数理解析研究所講究録
第2005巻 2016年 128-140

Exponential Runge-Kutta methods for stiff stochastic differential equations

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Abstract It is well known that the numerical solution of stiff stochastic differential equations (SDEs) leads to a stepsize reduction when explicit methods are used. However, there are some classes of explicit methods that are well suited to solving some types of stiff SDEs. One such class is the class of stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods. SROCK methods reduce to Runge-Kutta Chebyshev methods when applied to ordinary differential equations (ODEs). Another promising class of methods is the class of explicit methods that reduce to explicit exponential Runge-Kutta (RK) methods when applied to semilinear ODEs. In the present paper, such explicit methods are considered. As a result, the stochastic exponential Euler scheme will be derived for strong approximations to the solution of stiff Itô SDEs with a semilinear drift term. In addition, stochastic exponential RK methods will be derived for weak approximations.

1 Introduction

While it has been customary to treat the numerical solution of stiff ordinary differential equations (ODEs) by implicit methods, there are some classes of explicit methods that are well suited to solving some types of stiff problems. One such class is the class of Runge-Kutta Chebyshev (RKC) methods. They are useful for the stiff problems whose eigenvalues lie near the negative real axis. An original contribution is by van der Houwen and Sommeijer [17] who have constructed explicit s-stage Runge-Kutta (RK) methods whose stability functions are shifted Chebyshev polynomials $T_s(1+z/s^2)$. These have stability regions along the negative real axis of $[-2s^2, 0]$. Note that the methods need to increase the stage number s for stabilization. Another suitable class of methods is the class of explicit exponential RK methods for semilinear problems [6, 7]. Note that explicit exponential RK methods are A-stable.

Similarly, for stochastic differential equations (SDEs) stabilized explicit RK methods have been developed. An original contribution concerning RKC methods is by Abdulle and his colleagues [1, 2] who have developed a family of explicit stochastic orthogonal Runge-Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. Their methods reduce to the first order RKC methods when applied to ODEs. Note that these methods also need to increase the stage number for stabilization. Shi, Xiao and Zhang [16] have considered an exponential Euler scheme for the strong approximation to the solution of SDEs with multiplicative noise driven by a scalar Wiener process. Exponential integrators have been also considered for stochastic partial differential equations with a semilinear drift term and additive noise [8] or multiplicative noise [3].

The present paper will be composed of two parts. In the first part, we will introduce an explicit exponential Euler scheme proposed by Komori and Burrage [11] for strong approximations to the solution of multi-dimensional, non-commutative Itô SDEs with a semilinear drift term. In the second part, we will devote ourselves to deriving stochastic exponential

Runge-Kutta (SERK) methods for weak approximations to the solution of the same type of Itô SDEs.

2 Explicit exponential RK methods for ODEs

We consider autonomous semilinear ODEs given by

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t)), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{2. 1}$$

where $\boldsymbol{y}$ is an $\mathbb{R}^d$-valued function on $[0, \infty)$, A is a $d \times d$ matrix and $\boldsymbol{f}$ is an $\mathbb{R}^d$-valued nonlinear function on $\mathbb{R}^d$ or a constant vector. By the variation-of-constants formula, we have

$$\boldsymbol{y}(t_{n+1}) = \mathrm{e}^{Ah}\boldsymbol{y}_n + \int_{t_n}^{t_{n+1}} \mathrm{e}^{A(t_{n+1}-s)}\boldsymbol{f}(\boldsymbol{y}(s))\mathrm{d}s \tag{2. 2}$$

if $\boldsymbol{y}(t_n) = \boldsymbol{y}_n$. Here, $\boldsymbol{y}_n$ denotes a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (2. 1) for an equidistant grid point $t_n \stackrel{\text{def}}{=} nh$ $(n = 1, 2, \ldots, M)$ with step size h (M is a natural number). By interpolating $\boldsymbol{f}(\boldsymbol{y}(s))$ at $\boldsymbol{f}(\boldsymbol{y}_n)$ only, we obtain the simplest exponential scheme for (2. 1) [7]:

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h, \tag{2. 3}$$

where $\varphi_1(Z) \stackrel{\text{def}}{=} Z^{-1}(e^Z - I)$ and I stands for the $d \times d$ identity matrix. This is called the explicit exponential Euler scheme.

In addition, higher order exponential RK methods have been proposed in [6, 7]. The following is a second order exponential RK method [7]:

$$\begin{aligned}
\boldsymbol{Y}_1 &= \mathrm{e}^{c_2hA}\boldsymbol{y}_n + c_2h\varphi_1(c_2hA)\boldsymbol{f}(\boldsymbol{y}_n), \\
\boldsymbol{y}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{1}{c_2}\varphi_2(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) + h\frac{1}{c_2}\varphi_2(hA)\boldsymbol{f}(\boldsymbol{Y}_1),
\end{aligned} \tag{2. 4}$$

where c_2 is a parameter and $\varphi_2(Z) \stackrel{\text{def}}{=} Z^{-2}(e^Z - I - Z)$. The following is a third order exponential RK method [6]:

$$\begin{aligned}
\boldsymbol{Y}_1 &= \mathrm{e}^{c_2hA}\boldsymbol{y}_n + c_2h\varphi_1(c_2hA)\boldsymbol{f}(\boldsymbol{y}_n), \\
\boldsymbol{Y}_2 &= \mathrm{e}^{c_3hA}\boldsymbol{y}_n + h\{c_3\varphi_1(c_3hA) - a_{32}(hA)\}\boldsymbol{f}(\boldsymbol{y}_n) + ha_{32}(hA)\boldsymbol{f}(\boldsymbol{Y}_1), \\
\boldsymbol{y}_{n+1} &= \mathrm{e}^{hA}\boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{\gamma+1}{\gamma c_2 + c_3}\varphi_2(hA)\right\}\boldsymbol{f}(\boldsymbol{y}_n) \\
&\quad + h\frac{1}{\gamma c_2 + c_3}\varphi_2(hA)\{\gamma\boldsymbol{f}(\boldsymbol{Y}_1) + \boldsymbol{f}(\boldsymbol{Y}_2)\},
\end{aligned} \tag{2. 5}$$

where c_2, c_3 and γ are parameters satisfying

$$2(\gamma c_2 + c_3) = 3\left(\gamma c_2^2 + c_3^2\right)$$

and $a_{32}(Z) \stackrel{\text{def}}{=} \frac{c_2}{\gamma}\varphi_2(c_2Z) + \frac{c_3^2}{c_2}\varphi_2(c_3Z)$ (It should be noted that there is a typographical error in (5.9) of [6]).

