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CURVE COMPLEXES AND THE DM-COMPACTIFICATION OF MODULI SPACES OF RIEMANN SURFACES

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1. INTRODUCTION

Let $M_{g,n}$ be the moduli space of Riemann surfaces of genus g with n punctures. In this report, we study the DM (=Deligne-Mumford) compactification $\overline{M_{g,n}}$ of $M_{g,n}$. Our purpose is three-fold: (1) to construct a "natural" atlas of orbifold-charts on $\overline{M_{g,n}}$, making use of N. V. Ivanov's "scissored Teichmüller space" $P^{\varepsilon}_{g,n}$ [9], (2) to clarify the role of W. J. Harvey's curve complex $\mathcal{C}_{g,n}$ [7] in the compactification process, and finally (3) to point out a natural connection between Teichmüller spaces and crystallographic groups.

2. BASIC DEFINITIONS

We consider a pair (S, w) of a Riemann surface S and an orientation preserving homeomorphism $w : \Sigma_{g,n} \to S$, where $\Sigma_{g,n}$ is an oriented surface of type (g, n). Two such pairs (S, w) and (S', w') are *equivalent* $(S, w) \sim (S', w')$ if and only if there exists a biholomorphic map $t : S \to S'$ such that the following diagram homotopically commutes:

$$\begin{array}{ccc} \Sigma_{g,n} & \xrightarrow{w} & S \\ {\scriptstyle id.}\downarrow & & \downarrow{\scriptstyle t} \\ \Sigma_{g,n} & \xrightarrow[w']{} & S'. \end{array}$$

The *Teichmüller space* $T_{g,n}$ is defined by

$$T_{g,n} = \{(S, w)\}/ \sim .$$

We denote the mapping class group of $\Sigma_{g,n}$ by $\Gamma_{g,n}$, and define its action on $T_{g,n}$ by

$$[f]_*[S, w] = [S, w \circ f^{-1}],$$

where $[f] \in \Gamma_{g,n}$ and $[S, w] \in T_{g,n}$.

$T_{g,n}$ is a complex analytic space ([22], [3]), and is a bounded domain [4] of $\dim_{\mathbb{C}} T_{g,n} = 3g - 3 + n$.

We define the *length function* $L : T_{g,n} \to \mathbb{R}$ as follows: Let C be an essential simple closed curve on $\Sigma_{g,n}$. For any point $p = [S, w] \in T_{g,n}$, let $l_p(C)$ be the length of the simple closed geodesic $\hat{C}$ on S homotopic to $w(C)$. Define $L : T_{g,n} \to \mathbb{R}$ by

$$L(p) \stackrel{\text{def.}}{=} \min_{C \subset \Sigma_{g,n}} l_p(C).$$

The length function L is a piecewise real analytic function on $T_{g,n}$ (Fenchel-Nielsen, Abikoff [2]).

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3. IVANOV'S SCISSORED TEICHMÜLLER SPACE $P^{\varepsilon}_{g,n}$

Let $\varepsilon > 0$ be a sufficiently small number. In his cohomological study on the mapping class groups, N. V. Ivanov [9] introduced the following space, which we would like to call *Ivanov's scissored Teichmüller space* and to denote by $P^{\varepsilon}_{g,n}$:

$$P^{\varepsilon}_{g,n} \overset{\text{def.}}{=} \{p \in T_{g,n} \mid L(p) \geqq \varepsilon\}.$$

$P^{\varepsilon}_{g,n}$ is a real analytic manifold with corners. (The author was pointed out by Hiroshige Shiga that $P^{\varepsilon}_{g,n}$ is usually known as a *thick part* of $T_{g,n}$.)

To what extent should ε be small? To answer this question, let us recall the following

Theorem 3.1. *(Keen* [12], *Abikoff* [2]*) There is an universal constant M such that two distinct simple closed geodesics on S are disjoint, if their lengths are smaller than M.*

The number ε should be taken as $\varepsilon < M$.

