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Symmetry-breaking bifurcation of positive solutions to a one-dimensional Liouville type equation

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We consider the two-point boundary value problem for the one-dimensional Liouville type equation

$$(1)\qquad \begin{cases} u'' + \lambda|x|^l e^u = 0, & x \in (-1,1), \\ u(-1) = u(1) = 0, \end{cases}$$

where $\lambda > 0$ and $l > 0$.

Jacobsen and Schmitt [2] studied the exact multiplicity of radial solutions of the problem for the multi-dimensional Liouville type equation

$$(2)\qquad \begin{cases} \Delta u + \lambda|x|^l e^u = 0 & \text{in } B, \\ \qquad u = 0 & \text{on } \partial B, \end{cases}$$

where $l \geq 0$ and $B := \{x \in \mathbf{R}^n : |x| < 1\}$. They proved the following (i)–(iii):

(i) if $1 \leq N \leq 2$, then there exists $\lambda_* > 0$ such that (2) has exactly two radial solutions for $0 < \lambda < \lambda_*$, a unique radial solution for $\lambda = \lambda_*$ and no radial solution for $\lambda > \lambda_*$;

(ii) if $3 \leq N < 10 + 4l$, then (2) has infinitely many radial solutions for $\lambda = (l+2)(N-2)$ and a finite but large number of radial solutions when $|\lambda - (l+2)(N-2)|$ is sufficiently small;

(iii) if $N \geq 10 + 4l$, then (2) has a unique radial solution for $0 < \lambda < (l+2)(N-2)$ and no radial solution for $\lambda \geq (l+2)(N-2)$.

We note here that every solution of (2) is positive in B, by the strong maximum principle. Result (i)–(iii) were established by Joseph and Lundgren [3] for the case $l = 0$, that is, for the Liouville equation

$$(3)\qquad \begin{cases} \Delta u + \lambda e^u = 0 & \text{in } \Omega, \\ \qquad u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\Omega = B$. Gidas, Ni and Nirenberg's theorem ([1]) shows that every positive solution of (3) is radially symmetric when $\Omega = B$. However, when Ω is an annulus $A := \{x \in \mathbf{R}^N : a < |x| < b\}$, $a > 0$, problem (3) may has non-radial solutions. Indeed, Lin [4] proved that (3) has infinitely many symmetry-breaking bifurcation points when $N = 2$ and $\Omega = A$. Nagasaki and Suzuki [6] found that large non-radial solutions of (3) when $N = 2$ and $\Omega = A$. More precisely, for each sufficiently large $\mu > 0$, there exist (λ, u) such that $\lambda > 0$, u is a non-radial solution of (3) and $\int_A e^u dx = \mu$ when $N = 2$ and $\Omega = A$.

Recently, Miyamoto [5] considered the problem for the Liouville type equation (2) and proved the following result.

Theorem A ([5]). *Let n_0 be the largest integer that is smaller than $1+\frac{l}{2}$ and let $\alpha_n := 2\log\frac{2l+4}{l+2-2n}$. All the radial solutions of (2) with $N=2$ can be written explicitly as*

$$\lambda(\alpha)=2(l+2)^2(e^{-\alpha/2}-e^{-\alpha}),\quad U(r;\alpha)=\log\frac{e^\alpha}{(1+(e^{\alpha/2}-1)r^{l+2})^2}.$$

The radial solutions can be parameterized by the L^∞-norm, it has one turning point at $\lambda=\lambda(\alpha_0)=(l+2)/2$, and it blows up as $\lambda\downarrow 0$. For each $n\in\{1,2,\cdots,n_0\}$, $(\lambda(\alpha_n),U(r;\alpha_n))$ is a symmetry breaking bifurcation point from which an unbounded branch consisting of non-radial solutions of (2) with $N=2$ emanates, and $U(r;\alpha)$ is nondegenerate if $\alpha\neq\alpha_n$, $n=0,1,\cdots,n_0$. Each non-radial branch is in $(0,\lambda(\alpha_0)))\times\{u>0\}\subset\mathbf{R}\times H_0^2(B)$.

When $N=2$, radial solutions of problems (2) and (3) can be written explicitly, and hence, Lin [4] and Miyamoto [5] succeeded to show the existence of bifurcation points. That is difficult even if we know exact solutions, much more difficult if we do not know them usually. When $N\neq 2$, we do not find exact radial solutions of (2). However, the structure of eigenvalues and eigenfunctions of the linearized problem in the dimension 1 is well-known, and then, by the comparison function introduced in [7], we can find the Morse indices of even solutions of (1). Then we obtain the existence of a symmetry-breaking bifurcation point of (1).