3 Strong order stochastic exponential Euler scheme

Similarly to the case of ODEs, we are concerned with autonomous SDEs with the semilinear drift term given by

$$\mathrm{d}\boldsymbol{y}(t) = (A\boldsymbol{y}(t) + \boldsymbol{f}(\boldsymbol{y}(t)))\mathrm{d}t + \sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}(t))\mathrm{d}W_j(t), \qquad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0, \tag{3. 1}$$

where $\boldsymbol{g}_j$, $j = 1, 2, \ldots, m$ are $\mathbb{R}^d$-valued functions on $\mathbb{R}^d$, the $W_j(t)$, $j = 1, 2, \ldots, m$ are independent Wiener processes and $\boldsymbol{y}_0$ is independent of $W_j(t) - W_j(0)$ for $t > 0$. If a global Lipschitz condition is satisfied, the stochastic differential equation (SDE) has exactly one continuous global solution on the entire interval $[0, \infty)$ [4, p. 113].

Similarly to (2. 2), we have

$$\boldsymbol{y}(t_{n+1}) = \mathrm{e}^{Ah}\boldsymbol{y}_n + \int_{t_n}^{t_{n+1}} \mathrm{e}^{A(t_{n+1}-s)}\boldsymbol{f}(\boldsymbol{y}(s))\mathrm{d}s + \sum_{j=1}^{m} \int_{t_n}^{t_{n+1}} \mathrm{e}^{A(t_{n+1}-s)}\boldsymbol{g}_j(\boldsymbol{y}(s))\mathrm{d}W_j(s) \tag{3. 2}$$

if $\boldsymbol{y}(t_n) = \boldsymbol{y}_n$ (see also [3, 16]). By utilizing this, we can have

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{Ah}\boldsymbol{y}_n + \mathrm{e}^{Ah}\boldsymbol{f}(\boldsymbol{y}_n)h + \mathrm{e}^{Ah}\sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}_n)\triangle W_j \tag{3. 3}$$

as an approximation to $\boldsymbol{y}(t_{n+1})$, where $\triangle W_j \stackrel{\text{def}}{=} W_j(t_{n+1}) - W_j(t_n)$. For $m = 1$ (3. 3) is the same as an exponential Euler scheme proposed by Shi et al. [16] for SDEs with a scalar Wiener process. When (3. 3) is applied to ODEs, it is equivalent to the Lawson-Euler scheme [12, 16]. In addition, it has a similar type of approximations in both of the drift and diffusion terms. Thus, let us call it the stochastic Lawson-Euler scheme.

By utilizing (3. 2), we can also derive other schemes. One of them is

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h + \mathrm{e}^{Ah}\sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}_n)\triangle W_j.$$

Adamu [3] has proposed this scheme and has called it the SETD0 scheme (SETD stands for "stochastic exponential time differencing"). In addition, we can obtain the following scheme [11]:

$$\boldsymbol{y}_{n+1} = \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h + \varphi_1(Ah)\sum_{j=1}^{m} \boldsymbol{g}_j(\boldsymbol{y}_n)\triangle W_j. \tag{3. 4}$$

When (3. 4) is applied to ODEs, it is equivalent to the exponential Euler scheme. In addition, it has a similar type of approximations in both of the drift and diffusion terms. Thus, let us call it the stochastic exponential Euler scheme.

In general, when discrete approximations $\boldsymbol{y}_n$ are given by a numerical scheme, we say that the scheme is of strong order p if there exists a constant C such that

$$\left(E[||\boldsymbol{y}_M - \boldsymbol{y}(T)||^2]\right)^{1/2} \leq Ch^p$$

with $T = Mh$ and h sufficiently small [9, 15], where $||\cdot||$ stands for the Euclidean norm. If we assume $\boldsymbol{f}, \boldsymbol{g}_j \in C^2$ for $j = 1, 2, \ldots, m$, then, the exponential schemes mentioned above are of strong order a half for solving (3. 1) [3, 11, 16]. In [3], another approximation was considered and it finally led to the square root of a matrix exponential function.

In some problems approximate solutions need to be non-negative and they are often required to satisfy other boundary conditions. The projection method [5] is very useful to deal with such problems. However, we cannot use the SROCK methods together with the projection method because the methods need several intermediate stage values for stabilization. On the other hand, the pair of the stochastic exponential Euler scheme and the projection method performs very well for stiff biochemical problems [11].

4 Weak order SERK methods

We derive SERK methods of weak order one or two by utilizing some results in SRK methods. For this, we give a brief introduction to SRK methods in the first subsection. After it, we will derive and show SERK methods in the second and third subsections.