3.1. **Facets of $P^{\varepsilon}_{g,n}$.** Suppose a point $p_0 = [S_0, w_0]$ is on the boundary $\partial P^{\varepsilon}_{g,n}$ of $P^{\varepsilon}_{g,n}$, then we have

$$L(p_0) = \varepsilon.$$

There exist a finite number of simple closed curves

$$C_1, \cdots, C_k$$

on $\Sigma_{g,n}$ such that $l_{p_0}(C_i) = \varepsilon,\ i = 1, \cdots, k$. (Recall this means that the geodesics $\hat{C}_i$ have hyperbolic length ε on S_0, where $\hat{C}_i$ is the simple closed geodesic homotopic to $w_0(C_i),\ i = 1, \cdots, k$.) The geodesics $\hat{C}_1, \cdots, \hat{C}_k$ are disjoint, because $\varepsilon < M$, and we may assume that $C_1, \cdots, C_k$ are disjoint on $\Sigma_{g,n}$. We have

$$k \leqq 3g - 3 + n,$$

because $3g-3+n$ is the maximum number of the simple closed curves on $\Sigma_{g,n}$ which are essential, disjoint, and mutually non-isotopic.

Let σ be the set of these simple closed curves on $\Sigma_{g,n}$:

$$\sigma = \{C_1, \cdots, C_k\}.$$

Define the facet $F^{\varepsilon}(\sigma)$ corresponding to σ by

$$F^{\varepsilon}(\sigma) = \{p \in P^{\varepsilon}_{g,n} \mid l_p(C_i) = \varepsilon,\ i = 1, \cdots, k\}.$$

For all points $p = [S, w]$ on $F^{\varepsilon}(\sigma)$, we assume that other simple cosed geodesics on S have length greater than ε. (The point p_0 is on this facet.)

In general, for any set σ of essential, disjoint, and mutually non-isotopic simple closed curves on $\Sigma_{g,n}$, the corresponding facet $F^{\varepsilon}(\sigma)$ is a real analytic manifold homeomorphic to

$$\mathbb{R}^{2(3g-3+n)-k},$$

where $k = \#\sigma$. Facets are analogous to open faces of a finite polyhedron.

Here is an incidence relation: If $\sigma \subset \sigma'$, then we have

$$\overline{F^{\varepsilon}(\sigma)} \supset F^{\varepsilon}(\sigma').$$

If $\#\sigma < 3g - 3 + n$, the facet $F^{\varepsilon}(\sigma)$ is surrounded by an infinite number of facets. Thus in this case, a facet is itself an infinite polyhedron.

3.2. **Abelian subgroups** $\Gamma(\sigma)$. Let σ denote $\{C_1, \cdots, C_k\}$ as before. Let $\tau(C_i)$ be the right handed (i.e. negative) Dehn twist aboutC_i, and define $\Gamma(\sigma)$ to be the subgroup of $\Gamma_{g,n}$ generated by

$$\tau(C_i), \quad i = 1, \cdots, k.$$

The group $\Gamma(\sigma)$ is a free abelian group of rank k. Since the action of $\Gamma_{g,n}$ on $T_{g,n}$ preserves the Poincaré metric on Riemann surfaces (hence preserves the length function L), and

$$\tau(C_i)(C_j) = C_j, \quad i, j = 1, \cdots, k,$$

the twists $\tau(C_i)$ preserve $F^\varepsilon(\sigma)$. This action of $\Gamma(\sigma)$ on $F^\varepsilon(\sigma)$ is real analytic and properly discontinuous.

4. Complex of curves and $P^\varepsilon_{g,n}$

W. J. Harvey (1977) [7] introduced an abstract simplicial complex called the *complex of curves* $\mathcal{C}_{g,n} = \mathcal{C}(\Sigma_{g,n})$:

Definition 4.1. A *vertex* of $\mathcal{C}_{g,n}$ is an isotopy class of an essential simple closed curve on $\Sigma_{g,n}$, and a *simplex* σ of $\mathcal{C}_{g,n}$ is a set of vertices represented by a disjoint union of essential simple closed curves which are mutually non-isotopic.

Facets $F^\varepsilon(\sigma)$ are in one-to-one correspondence with the simplices σ of $\mathcal{C}_{g,n}$.