Let $m(U)$ be the Morse index of a solution U to (1), that is, the number of negative eigenvalues μ of

$$\begin{cases}\phi''+\lambda|x|^l e^{U(x)}\phi+\mu\phi=0, & x\in(-1,1),\\ \phi(-1)=\phi(1)=0,\end{cases}\tag{4}$$

A solution U of (1) is said to be degenerate if $\mu=0$ is an eigenvalue of (4). Otherwise, it is said to be nondegenerate.

The main result is as follows.

Theorem 1. *For each $\alpha>0$, there exists a unique $(\lambda(\alpha),U(x;\alpha))$ such that (1) with $\lambda=\lambda(\alpha)$ has a unique positive even solution $U=U(x;\alpha)$ such that $\|U\|_\infty=\alpha$. Moreover, there exist α_*, α_1, α_2 and α_3 such that $\alpha_*<\alpha_1\le\alpha_2\le\alpha_3$ and the following (i)–(vii) hold:*

(i) *if $0<\alpha<\alpha_*$, then $m(U)=0$ and $U(x;\alpha)$ is nondegenerate;*
(ii) *if $\alpha=\alpha_*$, then $m(U)=0$ and $U(x;\alpha)$ is degenerate;*
(iii) *if $\alpha_*<\alpha<\alpha_1$, then $m(U)=1$ and $U(x;\alpha)$ is nondegenerate;*
(iv) *if $\alpha=\alpha_1$, then $m(U)=1$ and $U(x;\alpha)$ is degenerate;*
(v) *if $\alpha=\alpha_2$, then $m(U)=1$, $U(x;\alpha)$ is degenerate and (U,λ) is a non-even bifurcation point, that is, for each $\varepsilon>0$, there exists (λ,u) such that u is a non-even positive solution of (1) and $|\lambda-\lambda(\alpha_2)|+\|u-U(\,\cdot\,,\alpha_2)\|_\infty<\varepsilon$;*
(vi) *if $\alpha=\alpha_3$, then $m(U)=2$ and $U(x;\alpha)$ is degenerate;*
(vii) *if $\alpha>\alpha_3$, then $m(U)=2$ and $U(x;\alpha)$ is nondegenerate.*

Here and Hereafter, we use the notation $\|U\|_\infty = \sup_{x\in[-1,1]} U(x)$.

For the proof of Theorem 1, see [8]. Here, we give a sufficient condition for the second eigenvalue of the linearized problem to be negative for the following problem

$$\text{(5)} \qquad \begin{cases} u'' + \lambda h(x) f(u) = 0, & x \in (-1,1), \\ u(-1) = u(1) = 0. \end{cases}$$

where $\lambda > 0$ and $h \in C^1([-1,0) \cup (0,1]) \cap C[-1,1]$, $h(-x) = h(x)$, $h(x) > 0$ and $h'(x) \geq 0$ for $x > 0$, $f \in C^1[0,\infty)$, $f(s) > 0$ and $f'(s) \geq 0$ for $s > 0$. Namely we will show the following result, which plays a crucial role in the proof of Theorem 1.

Proposition 1. *Assume that, for each sufficiently large $\alpha > 0$, there exist $\lambda(\alpha) > 0$ and $U(x;\alpha)$ such that $U(x;\alpha)$ is a positive even solution of* (5) *at $\lambda = \lambda(\alpha)$ and $\|U(\,\cdot\,;\alpha)\|_\infty = \alpha$. Assume moreover that there exist $s_0 > 0$ and $\delta > 0$ such that*

$$\text{(6)} \qquad \frac{l(x)(g(s)-1) - 4}{g(s) + l(x) + 3} \geq \delta, \quad x \in (0,1], \ s \geq s_0,$$

where $l(x) = xh'(x)/h(x)$ and $g(s) = sf'(s)/f(s)$. Let $\mu_2(\alpha)$ be the second eigenvalue of

$$\text{(7)} \qquad \begin{cases} \phi'' + \lambda(\alpha) h(x) f'(U(x;\alpha))\phi + \mu\phi = 0, & x \in (-1,1), \\ \phi(-1) = \phi(1) = 0. \end{cases}$$

Then $\mu_2(\alpha) < 0$ for all sufficiently large $\alpha > 0$.