4.1 SRK methods

In order to deal with weak approximations for (3. 1), let $\boldsymbol{g}_0(\boldsymbol{y})$ be $A\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{y})$ and let us consider the following SRK method with the stage number s and $r \leq s$ [10], which is based on the SRK framework proposed by Rößler [14]:

$$\begin{aligned} \boldsymbol{y}_{n+1} = \boldsymbol{y}_n &+ \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right) \\ &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right) \\ &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(3)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(4)} \sqrt{h} \boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right), \end{aligned} \tag{4. 1}$$

where

$$\begin{aligned} \boldsymbol{H}_i^{(0)} &= \boldsymbol{y}_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) \quad (1 \leq i \leq r), \\ \boldsymbol{H}_i^{(0)} &= \boldsymbol{y}_n + \sum_{k=1}^{i-1} A_{ik}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=r}^{i-1}\sum_{l=1}^{m} B_{ik}^{(0)} \triangle \hat{W}_l \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right) \quad (r < i \leq s), \\ \boldsymbol{H}_r^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{r} A_{rk}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right), \\ \boldsymbol{H}_i^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{i} A_{ik}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=r}^{i-1} B_{ik}^{(1)} \sqrt{h} \boldsymbol{g}_j\left(\boldsymbol{H}_k^{(j)}\right) \quad (r < i \leq s), \\ \hat{\boldsymbol{H}}_i^{(j)} &= \boldsymbol{y}_n + \sum_{k=1}^{s} A_{ik}^{(2)} h \boldsymbol{g}_0\left(\boldsymbol{H}_k^{(0)}\right) + \sum_{k=r}^{s}\sum_{\substack{l=1 \\ l \neq j}}^{m} B_{ik}^{(2)} \tilde{\eta}^{(j,l)} \boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right) \quad (r \leq i \leq s) \end{aligned}$$

Table 1: Butcher tableau for (4. 1) with $r = s - 2$

$A^{(0)}_{21}$										
$\vdots$	$\ddots$									
$A^{(0)}_{s-1,1}$	$\cdots$	$A^{(0)}_{s-1,s-2}$			$B^{(0)}_{s-1,s-2}$					
$A^{(0)}_{s,1}$	$\cdots$	$A^{(0)}_{s,s-2}$	$A^{(0)}_{s,s-1}$		$B^{(0)}_{s,s-2}$	$B^{(0)}_{s,s-1}$				
$A^{(1)}_{s-2,1}$	$\cdots$	$A^{(1)}_{s-2,s-2}$								
$A^{(1)}_{s-1,1}$	$\cdots$	$A^{(1)}_{s-1,s-2}$	$A^{(1)}_{s-1,s-1}$		$B^{(1)}_{s-1,s-2}$					
$A^{(1)}_{s,1}$	$\cdots$	$A^{(1)}_{s,s-2}$	$A^{(1)}_{s,s-1}$	$A^{(1)}_{s,s}$	$B^{(1)}_{s,s-2}$	$B^{(1)}_{s,s-1}$				
$A^{(2)}_{s-2,1}$	$\cdots$	$A^{(2)}_{s-2,s-2}$	$A^{(2)}_{s-2,s-1}$	$A^{(2)}_{s-2,s}$	$B^{(2)}_{s-2,s-2}$	$B^{(2)}_{s-2,s-1}$	$B^{(2)}_{s-2,s}$			
$A^{(2)}_{s-1,1}$	$\cdots$	$A^{(2)}_{s-1,s-2}$	$A^{(2)}_{s-1,s-1}$	$A^{(2)}_{s-1,s}$	$B^{(2)}_{s-1,s-2}$	$B^{(2)}_{s-1,s-1}$	$B^{(2)}_{s-1,s}$			
$A^{(2)}_{s,1}$	$\cdots$	$A^{(2)}_{s,s-2}$	$A^{(2)}_{s,s-1}$	$A^{(2)}_{s,s}$	$B^{(2)}_{s,s-2}$	$B^{(2)}_{s,s-1}$	$B^{(2)}_{s,s}$			
α_1	$\cdots$	α_{s-2}	α_{s-1}	α_s	$\beta^{(1)}_{s-2}$	$\beta^{(1)}_{s-1}$	$\beta^{(1)}_{s}$	$\beta^{(2)}_{s-2}$	$\beta^{(2)}_{s-1}$	$\beta^{(2)}_{s}$
					$\beta^{(3)}_{s-2}$	$\beta^{(3)}_{s-1}$	$\beta^{(3)}_{s}$	$\beta^{(4)}_{s-2}$	$\beta^{(4)}_{s-1}$	$\beta^{(4)}_{s}$

for $j = 1, 2, \ldots, m$ and where the α_i, $\beta_i^{(r_a)}$, $A_{ik}^{(r_b)}$, and $B_{ik}^{(r_b)}$ ($1 \leq r_a \leq 4$ and $0 \leq r_b \leq 2$) denote the parameters of the method. The random variables involved in the method are given by $\tilde{\eta}^{(j,j)} \stackrel{\text{def}}{=} ((\triangle\hat{W}_j)^2 - h)/(2\sqrt{h})$,

$$\tilde{\eta}^{(j,l)} \stackrel{\text{def}}{=} \begin{cases} (\triangle\hat{W}_j\triangle\hat{W}_l - \sqrt{h}\triangle\tilde{W}_j)/(2\sqrt{h}) & (j < l), \\ (\triangle\hat{W}_j\triangle\hat{W}_l + \sqrt{h}\triangle\tilde{W}_l)/(2\sqrt{h}) & (j > l), \end{cases}$$

the $\triangle\tilde{W}_l$ ($1 \leq l \leq m-1$) are independent two-point distributed random variables with $P(\triangle\tilde{W}_j = \pm\sqrt{h}) = 1/2$ and the $\triangle\hat{W}_j$ ($1 \leq j \leq m$) are independent three-point distributed random variables with $P(\triangle\hat{W}_j = \pm\sqrt{3h}) = 1/6$ and $P(\triangle\hat{W}_j = 0) = 2/3$ [9, p. 225]. If we assume $r = s - 2$, for example, (4. 1) is characterized by the Butcher tableau in Table 1.