Proposition 4.2. *The totality of the facets* $\{F^\varepsilon(\sigma)\}_{\sigma\in\mathcal{C}_{g,n}}$ *makes a complex (*facet complex*) analogous to a simplicial complex. The flag complex associated with the facet complex is isomorphic to the* barycentric subdivision *of the complex of curves* $\mathcal{C}_{g,n}$.

Proof. A flag in the facet complex $\overline{F^\varepsilon(\sigma)} \supset \overline{F^\varepsilon(\sigma')} \supset F^\varepsilon(\sigma'')$ corresponds to a flag in the complex of curves $\mathcal{C}_{g,n}$, $\sigma \subset \sigma' \subset \sigma''$. The latter corresponds to a simplex of the barycentric subdivision of $\mathcal{C}_{g,n}$. □

4.1. **Automorphisms of $\mathcal{C}_{g,n}$.** We need the following theorem:

Theorem 4.3. *(Ivanov* [10], *Korkmaz* [13], *Luo* [15]*) Except for a few sporadic cases (spheres with* $\leqq 4$ *punctures, tori with* $\leqq 2$ *punctures and a closed surface of genus* 2*), the following holds:*

$$Aut(\mathcal{C}_{g,n}) = \Gamma^*_{g,n},$$

where $\Gamma^*_{g,n}$ *stands for the* extended mapping class group *(containing orientation reversing homeomorphisms)* .

The scissored Teichmüller space $P^\varepsilon_{g,n}$ together with the Teichmüller metric becomes a metric (infinite) polyhedron. The following proposition is a corollary to the above theorem:

Proposition 4.4. *With the same exceptions as above, we have*

$$Isom_+(P^\varepsilon_{g,n}) = \Gamma_{g,n}.$$

Proof. An isomorphism of $P^\varepsilon_{g,n}$ induces on $\partial P^\varepsilon_{g,n}$ an automorphism of the facet complex, thus that of the barycentric subdivision of $\mathcal{C}_{g,n}$, and finally an automorphism of $\mathcal{C}_{g,n}$. The automorhism of $\mathcal{C}_{g,n}$ in turn corresponds (by Ivanov-Korkmaz-Luo's theorem) to an action of the mapping class group $\Gamma_{g,n}$, hence an (orientation preserving) isometry of $T_{g,n}$. □

Essentially the same arguments have been done in Papadopoulos [21] and Ohshika [20].

Proposition 4.5. *The subgroup of* $\Gamma_{g,n}$ *which preserves a facet* $F^{\varepsilon}_{g,n}$ *is precisely* $N\Gamma(\sigma)$, *the normalizer of* $\Gamma(\sigma)$ *in* $\Gamma_{g,n}$.

Proof. If a mapping class $[f] \in \Gamma_{g,n}$ peserves $F^{\varepsilon}_{g,n}$, then $[f]$ induces on $\Sigma_{g,n}$ a permutation of $\sigma = \{C_1, \cdots, C_k\}$, and *vice versa.* Such mapping classes form the normalizer $N\Gamma(\sigma)$ of $\Gamma(\sigma)$. □

4.2. **"Fringe"** $FR^{\varepsilon}(\sigma)$ **bounded by** $F^{\varepsilon}(\sigma)$. The *fringe* $FR^{\varepsilon}(\sigma)$ is defined by

$$FR^{\varepsilon}(\sigma) = \bigcup_{0<\delta<\varepsilon} F^{\delta}(\sigma).$$

Then we have

Corollary 4.6. *The subgroup of* $\Gamma_{g,n}$ *which preserves the fringe* $FR^{\varepsilon}(\sigma)$ *is the normalizer* $N\Gamma(\sigma)$. *The action of* $N\Gamma(\sigma)$ *on* $FR^{\varepsilon}(\sigma)$ *is properly discontinuous.*

Proof. $FR^{\varepsilon}(\sigma)$ is foliated by the facets $F^{\delta}(\sigma)$, and the corollary holds for each leaf $F^{\delta}(\sigma)$. □

Define the *augmented fringe* as follows:

$$\overline{FR^{\varepsilon}(\sigma)} = \bigcup_{0\leq\delta<\varepsilon} F^{\delta}(\sigma) \; (= FR^{\varepsilon}(\sigma) \sqcup F^{0}(\sigma)).$$

