\mathbf{w}^3, \mathbf{w}^3, \nabla\mathbf{w}^3, \nabla^2\mathbf{w}^3)\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}\kappa^3\|_{L_p((0,\infty),W_q^1(\dot\Omega))}\\ &\quad\le C\Big(\|e^{\varepsilon_0 t}(\mathbf{f}, \partial_t\mathbf{g}, \mathbf{g})\|_{L_p((0,\infty),L_q(\dot\Omega))} + \|e^{\varepsilon_0 t}(g, \mathbf{h})\|_{H^{1,1/2}_{q,p}(\dot\Omega\times\mathbf{R})} + \|e^{\varepsilon_0 t}\mathbf{k}\|_{H^{1,1/2}_{q,p}(\Omega_+\times\mathbf{R})}\Big).\end{aligned}$$

We thus obtain the required estimate in Theorem 4.5 by (4.17), (4.19), and (4.21). $\square$

By Theorem 4.4 and Theorem 4.5, we obtain Theorem 4.1.

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