Let $C_P^L(\mathbb{R}^d, \mathbb{R})$ be the family of L times continuously differentiable real-valued functions on $\mathbb{R}^d$, whose partial derivatives of order less than or equal to L have polynomial growth. Whenever we deal with weak convergence of order q, we will assume the following on SDEs [9, p. 474] (also see [4, p. 113]):

Assumption 4.1 *All moments of the initial value $\boldsymbol{y}_0$ exist and $\boldsymbol{g}_j$ ($j = 0, 1, \ldots, m$) are Lipschitz continuous with all their components belonging to $C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$.*

Then, we can give the definition of weak convergence of order q [9, p. 327]:

Definition 4.1 *When discrete approximations $\boldsymbol{y}_n$ are given by a numerical scheme, we say that the scheme is of weak (global) order q if for all $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$, constants $C > 0$ (independent of h) and $\delta_0 > 0$ exist, such that*

$$|E[G(\boldsymbol{y}(t_M))] - E[G(\boldsymbol{y}_M)]| \leq Ch^q, \quad h \in (0, \delta_0).$$

If we want to derive a scheme of weak order one from (4. 1), for example, we need to find a set of parameter values satisfying the following nine order conditions [14]:

$$1.\ \sum_{i=1}^{s}\alpha_i=1,\quad 2.\ \sum_{i=r}^{s}\beta_i^{(4)}=0,\quad 3.\ \sum_{i=r}^{s}\beta_i^{(3)}=0,\quad 4.\ \left(\sum_{i=r}^{s}\beta_i^{(1)}\right)^2=1,$$

$$5.\ \sum_{i=r}^{s}\beta_i^{(2)}=0,\quad 6.\ \sum_{i=r+1}^{s}\beta_i^{(1)}\left(\sum_{k=r}^{i-1}B_{ik}^{(1)}\right)=0,\quad 7.\ \sum_{i=r}^{s}\beta_i^{(4)}\left(\sum_{k=1}^{s}A_{ik}^{(2)}\right)=0,$$

$$8.\ \sum_{i=r}^{s}\beta_i^{(3)}\left(\sum_{k=r}^{s}B_{ik}^{(2)}\right)=0,\quad 9.\ \sum_{i=r}^{s}\beta_i^{(4)}\left(\sum_{k=r}^{s}B_{ik}^{(2)}\right)^2=0.$$

We will refer to these in the next subsection.

In the case of weak order two we have 59 order conditions including the above nine order conditions, and we need three stages at least to satisfy them [14]. Let us suppose $s=3$. In order to solve the order conditions in a simple way, we can assume

$$\begin{aligned}
&\beta_1^{(1)}=\frac{-1+2\left(B_{21}^{(1)}\right)^2}{2\varepsilon_1\left(B_{21}^{(1)}\right)^2},\quad \beta_2^{(1)}=\beta_3^{(1)}=\frac{1}{4\varepsilon_1\left(B_{21}^{(1)}\right)^2},\quad \beta_1^{(2)}=0,\\
&\beta_2^{(2)}=-\beta_3^{(2)}=\frac{1}{2B_{21}^{(1)}},\quad \beta_1^{(3)}=-\frac{1}{2\varepsilon_1 b_2^2},\quad \beta_2^{(3)}=\beta_3^{(3)}=\frac{1}{4\varepsilon_1 b_2^2},\quad \beta_1^{(4)}=0,\\
&\beta_2^{(4)}=-\beta_3^{(4)}=\frac{1}{2b_2},\quad B_{32}^{(0)}=0,\quad B_{31}^{(1)}=-B_{21}^{(1)},\quad B_{32}^{(1)}=0,\\
&B_{11}^{(2)}=B_{12}^{(2)}=B_{13}^{(2)}=0,\quad B_{23}^{(2)}=B_{22}^{(2)},\quad B_{31}^{(2)}=-B_{21}^{(2)},\quad B_{32}^{(2)}=B_{33}^{(2)}=-B_{22}^{(2)},\\
&A_{21}^{(1)}=A_{31}^{(1)},\quad A_{22}^{(1)}=A_{32}^{(1)}=A_{33}^{(1)}=0,\quad A_{1,k}^{(2)}=A_{2,k}^{(2)}=A_{3,k}^{(2)}\quad (1\le k\le 3)
\end{aligned}\tag{4. 2}$$

when $B_{21}^{(1)}$, $B_{21}^{(2)}$ and $B_{22}^{(2)}$ are given [10]. Here, $\varepsilon_1 \stackrel{\text{def}}{=} \pm 1$ and $b_2 \stackrel{\text{def}}{=} B_{21}^{(2)}+2B_{22}^{(2)}$. Then, only the following three order conditions remain to be solved [10]:

$$10.\ \sum_{i=2}^{3}\alpha_i\left(B_{i,1}^{(0)}\right)^2=\frac{1}{2},\quad 11.\ \sum_{i=2}^{3}\alpha_i B_{i,1}^{(0)}=\frac{\varepsilon_1}{2},\quad 12.\ \sum_{i=1}^{3}\beta_i^{(1)}A_{i,1}^{(1)}=\frac{\varepsilon_1}{2}.$$

4.2 SERK methods

As preparations, we start with a simple case. Let us assume $s=r=1$ in (4. 1) and consider

$$\begin{aligned}
&\boldsymbol{H}_1^{(0)}=\boldsymbol{y}_n,\qquad \boldsymbol{H}_1^{(j)}=\boldsymbol{y}_n+h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)\quad (1\le j\le m),\\
&\boldsymbol{y}_{n+1}=\boldsymbol{y}_n+h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)+\sum_{j=1}^{m}\triangle\tilde{W}_j\boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right),
\end{aligned}\tag{4. 3}$$

which means

$$A_{11}^{(1)}=\alpha_1=\beta_1^{(1)}=1,\quad \beta_1^{(2)}=\beta_1^{(3)}=\beta_1^{(4)}=0.$$