$N\Gamma(\sigma)$ acts on $\overline{FR^{\varepsilon}(\sigma)}$ continuouly, but *not* properly discontinuously, because the infinite subgroup $\Gamma(\sigma)$ ($\subset N\Gamma(\sigma)$) fixes the points of the added ideal boundary $F^{0}(\sigma)$. Abikoff[1] attached to $T_{g,n}$ all ideal boundaries, and considered the *augmented Teichmüller space*

$$\overline{T}_{g,n} = T_{g,n} \sqcup \bigcup_{\sigma\in\mathcal{C}_{g,n}} F^{0}(\sigma).$$

Yamada [24] identified $\overline{T}_{g,n}$ with the Weil-Petersson completion of $T_{g,n}$, and proved the geodesic convexity of the ideal boundaries $F^{0}(\sigma)$. It is well-known that the quotient space of $\overline{T}_{g,n}$ under the action of $\Gamma_{g,n}$ is the compactified moduli space $\overline{M_{g,n}}$. Note that the union of the augmented fringes $\bigcup_{\sigma\in\mathcal{C}_{g,n}} \overline{FR^{\varepsilon}(\sigma)}$ gives an open neighborhood of the singular divisors when divided out by the action of $\Gamma_{g,n}$.

5. Controlled deformation spaces

To analyse the orbifold structure of $\overline{M_{g,n}}$, the fringes $\overline{FR^{\varepsilon}(\sigma)}$ are not necessarily adequate, because they are *pairwise disjoint*:

$$\overline{FR^{\varepsilon}(\sigma)} \cap \overline{FR^{\varepsilon}(\sigma')} = \emptyset, \quad \text{if} \quad \sigma \neq \sigma'.$$

(Recall that the facets are like open faces of a polyhedron.) Namely the fringes do not make an open covering of the singular divisors $\bigcup_{\sigma\in\mathcal{C}} F^{0}(\sigma)$.

To remedy the deficiency, we introduce *controlled deformation spaces.* But before going to them, let us recall *Bers' deformation spaces.*

Let $\sigma \in \mathcal{C}_{g,n}$ be any simplex $\sigma = \{C_1, \cdots, C_k\} \in \mathcal{C}_{g,n}$. Let $\Sigma_{g,n}(\sigma)$ denote the surface with nodes obtained by pinching each C_i ($\in \sigma$) in $\Sigma_{g,n}$ to a point. Bers [5] introduced the *deformation space* $D(\sigma)$ associated with $\Sigma_{g,n}(\sigma)$. The following fact is known:

Proposition 5.1. *(See Kra* [14] *§9, Matsumoto* [18] *§6.)* $D(\sigma)$ *is homeomorphic to* $(T_{g,n}/\Gamma(\sigma)) \cup \bigcup_{i=1}^{k} \Pi_i$, *where* $\Pi_i = \mathbb{C}^{i-1} \times \{0\} \times \mathbb{C}^{3g-3+n-i}$, *and* $\bigcap_{i=1}^{k} \Pi_i$ *corresponds to* F^{0}.

(Π_i is mentioned as a "distinguished subset"in Bers [5].) Bers announced in 1970's that $D(\sigma)$ is a bounded domain (see [5]), but without proof. Recently, Hubbard and Koch [8] gave a proof.

Theorem 5.2. *The deformation space $D(\sigma)$ is a complex manifold of* $\dim_{\mathbb{C}} = 3g-3+n$.

Their arguments are a little bit complicated, but the geometry is conceptually clear. The space $F^0(\sigma)$ is the Teichmüller space of the nodal surface $\Sigma_{g,n}(\sigma)$ and serves as the "core"of $D(\sigma)$ (Masur [17]). It is thickened in the transverse direction by the "plumbing coordinates"(Marden [16], Earle and Marden [6]).

5.1. **The groups $W(\sigma)$.** Define

$$W(\sigma) = N\Gamma(\sigma)/\Gamma(\sigma).$$

The groups $W(\sigma)$ are not finite groups in general.

Proposition 5.3. *(i) $W(\sigma)$ is the mapping class group of the nodal surface $\Sigma_{g,n}(\sigma)$.*
(ii) $W(\sigma)$ acts on $D(\sigma)$ holomorphically and properly discontinuously.

