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THE ITERATED REMAINDERS OF THE RATIONALS

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ABSTRACT. Repeat taking remainders of Stone-Čech compactifications of the rationals
$\mathbb{Q}^{(1)}=\mathbb{Q}^{*}=\beta\mathbb{Q}\backslash\mathbb{Q},\ \mathbb{Q}^{(2)}=\beta\mathbb{Q}^{(1)}\backslash\mathbb{Q}^{(1)},\ \mathbb{Q}^{(3)}=\beta\mathbb{Q}^{(2)}\backslash\mathbb{Q}^{(2)},\ \mathbb{Q}^{(4)}\ \cdots$.
We point out that they have similar structures, but, are topologically different. In particular we prove here that $\mathbb{Q}^{(1)}\not\approx\mathbb{Q}^{(3)}$. This result will be generalized to show that $\mathbb{Q}^{(n)}\not\approx\mathbb{Q}^{(n+2)}$ for any $n\geqslant 1$ in the forthcoming paper [4].

1. INTRODUCTION

Consider the space of rationals $\mathbb{Q}$, and repeat taking its remainders of Stone-Čech compactifications $\mathbb{Q}^{(n+1)}=(\mathbb{Q}^{(n)})^{*}=\beta\mathbb{Q}^{(n)}\backslash\mathbb{Q}^{(n)}$ $(n\geqslant 0)$ where $\mathbb{Q}^{(0)}=\mathbb{Q}$, i.e.,

$$\mathbb{Q}^{(1)}=\mathbb{Q}^{*},\ \mathbb{Q}^{(2)}=\mathbb{Q}^{**},\ \mathbb{Q}^{(3)}=\mathbb{Q}^{***},\cdots .$$

Van Douwen [2] asked whether or not $\mathbb{Q}^{(n)}\approx\mathbb{Q}^{(n+2)}$ for $n\geqslant 1$, remarking that $\mathbb{Q}^{(m)}$ for even m is never homeomorphic to $\mathbb{Q}^{(n)}$ for odd n, because the former is σ-compact but the latter is not.

In this paper we point out that both $\mathbb{Q}^{(n)}$ and $\mathbb{Q}^{(n+2)}$ have a similar structure of "fiber bundle" for every $n\geqslant 1$, but they are topologically different. In particular we here show that $\mathbb{Q}^{(1)}\not\approx\mathbb{Q}^{(3)}$, which we can generalize in the forthcoming paper [4] to show that $\mathbb{Q}^{(n)}\not\approx\mathbb{Q}^{(n+2)}$ for any $n\geqslant 1$, answering van Douwen's question.

The precise connections of the remainders can be seen by the following construction. Viewing $\beta\mathbb{Q}$ as a compactification of $\mathbb{Q}^{(1)}$, let

$$\Phi_{0}:\beta\mathbb{Q}^{(1)}=\mathbb{Q}^{(1)}\cup\mathbb{Q}^{(2)}\rightarrow\mathbb{Q}\cup\mathbb{Q}^{(1)}=\beta\mathbb{Q}$$

be the Stone extension of the identity map $id:\mathbb{Q}^{(1)}\rightarrow\mathbb{Q}^{(1)}$. Denote by

$$\phi_{0}:\mathbb{Q}^{(2)}\rightarrow\mathbb{Q}^{(0)}$$

the restriction of Φ_{0}. Next let

$$\Phi_{1}:\beta\mathbb{Q}^{(2)}=\mathbb{Q}^{(2)}\cup\mathbb{Q}^{(3)}\rightarrow\mathbb{Q}^{(1)}\cup\mathbb{Q}^{(2)}=\beta\mathbb{Q}^{(1)}$$

be the Stone extension of the identity map $id:\mathbb{Q}^{(2)}\rightarrow\mathbb{Q}^{(2)}$, and let

$$\phi_{1}:\mathbb{Q}^{(3)}\rightarrow\mathbb{Q}^{(1)}$$

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denote the restriction of Φ_1. In this way, for every $n \geqslant 0$ we can generally get the Stone extension

$$\Phi_n : \beta\mathbb{Q}^{(n+1)} = \mathbb{Q}^{(n+1)} \cup \mathbb{Q}^{(n+2)} \to \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} = \beta\mathbb{Q}^{(n)}$$

of the identity map $id : \mathbb{Q}^{(n+1)} \to \mathbb{Q}^{(n+1)}$, and its restriction map

$$\phi_n : \mathbb{Q}^{(n+2)} \to \mathbb{Q}^{(n)}.$$

Since every Φ_n $(n \in \omega)$ is perfect, so is every ϕ_n. Hence every $\mathbb{Q}^{(n)}$ $(n \in \omega)$ is Lindeöf since both $\mathbb{Q}^{(0)} = \mathbb{Q}$, $\mathbb{Q}^{(1)}$ are Lindeöf. We can also see that $\mathbb{Q}^{(n)}$ is σ-compact for even n, but $\mathbb{Q}^{(n)}$ is not for odd n, because $\mathbb{Q}^{(0)}$ is σ-compact but $\mathbb{Q}^{(1)}$ is not since $\mathbb{Q}^{(1)}$ is a perfect pre-image of the irrationals $\mathbb{P}$ as we see below.