Because Conditions 1 to 9 are satisfied, (4. 3) is of weak order one. Here, note that $\triangle\tilde{W}_j$ is available for weak order one instead of $\triangle\hat{W}_j$. On the other hand, since the Euler scheme and (2. 3) are of order one for (2. 1),

$$\left\| \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h - \left(\boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)\right)\right\| = O(h^2)$$

as $h \to 0$. For this, the replacement of $\boldsymbol{y}_n + h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)$ with $\mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h$ in (4. 3) does not violate the weak order of convergence. Thus, we can obtain the following SERK scheme of weak order one:

$$\begin{aligned} \boldsymbol{H}_1^{(j)} &= \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h \quad (1 \le j \le m), \\ \boldsymbol{y}_{n+1} &= \mathrm{e}^{Ah}\boldsymbol{y}_n + \varphi_1(Ah)\boldsymbol{f}(\boldsymbol{y}_n)h + \sum_{j=1}^{m} \triangle\tilde{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right). \end{aligned} \tag{4. 4}$$

It is remarkable that (4. 4) reduces to (4. 3) if A goes to the zero matrix, whereas they have the same weak order. Taking this into account, now let us consider a way of finding SERK methods who achieve weak order q $(= 1, 2)$ when (4. 1) is of the same weak order q. The following lemma will be helpful for us to do this.

Lemma 4.1 *Assume that $\boldsymbol{y}_{n+1}$ is given by (4. 1) and another approximation $\hat{\boldsymbol{y}}_{n+1}$ is given by*

$$\begin{aligned} \hat{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(1)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(2)} \tilde{\eta}^{(j,j)} \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right) \\ &+ \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(3)} \triangle\hat{W}_j \boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right) + \sum_{i=r}^{s}\sum_{j=1}^{m} \beta_i^{(4)} \sqrt{h}\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right), \end{aligned} \tag{4. 5}$$

where $\tilde{\boldsymbol{H}}_i^{(j)}, \bar{\boldsymbol{H}}_i^{(j)}$ ($i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$) and $\tilde{\boldsymbol{y}}_{n+1}$ satisfy the deterministic conditions

$$\begin{aligned} &\left\|\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right\| = O(h^q), \quad \left\|\bar{\boldsymbol{H}}_i^{(j)} - \hat{\boldsymbol{H}}_i^{(j)}\right\| = O(h^{q+1/2}), \\ &\left\|\tilde{\boldsymbol{y}}_{n+1} - \left\{\boldsymbol{y}_n + \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right)\right\}\right\| = O(h^{q+1/2}), \end{aligned} \tag{4. 6}$$

the expectation condition

$$\left\| E\left[\tilde{\boldsymbol{y}}_{n+1} - \left\{\boldsymbol{y}_n + \sum_{i=1}^{s} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right)\right\}\right]\right\| = O(h^{q+1}) \tag{4. 7}$$

and the covariance conditions

$$\begin{aligned} &\left\| E\left[\triangle\hat{W}_j\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1}), \\ &\left\| E\left[\tilde{\eta}^{(j,j)}\left(\tilde{\boldsymbol{H}}_i^{(j)} - \boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1}) \end{aligned} \tag{4. 8}$$

as $h \to 0$ for a given $q = 1$ or 2 under the condition that $\boldsymbol{y}_n$ is given. Then, for all $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$

$$|E[G(\hat{\boldsymbol{y}}_{n+1}) - G(\boldsymbol{y}_{n+1})]| = O(h^{q+1})$$

as $h \to 0$ under the condition that $\boldsymbol{y}_n$ is given.

Proof. From (4. 1), (4. 5), (4. 6) and (4. 7), we have

$$\left\|E\left[\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right]\right\| \leq \left\|E\left[\sum_{i=r}^{s}\sum_{j=1}^{m}\beta_i^{(1)}\triangle\hat{W}_j\frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)}-\boldsymbol{H}_i^{(j)}\right)\right]\right\| + \left\|E\left[\sum_{i=r}^{s}\sum_{j=1}^{m}\beta_i^{(2)}\tilde{\eta}^{(j,j)}\frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)}-\boldsymbol{H}_i^{(j)}\right)\right]\right\| + O(h^{q+1}).$$

Here,

$$\left\|E\left[\triangle\hat{W}_j\frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)}-\boldsymbol{H}_i^{(j)}\right)\right]\right\| = \left\|E\left[\triangle\hat{W}_j\frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{y}_n\right)\left(\tilde{\boldsymbol{H}}_i^{(j)}-\boldsymbol{H}_i^{(j)}\right)\right]\right\| + O(h^{q+1})$$

because of (4. 1) and (4. 6). This and (4. 8) lead to

$$\left\|E\left[\triangle\hat{W}_j\frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)}-\boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1})$$

under the condition that $\boldsymbol{y}_n$ is given. Similarly,

$$\left\|E\left[\tilde{\eta}^{(j,j)}\frac{\partial \boldsymbol{g}_j}{\partial \boldsymbol{y}}\left(\boldsymbol{H}_i^{(j)}\right)\left(\tilde{\boldsymbol{H}}_i^{(j)}-\boldsymbol{H}_i^{(j)}\right)\right]\right\| = O(h^{q+1}).$$

Hence, we have

$$\left\|E\left[\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right]\right\| = O(h^{q+1}) \tag{4. 9}$$

under the condition that $\boldsymbol{y}_n$ is given.

On the other hand,

$$\left\|\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right\| = O(h^{q+1/2})$$

because of (4. 1), (4. 5) and (4. 6). For all $G \in C_P^{2(q+1)}(\mathbb{R}^d, \mathbb{R})$, thus,

$$\begin{aligned} G\left(\hat{\boldsymbol{y}}_{n+1}\right) - G\left(\boldsymbol{y}_{n+1}\right) &= \frac{\partial G}{\partial \boldsymbol{y}}\left(\boldsymbol{y}_{n+1}\right)\left(\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right) + O(h^{2q+1}) \\ &= \frac{\partial G}{\partial \boldsymbol{y}}\left(\boldsymbol{y}_{n}\right)\left(\hat{\boldsymbol{y}}_{n+1}-\boldsymbol{y}_{n+1}\right) + O(h^{q+1}). \end{aligned}$$

Consequently, because of (4. 9) we obtain

$$E\left[G\left(\hat{\boldsymbol{y}}_{n+1}\right) - G\left(\boldsymbol{y}_{n+1}\right)\right] = O(h^{q+1})$$

as $h \to 0$ under the condition that $\boldsymbol{y}_n$ is given. □

This lemma and Theorem 1.2 in [13] give us a way of finding SERK methods. That is, if $\boldsymbol{y}_{n+1}$ given by (4. 1) is of weak order q and $\hat{\boldsymbol{y}}_{n+1}$ given by an SERK method satisfies the assumption in the lemma, then $\hat{\boldsymbol{y}}_{n+1}$ is also of weak order q.