5.2. **Controlled deformation spaces.** Let M be a constant of Keen and Abikoff. We take an ε satisfying $0 < \varepsilon < M$. We insert $6g-6+2n$ numbers between ε and M:

$$\varepsilon < \varepsilon_1 < \eta_1 < \cdots < \varepsilon_{3g-3+n} < \eta_{3g-3+n} < M.$$

Let $\hat{\varepsilon}$ denote this sequence. We define the *controlled deformation space* $D_{\hat{\varepsilon}}(\sigma)$ as follows (σ being $\{C_1, \cdots, C_k\}$):

$$D_{\hat{\varepsilon}}(\sigma) = \{p = [S, w] \in D(\sigma) \mid l_p(C_i) < \varepsilon_k, \quad i = 1, \ldots, k,$$
$$\text{and other simple closed geodesics on } S \text{ are longer than } \eta_k\}$$

Why do we need the controlled deformation spaces $D_{\hat{\varepsilon}}(\sigma)$? Because Bers' deformation spaces $D(\sigma)$ do not naturally descend to $\overline{M_{g,n}}$, but $D_{\hat{\varepsilon}}(\sigma)$ do. For a proof of this fact, see [18], §7

Proposition 5.4. *(i) $D_{\hat{\varepsilon}}(\sigma)$ is a bounded domain of $\mathbb{C}^{3g-3+n}$.*
(ii) The group $W(\sigma)$ acts on $D_{\hat{\varepsilon}}(\sigma)$ holomorphically and properly discontinuously.
(iii) $D_{\hat{\varepsilon}}(\sigma)/W(\sigma)$ is an open subset of $\overline{M_{g,n}}$.
(iv) $D_{\hat{\varepsilon}}(\sigma)/W(\sigma)$ contains the "main part"of the quotient of the augmented fringe $\overline{FR^{\varepsilon}}(\sigma)/W(\sigma)$.
(v) The family $\{D_{\hat{\varepsilon}}(\sigma)/W(\sigma)\}_{\sigma\in\mathcal{C}_{g,n}}$ is an open covering of the "boundary"singular divisors $\bigcup_{\sigma\in\mathcal{C}_{},\backslash} F^0(\sigma)/\Gamma_{g,n}$.

Summarizing the above, we have our main theorem:

Theorem 5.5. *(Matsumoto [18]) The family $\{(D_{\hat{\varepsilon}}(\sigma), W(\sigma))\}_{\sigma\in\mathcal{C}_{g,n}}$ gives orbifold-charts containing the boundary singular divisors in $\overline{M_{g,n}}$.*

Remark 5.6. If $\sigma' = f(\sigma)$ by a mapping class $[f] \in \Gamma_{g,n}$, we consider that $(D_{\hat{\varepsilon}}(\sigma), W(\sigma))$ and $(D_{\hat{\varepsilon}}(\sigma'), W(\sigma'))$ are the identical charts. Thus the index set of the family of charts is actually $\mathcal{C}_{g,n}/\Gamma_{g,n}$.

6. Crystallographic groups

Definition 6.1. A *crystallographic group* in Euclidean m-space $\mathbb{E}^m$ is a group G of isometries of $\mathbb{E}^m$ whose translation vectors form a lattice $L \subset \mathbb{E}^m$.

The image of G under linearization $Isom(\mathbb{E}^m) \to O(\mathbb{E}^m)$ is called the *point group* of G and denoted by $\vec{G}$. This is a finite group. There is a canonical exact sequence

$$1 \to T \to G \to \vec{G} \to 1,$$

where T is the translation subgroup of G. See [11].