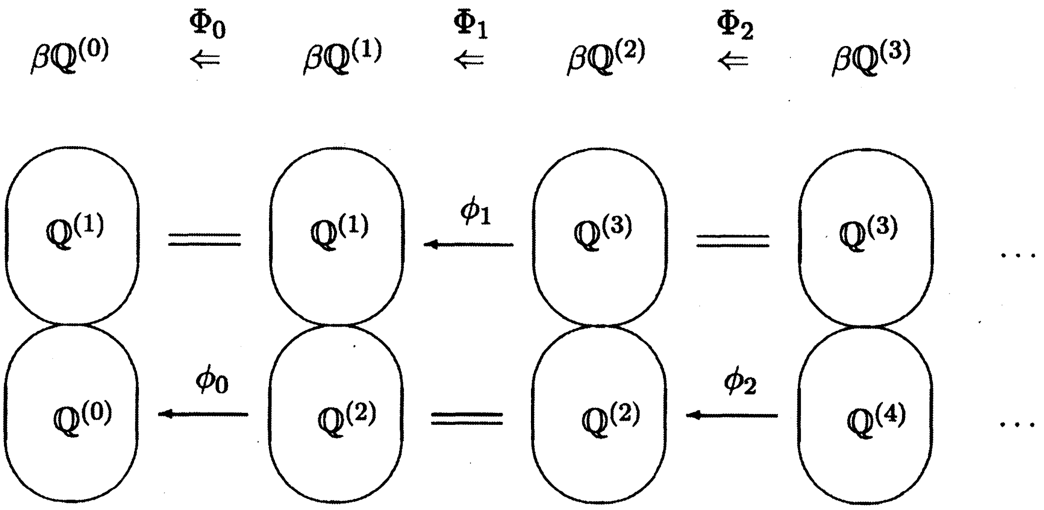

FIG. 1

A collection $\mathcal{B}$ of nonempty open sets of X is called a *π-base* for X if every nonempty open set in X includes some member of $\mathcal{B}$. The minimal cardinality of such a π-base is called the *π-weight* of X. Note that any dense subspace of X has the same π-weight as X, and any space of countable π-weight is separable. Consequently, any dense subset of a space of countable π-weight is also of countable π-weight, and hence separable. So, all of $\beta\mathbb{Q}^{(n)}$, $\mathbb{Q}^{(n)}$ $(n \in \omega)$ are of countable π-weight, and hence separable.

Recall that an onto map $g : X \to Y$ is called *irreducible* if every nonempty open subset U of X includes some fiber $g^{-1}(y)$, and it is well known and easy to see that

(1) every extension of a homeomorphism is irreducible, and

(2) the restriction of a closed irreducible map to any dense subset is irreducible.

Therefore we can see that all of the maps Φ_n, ϕ_n $(n \in \omega)$ are perfect irreducible. Consider the partition of the closed interval $[0,1] = Q \cup P$ where

$$Q = [0,1] \cap \mathbb{Q} \approx \mathbb{Q} \text{ and } P = [0,1] \backslash \mathbb{Q} \approx \mathbb{P},$$

and let $f : \beta\mathbb{Q} \to [0,1]$ be the Stone extension of the homeomorphism $\mathbb{Q} \approx Q$. Then the restriction $f_0 = f \restriction \mathbb{Q}^{(1)} : \mathbb{Q}^{(1)} \to P \approx \mathbb{P}$ is perfect irreducible. Thus we get the following sequence of perfect irreducible maps:

$$\mathbb{Q} \leftarrow \mathbb{Q}^{(2)} \leftarrow \mathbb{Q}^{(4)} \leftarrow \cdots ; \quad \mathbb{P} \leftarrow \mathbb{Q}^{(1)} \leftarrow \mathbb{Q}^{(3)} \leftarrow \mathbb{Q}^{(5)} \leftarrow \cdots .$$

All spaces are assumed to be completely regular and Hausdorff, and maps are always continuous, unless otherwise stated. "Partition" is synonymous with "disjoint union." For a subset A of some compact space K we use the notation A^* to denote the remainder $\mathrm{cl}_K A \backslash A$ when K is clear from the context. Our terminologies are based upon [3].