4.3 Examples of SERK methods

When we set $s = r = 2$ in (4. 1), we have

$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + \alpha_1 h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + \alpha_2 h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right) + \sum_{j=1}^{m} \beta_2^{(1)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\boldsymbol{H}_2^{(j)}\right), \tag{4. 10}$$

where

$$\boldsymbol{H}_1^{(0)} = \boldsymbol{y}_n, \qquad \boldsymbol{H}_2^{(0)} = \boldsymbol{y}_n + A_{21}^{(0)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right),$$
$$\boldsymbol{H}_2^{(j)} = \boldsymbol{y}_n + A_{21}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right) + A_{22}^{(1)} h \boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right)$$

for $j = 1, 2, \ldots, m$. When $\alpha_1 + \alpha_2 = \beta_2^{(1)} = 1$, this method is of weak order one because Conditions 1 to 9 are satisfied.

Taking this and (2. 4) into account, let us consider the following SERK method

$$\boldsymbol{y}_{n+1} = \tilde{\boldsymbol{y}}_{n+1} + \sum_{j=1}^{m} \beta_2^{(1)} \triangle \hat{W}_j \boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_2^{(j)}\right), \tag{4. 11}$$

where

$$\tilde{\boldsymbol{H}}_1^{(0)} = \boldsymbol{y}_n, \qquad \tilde{\boldsymbol{H}}_2^{(0)} = \mathrm{e}^{A_{21}^{(0)} hA} \boldsymbol{y}_n + A_{21}^{(0)} h \varphi_1\left(A_{21}^{(0)} hA\right) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right),$$
$$\tilde{\boldsymbol{H}}_2^{(j)} = \mathrm{e}^{hA} \boldsymbol{y}_n + h\left\{\varphi_1(hA) - \frac{1}{A_{21}^{(0)}}\varphi_2(hA)\right\} \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right)$$
$$+ h \frac{1}{A_{21}^{(0)}} \varphi_2(hA) \boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right), \qquad \tilde{\boldsymbol{y}}_{n+1} = \tilde{\boldsymbol{H}}_2^{(j)}$$

for $j = 1, 2, \ldots, m$. When $A_{21}^{(1)} = \alpha_1$ and $A_{22}^{(1)} = \alpha_2$ as well as

$$\alpha_1 = 1 - \frac{1}{2A_{21}^{(0)}}, \qquad \alpha_2 = \frac{1}{2A_{21}^{(0)}}, \tag{4. 12}$$

we have

$$\left\|\tilde{\boldsymbol{H}}_2^{(j)} - \boldsymbol{H}_2^{(j)}\right\| = O(h^3), \qquad \left\|\tilde{\boldsymbol{y}}_{n+1} - \left\{\boldsymbol{y}_n + \sum_{i=1}^{2} \alpha_i h \boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right)\right\}\right\| = O(h^3).$$

Moreover, if $\beta_2^{(1)} = 1$, (4. 11) is of weak order one because (4. 10) is of weak order one, whereas it is of order two for (2. 1).

Next, let us suppose $s=3$ and $r=1$ in (4. 1) and consider

$$\begin{aligned}\boldsymbol{y}_{n+1}=\boldsymbol{y}_n&+\alpha_1 h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)+\alpha_2 h\boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right)+\sum_{i=1}^{3}\sum_{j=1}^{m}\beta_i^{(1)}\triangle\hat{W}_j\boldsymbol{g}_j\left(\boldsymbol{H}_i^{(j)}\right)\\&+\sum_{j=1}^{m}\beta_2^{(2)}\tilde{\eta}^{(j,j)}\boldsymbol{g}_j\left(\boldsymbol{H}_2^{(j)}\right)+\sum_{j=1}^{m}\beta_3^{(2)}\tilde{\eta}^{(j,j)}\boldsymbol{g}_j\left(\boldsymbol{H}_3^{(j)}\right)\\&+\sum_{i=1}^{3}\sum_{j=1}^{m}\beta_i^{(3)}\triangle\hat{W}_j\boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_i^{(j)}\right)\\&+\sum_{j=1}^{m}\beta_2^{(4)}\sqrt{h}\boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_2^{(j)}\right),+\sum_{j=1}^{m}\beta_3^{(4)}\sqrt{h}\boldsymbol{g}_j\left(\hat{\boldsymbol{H}}_3^{(j)}\right),\end{aligned}\tag{4. 13}$$

where

$$\begin{aligned}\boldsymbol{H}_1^{(0)}&=\boldsymbol{y}_n,\qquad \boldsymbol{H}_2^{(0)}=\boldsymbol{y}_n+A_{21}^{(0)}h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)+\sum_{l=1}^{m}B_{21}^{(0)}\triangle\hat{W}_l\boldsymbol{g}_l\left(\boldsymbol{H}_1^{(l)}\right),\\\boldsymbol{H}_1^{(j)}&=\boldsymbol{y}_n+A_{11}^{(1)}h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right),\\\boldsymbol{H}_i^{(j)}&=\boldsymbol{y}_n+A_{i,1}^{(1)}h\boldsymbol{g}_0\left(\boldsymbol{H}_1^{(0)}\right)+B_{i,1}^{(1)}\sqrt{h}\boldsymbol{g}_j\left(\boldsymbol{H}_1^{(j)}\right),\\\hat{\boldsymbol{H}}_1^{(j)}&=\boldsymbol{y}_n+A_{12}^{(2)}h\boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right),\\\hat{\boldsymbol{H}}_i^{(j)}&=\boldsymbol{y}_n+A_{i,2}^{(2)}h\boldsymbol{g}_0\left(\boldsymbol{H}_2^{(0)}\right)+\sum_{k=1}^{3}\sum_{\substack{l=1\\l\neq j}}^{m}B_{ik}^{(2)}\tilde{\eta}^{(j,l)}\boldsymbol{g}_l\left(\boldsymbol{H}_k^{(l)}\right)\end{aligned}$$

for $i=2,3$ and $j=1,2,\ldots m$.