6.1. **Crystallographic groups in Teichmüller theory.** For simplicity, we consider a closed surface Σ_g (i.e. $n = 0$), and in what follows, we assume that σ is a maximal simplex of $\mathcal{C}_g$, i.e., $\sigma = \{C_i\}_{i=1,\dots,3g-3}$. Then the group $W(\sigma)$ is finite. In this case, the facet $F^\varepsilon(\sigma)$ is defined by

$$l_i = \varepsilon, \quad i = 1, \dots, 3g-3$$

by the Fenchel-Nielsen coordinates associated with σ,

$$(l_i, \tau_i), \quad i = 1, \dots, 3g-3.$$

By Wolpert's formula, the Weil-Petersson symplectic form is written as follows:

$$\omega_{WP} = \frac{1}{2}\sum_i dl_i \wedge d\tau_i.$$

We see $\omega_{WP}|F^\varepsilon(\sigma) = 0$, thus $F^\varepsilon(\sigma)$ is a *Lagrangian submanifold* of $\dim_{\mathbb{R}} = 3g-3$.

$F^\varepsilon(\sigma)$ is homeomorphic to $\mathbb{R}^{3g-3}$ on which $\Gamma(\sigma)$ acts as translations. The action of $N\Gamma(\sigma)$ on $F^\varepsilon(\sigma)$ preserves the Weil-Petersson metric $\langle\cdot,\cdot\rangle$. From Wolpert's lecture note [23], we have

$$\langle\lambda_i, \lambda_j\rangle = \frac{1}{2\pi}\delta_{ij} + O(l_i^{3/2} l_j^{3/2}), \quad \text{for} \quad \lambda_i = \mathrm{grad}\sqrt{l_i}.$$

On $F^\varepsilon(\sigma)$, we have

$$\langle\lambda_i, \lambda_j\rangle = \frac{1}{2\pi}\delta_{ij} + O(\varepsilon^3),$$

because $l_i = l_j = \varepsilon$ on $F^\varepsilon(\sigma)$. $F^\varepsilon(\sigma)$ has twist coordinates $\tau_1, \dots, \tau_{3g-3}$. Wolpert's *twist-length duality* [23] asserts that

$$2t_i = J\mathrm{grad}\ l_i,$$

where $2t_i$ is the Hamiltonian vector field (along τ_i) corresponding to dl_i.

Thus

$$t_i = \frac{1}{2}J\mathrm{grad}\ l_i = \sqrt{\varepsilon}J\mathrm{grad}\sqrt{l_i} = \sqrt{\varepsilon}J\lambda_i,$$

and

$$\langle\frac{t_i}{\sqrt{\varepsilon}}, \frac{t_j}{\sqrt{\varepsilon}}\rangle = \langle J\lambda_i J\lambda_j\rangle = \frac{1}{2\pi}\delta_{ij} + O(\varepsilon^3).$$

Therefore, the facet $F^\varepsilon(\sigma)$ together with the (normalized) Weil-Petersson metric

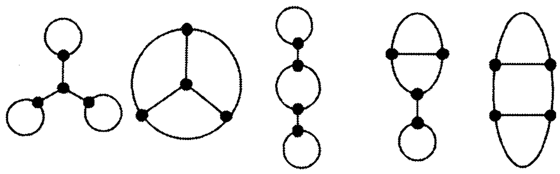
$$\frac{2\pi}{\varepsilon}\langle t_i, t_j\rangle = \delta_{ij} + O(\varepsilon^3)$$

converges to Euclidean space $\mathbb{E}^{3g-3}$ as $\varepsilon \to 0$, on which $N\Gamma(\sigma)$ acts as a crystallographic group.

In our case where σ is maximal, $W(\sigma)$ is a finite group. This group is nothing but the automorphism group of a finite trivalent graph (the pants graph, i.e., the dual graph of the pants decomposition associated with σ). Conversely, given any finite trivalent graph, a crystallographic group appears exactly in the same manner as above.

The group $W(\sigma)$ is somewhat similar to the "Weyl group", and a pants graph has an atomosphere of a "*root system*". Details of this report will appear in [19].