In the case where $h(x) = |x|^l$, $l > 0$ and $f(s) = e^s$, it follows that $l(x) = xh'(x)/h(x) = l$ for $x \in (0,1]$ and $g(s) = sf'(s)/f(s) = s$, and hence (6) is satisfied.

We conclude that if U is a positive even solution of (1) and $\|U\|_\infty \leq 1$, then $m(U) = 0$. Indeed, let μ_1 be the first eigenvalue of (4) and let ϕ_1 be an eigenfunction corresponding to μ_1. We may assume that $\phi_1(x) > 0$ on $(-1,1)$. Integrating the equality

$$(\phi_1(x)U'(x) - \phi_1'(x)U(x))' = \mu_1\phi_1(x)U(x) + \lambda|x|^l e^{U(x)}\phi_1(x)(U(x)-1)$$

on $[-1,1]$, we have

$$\mu_1 \int_{-1}^{1} \phi_1(x)U(x)dx = \lambda \int_{-1}^{1} |x|^l e^{U(x)}\phi_1(x)(1-U(x))dx > 0.$$

Consequently, we have $\mu_1 > 0$, which means $m(U) = 0$. By applying Proposition 1, we can conclude that $m(U(\,\cdot\,;\alpha)) = 0$ for $0 < \alpha \leq 1$ and $m(U(\,\cdot\,;\alpha)) \geq 2$ for all sufficiently large $\alpha > 1$. Then, using the Leray-Schauder degree, we can find a bifurcation point.

To prove Proposition 1, we need the following two lemmas.

Lemma 1. *Let ϕ_2 be an eigenfunction corresponding to the second eigenvalue $\mu_2(\alpha)$ of* (7)*. Then ϕ_2 is odd, $\phi_2(0) = \phi_2(1) = 0$ and $\phi_2(x) \neq 0$ for $x \in (0,1)$.*

Proof. Let M_1 be the first eigenvalue of

$$\begin{cases} \Phi'' + \lambda(\alpha) h(x) f'(U(x;\alpha))\Phi + M\Phi = 0, & x \in (0,1), \\ \Phi(0) = \Phi(1) = 0, \end{cases}$$

and let Φ_1 be an eigenfunction corresponding to M_1. Then $\Phi_1(0) = \Phi_1(1) = 0$ and $\Phi_1(x) \neq 0$ on $(0,1)$. Set

$$\Phi(x) = \begin{cases} \Phi_1(x), & x \in [0,1], \\ -\Phi_1(-x), & x \in [-1,0). \end{cases}$$

Noting the fact that $\lim_{x\to -0} \Phi''(x) = \lim_{x\to -0}(-\Phi_1''(-x)) = -\Phi_1''(0) = 0$, we easily check that Φ is a solution of

$$\begin{cases} \Phi'' + \lambda(\alpha)h(x)f'(U(x;\alpha))\Phi + M_1\Phi = 0, & x \in (-1,1), \\ \Phi(-1) = \Phi(1) = 0, \end{cases}$$

and Φ is odd, $\Phi(x) \neq 0$ on $(0,1)$ and $\Phi(0) = 0$. Therefore, M_1 is an eigenvalue of (7) and Φ is an eigenfunction corresponding to M_1. Since Φ has exactly one zero in $(-1,1)$, M_1 must be μ_2 and hence $\phi_2(x)$ must be $c\Phi(x)$ for some $c \neq 0$. □

Lemma 2. *Assume that $w \in C[a,b]$ is positive and concave on (a,b). Let $\rho \in (0,1/2)$. Then $w(x) \geq \rho \max_{\xi\in[a,b]} w(\xi)$ for $x \in [(1-\rho)a + \rho b, \rho a + (1-\rho)b]$.*

Proof. We take $c \in [a,b]$ for which $w(c) = \max_{\xi\in[a,b]} w(\xi)$. Then $w(c) > 0$. Since w is positive and concave on (a,b), we have

$$w(x) \geq \frac{w(c)(x-a)}{c-a} \geq \frac{w(c)(x-a)}{b-a} =: l_1(x), \quad x \in [a,c],$$

and

$$w(x) \geq \frac{w(c)(b-x)}{b-c} \geq \frac{w(c)(b-x)}{b-a} =: l_2(x), \quad x \in [c,b].$$