Corresponding to this and (2. 4), let us suppose the following SERK method

$$\begin{aligned}\boldsymbol{y}_{n+1}=\tilde{\boldsymbol{y}}_{n+1}&+\sum_{i=1}^{3}\sum_{j=1}^{m}\beta_i^{(1)}\triangle\tilde{W}_j\boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_i^{(j)}\right)\\&+\sum_{j=1}^{m}\beta_2^{(2)}\tilde{\eta}^{(j,j)}\boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_2^{(j)}\right)+\sum_{j=1}^{m}\beta_3^{(2)}\tilde{\eta}^{(j,j)}\boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_3^{(j)}\right)\\&+\sum_{i=1}^{3}\sum_{j=1}^{m}\beta_i^{(3)}\triangle\tilde{W}_j\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_i^{(j)}\right)\\&+\sum_{j=1}^{m}\beta_2^{(4)}\sqrt{h}\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_2^{(j)}\right),+\sum_{j=1}^{m}\beta_3^{(4)}\sqrt{h}\boldsymbol{g}_j\left(\bar{\boldsymbol{H}}_3^{(j)}\right),\end{aligned}\tag{4. 14}$$

where

$$\tilde{\boldsymbol{H}}_1^{(0)} = \boldsymbol{y}_n, \qquad \tilde{\boldsymbol{H}}_1^{(j)} = \mathrm{e}^{A_{11}^{(1)}hA}\boldsymbol{y}_n + A_{11}^{(1)}h\varphi_1\left(A_{11}^{(1)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right),$$

$$\tilde{\boldsymbol{H}}_2^{(0)} = \mathrm{e}^{A_{21}^{(0)}hA}\boldsymbol{y}_n + A_{21}^{(0)}h\varphi_1\left(A_{21}^{(0)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + \sum_{l=1}^{m} B_{21}^{(0)}\triangle\tilde{W}_l\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right),$$

$$\tilde{\boldsymbol{H}}_i^{(j)} = \mathrm{e}^{A_{i\,1}^{(1)}hA}\boldsymbol{y}_n + A_{i\,1}^{(1)}h\varphi_1\left(A_{i\,1}^{(1)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + B_{i\,1}^{(1)}\sqrt{h}\boldsymbol{g}_j\left(\tilde{\boldsymbol{H}}_1^{(j)}\right),$$

$$\bar{\boldsymbol{H}}_1^{(j)} = \mathrm{e}^{A_{12}^{(2)}hA}\boldsymbol{y}_n + A_{12}^{(2)}hA\varphi_1\left(A_{12}^{(2)}hA\right)\sum_{l=1}^{m} B_{21}^{(0)}\triangle\tilde{W}_l\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right)$$
$$+ A_{12}^{(2)}h\varphi_1\left(A_{12}^{(2)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right),$$

$$\bar{\boldsymbol{H}}_i^{(j)} = \mathrm{e}^{A_{i\,2}^{(2)}hA}\boldsymbol{y}_n + A_{i\,2}^{(2)}hA\varphi_1\left(A_{i\,2}^{(2)}hA\right)\sum_{l=1}^{m} B_{21}^{(0)}\triangle\tilde{W}_l\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right)$$
$$+ A_{i\,2}^{(2)}h\varphi_1\left(A_{i\,2}^{(2)}hA\right)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right) + \sum_{k=1}^{3}\sum_{\substack{l=1\\ l\neq j}}^{m} B_{ik}^{(2)}\tilde{\eta}^{(j\,l)}\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_k^{(l)}\right)$$

$$\tilde{\boldsymbol{y}}_{n+1} = \mathrm{e}^{hA}\boldsymbol{y}_n + \frac{1}{A_{21}^{(0)}}hA\varphi_2(hA)\sum_{l=1}^{m} B_{21}^{(0)}\triangle\tilde{W}_l\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right)$$
$$+ h\left\{\varphi_1(hA) - \frac{1}{A_{21}^{(0)}}\varphi_2(hA)\right\}\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_1^{(0)}\right) + h\frac{1}{A_{21}^{(0)}}\varphi_2(hA)\boldsymbol{f}\left(\tilde{\boldsymbol{H}}_2^{(0)}\right)$$

for $i = 2, 3$ and $j = 1, 2, \quad m$

From these,

$$\left\|\tilde{\boldsymbol{H}}_1^{(j)} - \left\{\boldsymbol{H}_1^{(j)} + \frac{1}{2}\left(A_{11}^{(1)}h\right)^2 A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\}\right\| = O(h^3),$$

$$\left\|\tilde{\boldsymbol{H}}_i^{(j)} - \left\{\boldsymbol{H}_i^{(j)} + \frac{1}{2}\left(A_{i\,1}^{(1)}h\right)^2 A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\}\right\| = O(h^{5/2})$$

for $i = 2, 3$ and $j = 1, 2, \quad m$ Let us assume (4 12) Then, we have

$$\left\|\tilde{\boldsymbol{y}}_{n+1} - \left\{\boldsymbol{y}_n + \sum_{i=1}^{2}\alpha_i h\boldsymbol{g}_0\left(\boldsymbol{H}_i^{(0)}\right) + \frac{1}{6A_{21}^{(0)}}h^2A\left(A + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}_n)\right)\sum_{l=1}^{m} B_{21}^{(0)}\triangle\tilde{W}_l\boldsymbol{g}_l\left(\tilde{\boldsymbol{H}}_1^{(l)}\right)\right\}\right\| = O(h^3)$$