Here are the trivalent graphs for $g = 3$ (with 4 vertices and 6 edges) and the corresponding point groups $\overrightarrow{N\Gamma(\sigma)}$ (n.b. not their groups $W(\sigma)$):



D_6 $\quad$ S_4 $\quad$ $(\mathbb{Z}_2)^2$ $\quad$ $(\mathbb{Z}_2)^2$ $\quad$ $(\mathbb{Z}_2)^4$

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References

[1] W. Abikoff, *Degenerating families of Riemann surfaces*, Ann. of Math., **105** (1977), 29–44.
[2] W. Abikoff, *The Real Analytic Theory of Teichmüller Spaces*, L.N.M. **820**, Springer-Verlag, (1980).
[3] L. V. Ahlfors, *The complex analytic structure of the space of closed Riemann surfaces* in Analytic Functions, Princeton Conference, (R. Nevanlinna et al. eds) Princeton U. P. Princeton, New Jersey (1966), 45–66.
[4] L. Bers, *Correction to "Spaces of Riemann surfaces as bounded domains"*, Bull. Amer. Math. Soc., **67** (1961), 465–466.
[5] L. Bers, *On spaces of Riemann surfaces with nodes*, Bull. Amer. Math. Soc., **80** (1974), 1219–1222.
[6] C. J. Earle and A. Marden, *Holomorphic plumbing coordinates*, Contemporary Math., **575** (2012), 41–52.
[7] W. J. Harvey, *Geometric structure of surface mapping class groups*, in Homological Group Theory, Proc. of a symposium held at Durham in Sept. 1977, (ed. by C. T. C. Wall), London Math. Soc. Lecture Note Series, **36** (1977), 255–269,
[8] J. H. Hubbard and S. Koch, *An analytic construction of the Deligne-Mumford compactification of the moduli space of curves*, J. Diff. Geom., **98** (2014), 261–313.
[9] N. V. Ivanov, *Complexes of curves and the Teichmüller modular groups*, Russian Math. Surveys **42:3** (1987), 55–107.
[10] N. V. Ivanov, *Automorphisms of complexes of curves and of Teichmüller spaces*, Int. Math. Research Notices, **14** (1997), 651–666.
[11] B. Iversen, *Lectures on crystallographic groups*, Lecture Notes Series 1990/91 No. 60, Matematisk Institut, Aarhus Universitet.
[12] L. Keen, *Collars on Riemann surfaces*, in Discontinuous Gropus and Riemann Surfaces, Proc. of the 1973 Conf. at Univ. of Maryland (ed. by L. Greenberg), Ann. Math. Studies **79** (1974), 263–268.
[13] M. Korkmaz, *Automorphisms of complexes of curves on punctured spheres and on punctured tori*, Topology and Its Applications, **95** (1999), 85–111.
[14] I. Kra, *Horocyclic coordinates for Riemann surfaces and moduli spaces. I: Teichmüller and Riemann spaces of Kleinian groups*, J. Amer. Math. Soc., **3** (1990), 499–578.

[15] F. Luo, *Automorphisms of the complex of curves*, Topology, **39** (2000), 283–298.

[16] A. Marden *Geometric Complex Coordinates for Teichmüller Space*, in Mathematical Aspects of String Theory (ed.by S. T. Yau), World Scientific, (1987), 341–357.

[17] H. Masur, *The extension of the Weil-Petersson metric to the boundary of Teichmüller space*, Duke Math. Jour., **43** (1976), 623–635.

[18] Y. Matsumoto, *On the universal degenerating family of Riemann surfaces*, IRMA Lectures in Math. and Theoretical Physics, **20** (2012), 71–102.

[19] Y. Matsumoto, *Compactification of moduli spaces and crystallographic groups*, in preparation.

[20] K. Ohshika, *Reduced Bers boundaries of Teichmüller spaces*, Ann. de L'Institut Fourier, **64** (2014), 145–176.

[21] A. Papadopoulos, *A regidity theorem for the mapping class group action on the space of unmeasured foliations on a surface*, Proc. Amer. Math. Soc., **136** (2008), 4453–4460.

[22] A. Weil, *Modules des surfaces de Riemann*, Séminaire Bourbaki, 10ᵉ année, Textes des Conférences, (1958), Exp. No. 168, reprint Séminaire Bourbaki **4** (1995), Exp. 168, 413–419.

[23] S. Wolpert, *Families of Riemann Surfaces and Weil-Petersson Geometry*, Regional Conference Series in Mathematics, No.**113**, Amer. Math. Soc. (2010).

[24] S. Yamada, *On the geometry of Weil-Petersson Completion of Teichmüller spaces*, Math. Research Letters, **11** (2004), 327–344.

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