2. Similar Structures

We first show that both $\mathbb{Q}^{(n)}$ and $\mathbb{Q}^{(n+2)}$ have a similar structure for every $n \geqslant 1$. In general, for any space Y let us denote by $\mathbf{H}(Y)$ the collection of all homeomorphisms $h : Y \approx Y$. Let X be a nowhere compact, dense-in-itself space, where *nowhere compact* (or *nowhere locally compact*) means that X contains no compact neighborhood, or equivalently, that X is a dense subset of some/any compact space K such that the remainder $K \backslash X$ is also dense in K. Let cX be some compactification of X and let $\mathcal{H}_\star \subseteq \mathbf{H}(X)$ denote the collection of all $h \in \mathbf{H}(X)$ such that

$(\star)$ h is extendable to $c(h) \in \mathbf{H}(cX)$.

(Of course, $\mathcal{H}_\star = \mathbf{H}(cX)$ if $cX = \beta X$.) Let $X^{(1)} = cX \backslash X$ be the remainder, and for every $h \in \mathcal{H}_\star$ define $h^{(1)} \in \mathbf{H}(X^{(1)})$ to be the restriction of $c(h)$ to $X^{(1)}$. Next consider the Stone-Čech compactification $\beta X^{(1)}$ of $X^{(1)}$ and the Stone extension $\beta h^{(1)} \in \mathbf{H}(\beta X^{(1)})$ of $h^{(1)}$. Let $X^{(2)} = \beta X^{(1)} \backslash X^{(1)}$ be the remainder, and define $h^{(2)} \in \mathbf{H}(X^{(2)})$ to be the restriction of $\beta h^{(1)}$ to the remainder $X^{(2)}$; hence

$$h : X \approx X, \quad h^{(1)} : X^{(1)} \approx X^{(1)}, \quad h^{(2)} : X^{(2)} \approx X^{(2)}.$$

Note that $X^{(1)}$ is dense in βX, and $X^{(2)}$ is dense in $\beta X^{(1)}$, since we assume that X is nowhere compact. Viewing that βX is a compactification of $X^{(1)}$, we can consider the Stone extension $\Phi : \beta X^{(1)} \to \beta X$ of the identity map $id_{X^{(1)}} : X^{(1)} = X^{(1)}$. Let $\phi : X^{(2)} \to X$ be the restriction of Φ. Then both Φ and ϕ are perfect irreducible maps. We can show that the correspondence $\mathbf{H}(X) \supseteq \mathcal{H}_\star \ni h \mapsto h^{(2)} \in \mathbf{H}(X^{(2)})$ is compatible with the perfect irreducible map ϕ, i.e.,

Lemma 2.1. $h \circ \phi = \phi \circ h^{(2)} : X^{(2)} \to X.$

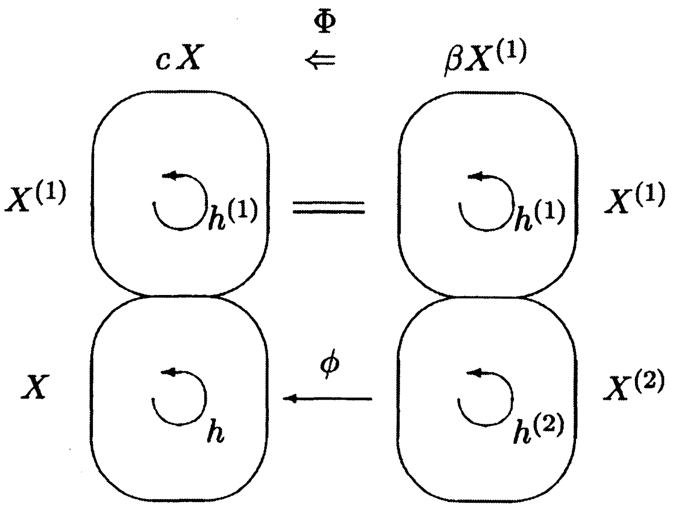

FIG. 2

Proof. To show this equality, it suffices to prove the equality

$$c(h) \circ \Phi = \Phi \circ \beta h^{(1)} : \beta X^{(1)} \to cX,$$

which follows from the obvious equality

$$h^{(1)} \circ id_{X^{(1)}} = id_{X^{(1)}} \circ h^{(1)} : X^{(1)} \to X^{(1)}$$

on the dense subset $X^{(1)}$ of $\beta X^{(1)}$. □

Corollary 2.2. *If $h(x) = y$ for $x, y \in X$, then $h^{(2)}(\phi^{-1}(x)) = \phi^{-1}(y)$.*

Proof. The inclusion $h^{(2)}(\phi^{-1}(x)) \subseteq \phi^{-1}(y)$ follows from 2.1. Since h is a homeomorphism, we can replace h by h^{-1} to get the reverse inclusion. □

Taking $X = \mathbb{Q}$, $cX = \beta\mathbb{Q}$, $\mathcal{H}_\star = \mathbf{H}(\mathbb{Q})$, we can deduce from 2.1 that

(2−1) $\qquad h \circ \phi_0 = \phi_0 \circ h^{(2)} : \mathbb{Q}^{(2)} \to \mathbb{Q}$ for every $h \in \mathbf{H}(\mathbb{Q})$.