In addition, because

$$\left\|\bar{\boldsymbol{H}}_i^{(j)} - \hat{\boldsymbol{H}}_i^{(j)} - \left(\frac{1}{2}A_{i\,2}^{(2)} - A_{21}^{(0)}\right)A_{i\,2}^{(2)}h^2A\left(A\boldsymbol{y}_n + \boldsymbol{f}(\boldsymbol{y}_n)\right)\right\| = O(h^{5/2})$$

for $i = 1, 2, 3$, let us assume

$$A_{12}^{(2)} = A_{22}^{(2)} = A_{32}^{(2)} = 2A_{21}^{(0)} \tag{4 15}$$

Then, if (4. 13) is of weak order two, (4. 14) is also weak order two.

Finally, let us find a solution for (4. 13) to achieve weak order two. The substitution of $\alpha_3 = 0$ into Conditions 10 and 11 yields $B_{21}^{(0)} = \varepsilon_1$ and $\alpha_2 = \frac{1}{2}$, which means $A_{21}^{(0)} = 1$ due to (4. 12). Taking into account that $B_{21}^{(0)}$, $\beta_i^{(1)}$ and $\beta_i^{(3)}$ $(i = 1, 2, 3)$ are multiplied by $\triangle\hat{W}_j$ $(1 \le j \le m)$ in (4. 13), we can suppose $\varepsilon_1 = 1$ without loss of generality. Because of (4. 2), Condition 12 automatically holds if $A_{11}^{(1)} = A_{21}^{(1)} = 1/2$ or we have $B_{21}^{(1)} = \pm\sqrt{\gamma_0}$ from Condition 12 if $\gamma_0 \stackrel{\text{def}}{=} (A_{21}^{(1)} - A_{11}^{(1)})/(1 - 2A_{11}^{(1)}) > 0$.

5 Concluding remarks

Corresponding to (2. 3), we have derived the stochastic exponential Euler scheme for strong approximations to the solution of (3. 1). We have also derived SERK methods, which are of weak order one or two and which reduce to (2. 3) or (2. 4) if $\boldsymbol{g}_j$, $j = 1, 2, \ldots, m$ vanish.

The following are other remarks:

- Similarly, by utilizing Lemma 4.1 we can construct an SERK method, which is of weak order two and which reduce to (2. 5) if $\boldsymbol{g}_j$, $j = 1, 2, \ldots, m$ vanish.
- Using a scalar test SDE with complex coefficients, we can show that our weak first order SERK methods are A-stable in MS. If the diffusion coefficients are real values in the test SDE, our weak second order SERK methods are also A-stable in MS.
- One of our future works is to perform numerical experiments to check the performance of our methods.

References

[1] A. Abdulle and S. Cirilli. S-ROCK: Chebyshev methods for stiff stochastic differential equations. *SIAM J. Sci. Comput.*, **30**(**2**):997–1014, 2008.

[2] A. Abdulle and T. Li. S-ROCK methods for stiff Itô SDEs. *Commun. Math. Sci.*, **6**(**4**):845–868, 2008.

[3] I. A. Adamu. *Numerical approximation of SDEs and stochastic Swift-Hohenberg equation.* PhD thesis, Heriot-Watt University, 2011.

[4] L. Arnold. *Stochastic Differential Equations: Theory and Applications.* John Wiley & Sons, New York, 1974.

[5] Y. Chen and X. Ye. Projection onto a simplex. e-print, 2011. arXiv:1101.6081v2.

[6] M. Hochbruck and A. Ostermann. Explicit exponential Runge-Kutta methods for semilinear parabolic problems. *SIAM J. Numer. Anal.*, **43**(**3**):1069–1090, 2005.

[7] M. Hochbruck and A. Ostermann. Exponential integrators. *Acta Numer.*, **19**:209–286, 2010.

[8] A Jentzen and E Kloeden Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise In *Proc R Soc Lond Ser A Math Phys Eng Sci 465*, pages 649–667, 2009

[9] P E Kloeden and E Platen *Numerical Solution of Stochastic Differential Equations* Springer, New York, 1999 Corrected Third Printing

[10] Y Komori and E Buckwar Stochastic Runge-Kutta methods with deterministic high order for ordinary differential equations *BIT*, **53(3)** 617–639, 2013

[11] Y Komori and K Burrage Exponential Euler-Maruyama scheme for simulation of stiff biochemical reaction systems Technical Report CSSE-40, Kyushu Institute of Technology, 2013 Preprint of a paper submitted for publication

[12] J D Lawson Generalized Runge-Kutta processes for stable systems with large Lipschitz constants *SIAM J Numer Anal*, **4(3)** 372–380, 1967

[13] A Roßler Rooted tree analysis for order conditions of stochastic Runge-Kutta methods for the weak approximation of stochastic differential equations *Stochastic Anal and Appl*, **24(1)** 97–134, 2006

[14] A Roßler Second order Runge-Kutta methods for Itô stochastic differential equations *SIAM J Numer Anal*, **47(3)** 1713–1738, 2009

[15] A Roßler Runge-Kutta methods for the strong approximation of solutions of stochastic differential equations *SIAM J Numer Anal*, **48(3)** 922–952, 2010

[16] C Shi, Y Xiao, and C Zhang The convergence and MS stability of exponential Euler method for semilinear stochastic differential equations *Abstr Appl Anal*, **2012**, 2012 35040701, 19 pages

[17] P J van der Houwen and B P Sommeijer On the internal stability of explicit m-stage Runge-Kutta methods for large m-values *Z Angew Math Mech*, **60** 479–485, 1980

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