Hence $w(x) \geq \min\{l_1(x), l_2(x)\}$ on $[a,b]$. We conclude that if $x \in [(1-\rho)a+\rho b, (a+b)/2]$, then

$$\min\{l_1(x), l_2(x)\} = l_1(x) \geq l_1((1-\rho)a + \rho b) = \rho w(c),$$

and if $x \in [(a+b)/2, \rho a + (1-\rho)b]$, then

$$\min\{l_1(x), l_2(x)\} = l_2(x) \geq l_2(\rho a + (1-\rho)b) = \rho w(c).$$

The proof is complete. □

Now we are ready to show Proposition 1.

Proof of Proposition 1. Let $\alpha > 0$ be sufficiently large. We use the following comparison function $y(x)$ introduced in [7]:

$$y(x) = xU(x;\alpha) - (x-1)^2U'(x;\alpha).$$

This function $y(x)$ satisfies $y(0) = y(1) = 0$, $y(x) > 0$ on $(0,1)$, and

$$y'' + \lambda(\alpha)h(x)f'(U(x;\alpha))y = \lambda(\alpha)x^{-1}h(x)H(x;\alpha)f(U(x;\alpha))$$

for $x \in (0,1]$, where

$$H(x;\alpha) = (1-x)^2l(x) + x(3x-4) + x^2g(U(x;\alpha)).$$

Let $\phi_2(x;\alpha)$ be an eigenfunction corresponding to $\mu_2(\alpha)$. From Lemma 1, it follows that $\phi_2(0;\alpha) = \phi_2(1;\alpha) = 0$ and $\phi_2(x;\alpha) \neq 0$ for $x \in (0,1)$. Without loss of generality, we may assume that $\phi_2(x;\alpha) > 0$ for $x \in (0,1)$ and $\max_{\xi\in[0,1]} \phi_2(\xi;\alpha) = 1$. We observe that

$$(y'\phi_2 - y\phi_2')' = \mu_2(\alpha)\phi_2 y + \lambda(\alpha)x^{-1}h(x)H(x;\alpha)f(U(x;\alpha))\phi_2, \quad x \in (0,1].$$

Integrating this equality on $(0,1)$, we obtain

$$(8) \qquad \mu_2(\alpha)\int_0^1 \phi_2(x;\alpha)y(x)dx + \lambda(\alpha)\int_0^1 x^{-1}h(x)H(x;\alpha)f(U(x;\alpha))\phi_2(x;\alpha)dx = 0.$$

Since

$$\begin{aligned} H(x) &= [g(U(x;\alpha)) + l(x) + 3]\left(x - \frac{l(x)+2}{g(U(x;\alpha)) + l(x) + 3}\right)^2 \\ &\quad + \frac{l(x)[g(U(x;\alpha)) - 1] - 4}{g(U(x;\alpha)) + l(x) + 3} \\ &\geq \frac{l(x)[g(U(x;\alpha)) - 1] - 4}{g(U(x;\alpha)) + l(x) + 3}, \end{aligned}$$

we have

$$\begin{aligned} (9) \qquad &\int_0^1 x^{-1}h(x)H(x;\alpha)f(U(x;\alpha))\phi_2(x;\alpha)dx \\ &\qquad \geq \int_0^1 x^{-1}h(x)\frac{l(x)[g(U(x;\alpha)) - 1] - 4}{g(U(x;\alpha)) + l(x) + 3}f(U(x;\alpha))\phi_2(x;\alpha)dx. \end{aligned}$$

Since $U''(x;\alpha) = -\lambda(\alpha)h(x)f(U(x;\alpha)) < 0$ on $(0,1]$, we find that $U'(x;\alpha)$ is decreasing in $x \in (0,1]$. From $U'(0;\alpha) = 0$ it follows that $U'(x;\alpha) < 0$ for $x \in (0,1]$, which implies that $U(x;\alpha)$ is also decreasing in $x \in (0,1]$. Then there exists $x(\alpha) \in (0,1)$ such that $U(x;\alpha) \geq s_0$ for $x \in [0, x(\alpha)]$ and $U(x;\alpha) < s_0$ for $x \in (x(\alpha), 1]$. Since $U(x;\alpha)$ is concave on $(0,1)$, we conclude that