Let $[0,1] = Q \cup P$, $Q \approx \mathbb{Q}$, $P \approx \mathbb{P}$ be as at the end of §1, and take $X = P$, $cX = [0,1]$; then $X^{(1)} = Q$, $X^{(2)} = Q^{(1)}$, and the corresponding map ϕ in Fig. 2 is identical to the map $f_0 : \mathbb{Q}^{(1)} \to P$ at the end of §1. Note that $\mathcal{H}_\star \subseteq \mathbf{H}(P)$ is the collection of all homeomorphisms of P extendable to homeomorphisms of $[0,1]$. Then we can deduce from 2.1 that

(2−2) $\qquad h \circ f_0 = f_0 \circ h^{(2)} : \mathbb{Q}^{(1)} \to P$ for every $h \in \mathcal{H}_\star$.

Note that for every pair of irrationals $p_1 < p_2$ in $P = [0,1]\backslash\mathbb{Q}$ we can find an $h \in \mathcal{H}_\star$ such that $h(p_1) = p_2$; for example, we can take as $c(h)$ in ($\star$) a strictly increasing function $c(h) : [0,1] \to [0,1]$ such that $c(h)(Q) = Q$, $c(h)(0) = 0$, $c(h)(p_1) = p_2$, $c(h)(1) = 1$. For $m \geqslant 1$ define g_{2m} and f_{2m-1} by

$$g_{2m} = \phi_0 \circ \phi_2 \circ \cdots \circ \phi_{2m-2} : \mathbb{Q}^{(2m)} \to \mathbb{Q},$$

$$f_{2m-1} = f_0 \circ \phi_1 \circ \phi_3 \circ \cdots \circ \phi_{2m-3} : \mathbb{Q}^{(2m-1)} \to P.$$

Then, using 2.1 we can extend the above (2-1), (2-2) to the followings, respectively, for $m \geqslant 1$.

$$(2-3) \qquad h \circ g_{2m} = g_{2m} \circ h^{(2m)} : \mathbb{Q}^{(2m)} \to \mathbb{Q} \quad \text{for every } h \in \mathbf{H}(\mathbb{Q}),$$

$$(2-4) \qquad h \circ f_{2m-1} = f_{2m-1} \circ h^{(2m-1)} : \mathbb{Q}^{(2m-1)} \to P \quad \text{for every } h \in \mathcal{H}_{\star},$$

where $h^{(n)} \in \mathbf{H}(\mathbb{Q}^{(n)})$. Combining these results with 2.2 we can summarize that

Theorem 2.3. *Let $m \geqslant 1$. Then every $\mathbb{Q}^{(2m)}$ admits a perfect irreducible projection g_{2m} onto $\mathbb{Q}$, and every $\mathbb{Q}^{(2m-1)}$ admits a perfect irreducible projection f_{2m-1} onto $P \approx \mathbb{P}$, with the additional property that they are "fiber-wise" homogeneous in the following sense:*
(1) For any $q_1 < q_2 \in \mathbb{Q}$ there exists a homeomorphism of $\mathbb{Q}^{(2m)}$, induced by a homeomorphism of $\mathbb{Q}$, carrying the fiber $g_{2m}^{-1}(q_1)$ to $g_{2m}^{-1}(q_2)$.
(2) For any $p_1 < p_2 \in P$ there exists a homeomorphism of $\mathbb{Q}^{(2m-1)}$, induced by a homeomorphism of P, carrying the fiber $f_{2m-1}^{-1}(p_1)$ to $f_{2m-1}^{-1}(p_2)$.
Moreover, under CH (=the Continuum Hypothesis) every fiber $g_{2m}^{-1}(q)$ of $q \in \mathbb{Q}$ as well as every fiber $f_{2m-1}^{-1}(p)$ of $p \in P$ is homeomorphic to $\omega^* = \beta\omega\backslash\omega$.

This last assertion follows from the well-known

Fact 2.4. *(see 1.2.6 in [8] or 3.37 in [9]) (CH) Let Y be a 0-dimensional, locally compact, σ-compact, non-compact space of weight at most* **c**. *Then $Y^* = \beta Y\backslash Y$ and ω^* are homeomorphic.*

Indeed, put $Z = g_{2m}^{-1}(q)$ and $Y = \beta\mathbb{Q}^{(2m-1)}\backslash Z$. Then Z is a zero-set of the 0-dimensional $\beta\mathbb{Q}^{(2m-1)}$ included in the remainder $\mathbb{Q}^{(2m)} = \beta\mathbb{Q}^{(2m-1)}\backslash\mathbb{Q}^{(2m-1)}$, so that $Y^* = \beta Y\backslash Y = Z$. Since Y is a cozero-set and separable, Y satisfies the condition in 2.4. Hence $Z \approx \omega^*$. Similarly we can prove that $f_{2m-1}^{-1}(p) \approx \omega^*$.

3. Remote Points and Extremally Disconnected Points

To analyze further the structure of $\mathbb{Q}^{(n)}$'s, we need the notion of remote points and extremally disconnected points. A point $p \in \beta X\backslash X$ is called a *remote* point of X if $p \notin \mathrm{cl}_{\beta X}F$ for every nowhere dense closed subset F of X. Van Douwen [2], Chae, Smith [1], showed

Fact 3.1. *Every non-pseudocompact space of countable π-weight has $2^{\mathbf{c}}$ many remote points.*

An easy consequence of this fact is

Fact 3.2. *Let X be a non-compact, Lindelöf space of countable π-weight. Then remote points of X form a G_δ-dense subset of $X^* = \beta X\backslash X$.*

Proof. Choose any point $p \in X^*$ and a zero-set Z of βX containing p. Since X is Lindelöf, we can suppose that Z misses X. Put $Y = \beta X \backslash Z$; then $\beta Y = \beta X$, and Y is of countable π-weight since X is. Hence 3.1 implies that $Y^* = Z$ contains remote points of Y, which are also remote points of X. □

A space T is said to be *extremally disconnected at a point* $p \in T$ (see [2]) if $p \notin \mathrm{cl}_T U_1 \cap \mathrm{cl}_T U_2$ for every pair of disjoint open sets U_1, U_2 in T. Let us call such a point p as an *extremally disconnected point* of T, or simply, an *e.d.* point of T, and denote the set of all such e.d. points by $\mathrm{Ed}(T)$. A space T is *extremally disconnected* if every point of T is an e.d. point, i.e., $\mathrm{Ed}(T) = T$. If S is dense in T, we always have $\mathrm{cl}_T U = \mathrm{cl}_T(U \cap S)$ for every open set U of T; hence a point $p \in S$ is an e.d. point of S if and only if it is an e.d. point of T, i.e., $\mathrm{Ed}(S) = S \cap \mathrm{Ed}(T)$.

Fact 3.3. ([2]) (1) *Any remote point of X is an e.d. point of βX.*
(2) *Suppose X is first countable and hereditarily separable, and $p \in \beta X \backslash X$. Then p is a remote point of X if and only if p is an e.d. point of βX.*

Let us call a point $p \in T$ a *common boundary* point of T if p is not an e.d. point of T, i.e., if $p \in \mathrm{cl}_T U_1 \cap \mathrm{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T. Similarly, we call a subset $A \subseteq T$ a *common boundary* set in T if $A \subseteq \mathrm{cl}_T U_1 \cap \mathrm{cl}_T U_2$ for some pair of disjoint open sets U_1, U_2 in T. We abbreviate "common boundary" to "co-boundary." (Such p, A are called "2-point," "2-set," respectively, in [2].) Note that any co-boundary set in T is nowhere dense in T, but the converse need not be true. Let $\mathrm{Cob}(T) = T \backslash \mathrm{Ed}(T)$ denote the set of all co-boundary points of T. Note also that if A is a co-boundary set, then every point of A is obviously a co-boundary point, but the converse need not be true except the case A is a countable discrete subset:

Lemma 3.4. *Suppose A is a countable discrete subset consisting of co-boundary points of T. Then A, and hence also $\mathrm{cl}_T A$, is a co-boundary set in T. Therefore, if T is compact, $\mathrm{Cob}(T)$ is always countably compact in the strong sense that every countable discrete subset has compact closure in $\mathrm{Cob}(T)$.*

Proof. Let $A = \{a_n\}_{n\in\omega} \subseteq \mathrm{Cob}(T)$ be discrete in T, and choose disjoint open sets $\{W_n\}_{n\in\omega}$ in T such that $a_n \in W_n$. In each W_n choose disjoint open sets U_n, V_n with $a_n \in \mathrm{cl}_T U_n \cap \mathrm{cl}_T V_n$. Put $U = \bigcup_{n\in\omega} U_n$ and $V = \bigcup_{n\in\omega} V_n$. Then these disjoint open sets U, V satisfy $A \subseteq \mathrm{cl}_T U \cap \mathrm{cl}_T V$, and hence $\mathrm{cl}_T A \subseteq \mathrm{cl}_T U \cap \mathrm{cl}_T V$. □

For an open set $U \subseteq X$ its maximal open extension $\mathrm{Ex}(U) \subseteq \beta X$ is defined by

$$\mathrm{Ex}(U) = \beta X \backslash \mathrm{cl}_{\beta X}(X \backslash U).$$

Suppose W is an open set in βX; then

$$\mathrm{cl}_{\beta X} W = \mathrm{cl}_{\beta X}(W \cap X) = \mathrm{cl}_{\beta X} \mathrm{Ex}(W \cap X).$$