$$U(x;\alpha) \geq \alpha(1-x), \quad x \in [0,1],$$

which shows that if $x \in [0, (\alpha - s_0)/\alpha]$, then $U(x;\alpha) \geq s_0$. Therefore, $x(\alpha) \geq (\alpha - s_0)/\alpha$, which implies

$$(10) \qquad \lim_{\alpha\to\infty} x(\alpha) = 1.$$

We take $s_1 \geq s_0$ for which $x(\alpha) \geq 3/4$ for $\alpha \geq s_1$. If $\alpha \geq s_1$, then (6) implies

$$\begin{aligned} (11) \qquad &\int_0^{x(\alpha)} x^{-1}h(x)\frac{l(x)[g(U(x;\alpha)) - 1] - 4}{g(U(x;\alpha)) + l(x) + 3}f(U(x;\alpha))\phi_2(x;\alpha)dx \\ &\qquad \geq \delta f(s_0)\int_0^{x(\alpha)} x^{-1}h(x)\phi_2(x;\alpha)dx \\ &\qquad \geq \delta f(s_0)\int_{1/4}^{3/4} x^{-1}h(x)\phi_2(x;\alpha)dx. \end{aligned}$$

Recalling $\max_{\xi\in[0,1]} \phi_2(\xi) = 1$, we have

$$\text{(12)} \quad \int_{x(\alpha)}^{1} x^{-1}h(x)\frac{l(x)[g(U(x;\alpha))-1]-4}{g(U(x;\alpha))+l(x)+3} f(U(x;\alpha))\phi_2(x;\alpha)dx$$
$$\geq -\int_{x(\alpha)}^{1} x^{-1}h(x)\frac{(l(x)+4)f(U(x;\alpha))\phi_2(x;\alpha)}{g(U(x;\alpha))+l(x)+3}dx$$
$$\geq -f(s_0)\int_{x(\alpha)}^{1} x^{-1}h(x)\frac{l(x)+4}{l(x)+3}dx.$$

Now we will show that there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. Assume to the contrary that there exists $\{\alpha_n\}_{n=1}^{\infty}$ such that $\mu_2(\alpha_n) \geq 0$ and $\alpha_n \geq s_1$ for $n \in \mathbf{N}$ and $\lim_{n\to\infty} \alpha_n = \infty$.

Since $\phi_2(x;\alpha_n) > 0$ and

$$\phi_2''(x;\alpha_n) = -h(x)f'(U(x;\alpha_n))\phi_2(x;\alpha_n) - \mu_2(\alpha_n)\phi_2(x;\alpha_n) \leq 0, \quad x \in (0,1),$$

we find that $\phi_2(x;\alpha_n)$ is concave on $(0,1)$. From Lemma 2 with $\rho = 1/4$, $a = 0$ and $b = 1$, it follows that

$$\phi_2(x;\alpha_n) \geq \frac{1}{4}\max_{\xi\in[0,1]} \phi_2(\xi;\alpha_n) = \frac{1}{4}, \quad x \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

By (11), we have

$$\text{(13)} \quad \int_{0}^{x(\alpha)} x^{-1}h(x)\frac{l(x)[g(U(x;\alpha))-1]-4}{g(U(x;\alpha))+l(x)+3} f(U(x;\alpha))\phi_2(x;\alpha)dx$$
$$\geq \frac{\delta f(s_0)}{4}\int_{1/4}^{3/4} x^{-1}h(x)dx.$$

Combining (8) with (9), (12) and (13), we have

$$0 \geq -\mu_2(\alpha_n)\int_0^1 \phi_2(x;\alpha_n)y(x)dx$$
$$\geq \lambda(\alpha_n)f(s_0)\left[\frac{\delta}{4}\int_{1/4}^{3/4} x^{-1}h(x)dx - \int_{x(\alpha_n)}^{1} x^{-1}h(x)\frac{l(x)+4}{l(x)+3}dx\right],$$

which implies

$$\int_{x(\alpha_n)}^{1} x^{-1}h(x)\frac{l(x)+4}{l(x)+3}dx \geq \frac{\delta}{4}\int_{1/4}^{3/4} x^{-1}h(x)dx > 0, \quad n \in \mathbf{N}.$$

This contradicts the fact (10). Consequently, there exists $s_2 \geq s_1$ such that $\mu_2(\alpha) < 0$ for $\alpha \geq s_2$. This completes the proof. □

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