Therefore we see

Fact 3.5. *Suppose $p \in \beta X \backslash X$. Then p is a co-boundary point of βX if and only if $p \in \mathrm{cl}_{\beta X}\mathrm{Ex}(U) \cap \mathrm{cl}_{\beta X}\mathrm{Ex}(V)$ for some disjoint open sets U, V in X.*

We denote the boundary of a subset W in Y by $\mathrm{Bd}_Y W$ so that $\mathrm{Bd}_Y W = \mathrm{cl}_Y W \backslash W$ if W is open in Y. Van Douwen [2] proved the equality

$$(*) \qquad \mathrm{Bd}_{\beta X}\mathrm{Ex}(U) = \mathrm{cl}_{\beta X}\mathrm{Bd}_X(U)$$

for every open set U of X. (Note that 3.3 (1) follows from this equality since $\mathrm{Bd}_X(U)$ is a nowhere dense subset of X.) Using this $(*)$ and 3.5 we get an "inner" characterization of co-boundary points, hence of e.d. points also, of βX for a normal space X:

Lemma 3.6. *Assume X is normal, and $p \in \beta X \backslash X$. Then p is a co-boundary point of βX if and only if $p \in \mathrm{cl}_{\beta X} F$ for some co-boundary set F in X. In other words, p is an e.d. point of βX if and only if*

$p \notin \mathrm{cl}_{\beta X} F$ for every co-boundary set F in X.

Proof. By 3.5 it suffices to show the equality

$$\mathrm{cl}_{\beta X}\mathrm{Ex}(U) \cap \mathrm{cl}_{\beta X}\mathrm{Ex}(V) = \mathrm{cl}_{\beta X}(\mathrm{cl}_X U \cap \mathrm{cl}_X V)$$

for disjoint open sets U, V in X, since $\mathrm{cl}_X U \cap \mathrm{cl}_X V$ is a co-boundary set in X. Using $(*)$ we get

$$\mathrm{cl}_{\beta X}\mathrm{Ex}(U) \cap \mathrm{cl}_{\beta X}\mathrm{Ex}(V) = \mathrm{Bd}_{\beta X}\mathrm{Ex}(U) \cap \mathrm{Bd}_{\beta X}\mathrm{Ex}(V)$$
$$= (\mathrm{cl}_{\beta X}\mathrm{Bd}_X U) \cap (\mathrm{cl}_{\beta X}\mathrm{Bd}_X V).$$

Since X is normal, this set is equal to $\mathrm{cl}_{\beta X}(\mathrm{Bd}_X U \cap \mathrm{Bd}_X V)$, where $\mathrm{Bd}_X U \cap \mathrm{Bd}_X V = \mathrm{cl}_X U \cap \mathrm{cl}_X V$. □

Lemma 3.7. *Suppose A is a closed subset of a normal space X. Then $A \subseteq \mathrm{Ed}(X)$ implies $\mathrm{cl}_{\beta X} A \subseteq \mathrm{Ed}(\beta X)$.*

Proof. Let A be a closed subset of a normal space X, and that $A \subseteq \mathrm{Ed}(X)$. Let F be any co-boundary closed set in X. By 3.6 it suffices to show that $\mathrm{cl}_{\beta X} F \cap \mathrm{cl}_{\beta X} A = \emptyset$. Since $F \subseteq \mathrm{Cob}(X)$ and $A \subseteq \mathrm{Ed}(X)$, we know that F, A are disjoint closed subsets of X. Hence the normality of X implies that $\mathrm{cl}_{\beta X} F \cap \mathrm{cl}_{\beta X} A = \emptyset$. □

The next lemma shows how co-boundary points or e.d. points behave w.r.t. closed irreducible maps. Let g be a map from X onto Y. For a subset $U \subseteq X$ define $g^\circ(U) \subseteq Y$, a *small image* of U, by

$$y \in g^\circ(U) \text{ if and only if } g^{-1}(y) \subseteq U,$$

i.e., $g^\circ(U) = Y \backslash g(X \backslash U) \subseteq g(U)$; so, g is irreducible if $g^\circ(U) \neq \emptyset$ for every non-empty open set U. Note an obvious useful formula

$$g^\circ(U \cap V) = g^\circ(U) \cap g^\circ(V)$$

for any sets $U, V \subseteq X$, which especially implies that $g^\circ(U) \cap g^\circ(V) = \emptyset$ whenever $U \cap V = \emptyset$. Suppose g is closed irreducible. Then it is well known that $g^\circ(U)$ is non-empty and open whenever U is, and

$$\mathrm{cl}_Y\, g^\circ(U) = \mathrm{cl}_Y\, g(U) = g(\mathrm{cl}_X U)$$

for every open subset $U \subseteq X$.

Lemma 3.8. *Let $g : X \to Y$ be any closed irreducible map. Then g maps co-boundary points to co-boundary points, i.e., $g(\mathrm{Cob}(X)) \subseteq \mathrm{Cob}(Y)$. Furthermore, for every $x \in X$*

$g(x) \in \mathrm{Cob}(Y)$ *if and only if* $x \in \mathrm{Cob}(X)$ *or* $|g^{-1}(g(x))| > 1$, *i.e.,*

$g(x) \in \mathrm{Ed}(Y)$ *if and only if* $x \in \mathrm{Ed}(X)$ *and* $g^{-1}(g(x)) = \{x\}$.

Consequently, $g^{-1}(\mathrm{Ed}(Y)) \subseteq \mathrm{Ed}(X)$, *and the restriction of g to*

$$g^{-1}(\mathrm{Ed}(Y)) \to \mathrm{Ed}(Y)$$

is a homeomorphism.

Proof. Let U_1, U_2 be any disjoint open sets in X. Then

$$g(\mathrm{cl}_X U_1 \cap \mathrm{cl}_X U_2) \subseteq g(\mathrm{cl}_X U_1) \cap g(\mathrm{cl}_X U_2) = \mathrm{cl}_Y g^\circ(U_1) \cap \mathrm{cl}_Y g^\circ(U_2),$$

and $g^\circ(U_1)$, $g^\circ(U_2)$ are disjoint open. Hence g maps co-boundary points to co-boundary points. Similarly, we can show that

$|g^{-1}(g(x))| > 1$ implies $g(x) \in \mathrm{Cob}(Y)$.

Indeed, if we take two points $x_1 \neq x_2$ in $g^{-1}(g(x))$, we can choose disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$ (using the Hausdorffness of X), getting $g(x) \in g(\mathrm{cl}_X U_1) \cap g(\mathrm{cl}_X U_2) = \mathrm{cl}_Y g^\circ(U_1) \cap \mathrm{cl}_Y g^\circ(U_2)$. So, to complete our proof, assume $g(x) \in \mathrm{Cob}(Y)$ and $|g^{-1}(g(x))| = 1$; then we need to show $x \in \mathrm{Cob}(X)$. The condition $g(x) \in \mathrm{Cob}(Y)$ implies that $g(x) \in \mathrm{cl}_Y V_1 \cap \mathrm{cl}_Y V_2$ for some disjoint open sets V_1, V_2 in Y. Since g is a closed map, $g(x) \in \mathrm{cl}_Y V_i$ implies $g^{-1}(g(x)) \cap \mathrm{cl}_X g^{-1}(V_i) \neq \emptyset$ for $i = 1, 2$. Hence the condition $g^{-1}(g(x)) = \{x\}$ implies $x \in \mathrm{cl}_X g^{-1}(V_1) \cap \mathrm{cl}_X g^{-1}(V_2)$, showing $x \in \mathrm{Cob}(X)$. □

4. Topological Difference of $\mathbb{Q}^{(1)}$ and $\mathbb{Q}^{(3)}$

Now let us apply the general theory in §3 to our spaces

$$\beta\mathbb{Q}^{(n)} = \mathbb{Q}^{(n)} \cup \mathbb{Q}^{(n+1)} \ (n \geqslant 0).$$

Recall that every $\mathbb{Q}^{(n)}$ is of countable π-weight and Lindelöf, hence normal. Put $C_n = \mathrm{Cob}(\mathbb{Q}^{(n)})$ and $E_n = \mathrm{Ed}(\mathbb{Q}^{(n)})$; then this gives a partition of $\mathbb{Q}^{(n)}$

$$\mathbb{Q}^{(n)} = C_n \cup E_n.$$

It is obvious that $E_0 = \emptyset$, i.e., $\mathbb{Q}^{(0)} = C_0$. Lemma 3.4 implies that each C_n $(n \geqslant 1)$ is dense in $\mathbb{Q}^{(n)}$, and Fact 3.2 with 3.3 (1) implies that each

E_n $(n \geqslant 1)$ is dense in $\mathbb{Q}^{(n)}$. Note in particular that E_1 coincides with the set of all remote points of $\mathbb{Q}$, by 3.3 (2).

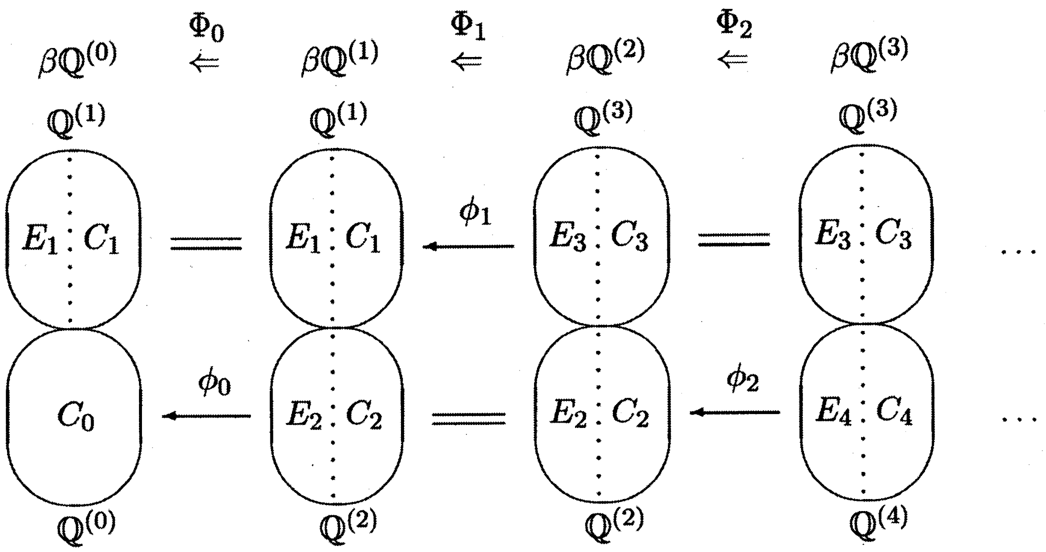

FIG. 3

Property 4.1. *Let A be any countable discrete subset of E_2 which is closed in $\mathbb{Q}^{(2)}$. Then*
(1) $\operatorname{cl} A \subseteq E_2 \cup C_1$ *in* $\beta\mathbb{Q}^{(1)}$, *while* (2) $\operatorname{cl} A \subseteq E_2 \cup E_3$ *in* $\beta\mathbb{Q}^{(2)}$.

Proof. (2) follows from 3.7. To prove (1), let A be as above. Then, since $\phi_0 : \mathbb{Q}^{(2)} \to \mathbb{Q}^{(0)}$ is perfect, $\phi_0(A)$ is also a countable discrete closed subset of $\mathbb{Q}^{(0)} = C_0$. Since $C_0 \cup C_1 = \operatorname{Cob}(\beta\mathbb{Q}^{(0)})$ is countably compact in the strong sense as stated in 3.4, we have $\operatorname{cl} \phi_0(A) \subseteq C_0 \cup C_1$ in $\beta\mathbb{Q}^{(0)}$. Pulling back by the map Φ_0, we get $\operatorname{cl} A \subseteq \mathbb{Q}^{(2)} \cup C_1$ in $\beta\mathbb{Q}^{(1)}$. This is the same as the assertion (1) since $A \subseteq E_2$. □

Now we can prove the following strong assertion which in particular implies that $\mathbb{Q}^{(1)} \not\approx \mathbb{Q}^{(3)}$.

Theorem 4.2. $\mathbb{Q}^{(1)}$ *admits no perfect irreducible map onto* $\mathbb{Q}^{(3)}$.

Proof. Suppose there existed a perfect irreducible map $\psi : \mathbb{Q}^{(1)} \to \mathbb{Q}^{(3)}$. Then, since $\beta\mathbb{Q}^{(2)}$ can be seen as a compactification of $\mathbb{Q}^{(3)}$, ψ extends to a perfect irreducible map

$$\Psi : \beta\mathbb{Q}^{(1)} = \mathbb{Q}^{(1)} \cup \mathbb{Q}^{(2)} \to \beta\mathbb{Q}^{(2)} = \mathbb{Q}^{(3)} \cup \mathbb{Q}^{(2)}.$$

Lemma 3.8 implies then that

$$E_2 \cup E_1 \supseteq \Psi^{-1}(E_2 \cup E_3) \approx E_2 \cup E_3.$$

Choose any countable discrete subset $B \subseteq E_2 \subseteq \mathbb{Q}^{(2)} \subseteq \beta\mathbb{Q}^{(2)}$ which is closed in $\mathbb{Q}^{(2)}$. (We can do this because E_2 is dense in $\mathbb{Q}^{(2)}$, and $\mathbb{Q}^{(2)}$ is

Lindelöf.) Put $A = \Psi^{-1}(B)$, then this A is also a countable discrete subset of E_2 which is closed in $\mathbb{Q}^{(2)}$. Property 4.1 (2) shows $\mathrm{cl}\, B \subseteq E_2 \cup E_3$ in $\beta\mathbb{Q}^{(2)}$, and so, pulling back by Ψ, we get

$$\mathrm{cl}\, A \subseteq \Psi^{-1}(E_2 \cup E_3) \subseteq E_2 \cup E_1$$

in $\beta\mathbb{Q}^{(1)}$. But this contradicts 4.1 (1). □

We will be able to show in [4] that for any $n \geqslant 1$, $\mathbb{Q}^{(n)}$ admits no perfect irreducible map onto $\mathbb{Q}^{(n+2)}$ by analyzing further the behavior of limit points of countable discrete subsets in $\mathbb{Q}^{(m)}$. Some of the basic techniques in this paper can be found also in [5, 6, 7].

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