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いくつかのロバン型境界条件と有限要素法への応用について

On Some Robin-type Boundary Conditions and Their Applications in Finite Element Method

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1 Introduction

Consider as a model case the Poisson equation

$$-\Delta u = f \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$ and f is a given function in Ω. The Robin boundary condition for (1.1), also known as a third boundary condition, is a simple linear combination of Dirichlet and Neumann conditions as in

$$\frac{\partial u}{\partial n} + \alpha u = h \quad \text{on} \quad \Gamma := \partial\Omega, \tag{1.2}$$

where $\frac{\partial u}{\partial n}$ is a derivative of u in the normal direction, n being the unit outer normal on Γ, α is a constant parameter and h is a given function on Γ.

Let us focus on two features of the Robin boundary condition with $\alpha > 0$. First, by making $\alpha \to \infty$ we see that (1.2) approaches the Dirichlet condition $u = 0$ on Γ at least formally. This fact can be mathematically justified, and the technique to approximate the Dirichlet condition by the Robin one, in a variational form, is referred to as a penalty method [1]. Second, (1.2) can be seen as a transmission condition between Ω and Γ, where two equations $-\Delta u = f$ and $\alpha u = h$ are interacting with each other through the Neumann term $\frac{\partial u}{\partial n}$. Utilization of a Robin boundary condition as a transmission condition is found, e.g., in domain decomposition methods [12] or fluid-structure interaction problems [5].

In this paper, we propose some applications of these two perspectives regarding Robin boundary conditions, to problems arising in finite element

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methods. In the first part (Section 2), we consider the incompressible Stokes equations with the slip boundary condition in a smooth domain which need not be polygonal. The smooth boundary is approximated by straight polygonal lines or polyhedral faces, which is usual in finite element approximation. Then we need a delicate treatment of the outer unit normal on the approximated boundary, because otherwise we would encounter a variational crime (also known as a Babuška's paradox [15]). We will show that a Robin-type approximation to the slip boundary condition enables us to avoid a variational crime, achieving the optimal rate of convergence if the Robin (penalty) parameter α is properly chosen. Our scheme can be easily implemented in finite element libraries such as `FreeFEM++` [7] or `FEniCS` [10].

In the second part (Section 3), we consider (1.1) with a new transmission condition, where (1.2) is modified to involve a second-order derivative on the boundary, i.e., the Laplace-Beltrami operator. This is called a generalized Robin boundary condition, and it is related with a dynamic boundary condition for heat equations [11], fluid-structure interaction problems [2], or artificial boundary conditions [13]. We will show that, instead of the standard Sobolev space $H^1(\Omega)$, the space of $H^1(\Omega)$-functions which admit $H^1(\Gamma)$-traces is well-suited for analysis of this generalized Robin problem. In particular we prove well-posedness and convergence of the finite element method for this problem in the function space mentioned above.

2 Penalty method to the slip boundary problem[2]

2.1 Slip boundary condition

We are concerned with the following incompressible Stokes equations:

$$\begin{cases} u - \nu\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u \cdot n = g & \text{on } \Gamma, \\ \sigma_\tau(u) = h & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d\, (d = 2, 3)$, and its boundary Γ is of $C^{2,1}$-class; ν, u, p are a viscosity constant, velocity and pressure, respectively; f, g, h are given data; $n \in W^{2,\infty}(\Gamma)$ is the outer unit normal and, for a generic vector A, we denote its normal and tangential components by $A \cdot n$ and $A_\tau = A - (A \cdot n)n$; let

[2]This study is based on a joint work with I. Oikawa (Waseda Univ.) and G. Zhou (Univ. of Tokyo).

$\sigma(u,p) = -pI + \nu(\nabla u + \nabla u^T)$ be the fluid stress tensor, and set $\sigma_\tau(u) := (\sigma(u,p)n)_\tau$ to be the tangential component of the traction vector.

The weak formulation for the above problem is well known and stated as follows. Let u_g be an element of $H^1(\Omega)^d$ such that $u_g \cdot n = g$ on Γ and $\operatorname{div} u_g = 0$ (this extension is possible if $\int_\Omega g\, ds = 0$, which we assume throughout this section). Find $(u,p) \in V \times \mathring{Q}$ such that, $u - u_g \in V_\tau$ and

$$\begin{cases} a(u,v) + b(v,p) = (f,v) + \langle h, v\rangle, & \forall v \in V_\tau, \\ b(u,q) = 0, & \forall q \in \mathring{Q}. \end{cases} \tag{2.2}$$

where $V = H^1(\Omega)^d$, $V_\tau = \{v \in V \,|\, v\cdot n = 0 \text{ on } \Gamma\}$ and $\mathring{Q} = \{q \in L^2(\Omega) \,|\, \int_\Omega q\, dx = 0\}$. We furthermore set $\mathring{V} := H_0^1(\Omega)^d$. For bilinear forms, given $G \subset \mathbb{R}^d$, we define

$$a_G(u,v) = \int_G u \cdot v\, dx + \frac{\nu}{2}\int_G \nu(\nabla u + \nabla u^T) : (\nabla v + \nabla v^T)\, dx,$$
$$b_G(v,q) = -\int_G \operatorname{div} v\, q\, dx.$$

The inner products in $L^2(G)^d$ and $L^2(\partial G)^d$ are denoted by $(\cdot,\cdot)_G$ and $\langle\cdot,\cdot\rangle_{\partial G}$ respectively. The subscripts G and ∂G are omitted if $G = \Omega$.

As a result of Korn's inequality and the famous inf-sup condition, we obtain

Theorem 2.1. *There exists a unique solution (u,p) of (2.2).*

By the Green formula for the Stokes equations and $u \cdot n = g$, we have

$$\begin{cases} a(u,v) + b(v,p) + c(v\cdot n, \lambda) = (f,v)_\Omega + \langle h, v\rangle_\Gamma, & \forall v \in V, \\ b(u,q) = 0, & \forall q \in \mathring{Q}, \\ c(u\cdot n - g, \mu) = 0 & \forall \mu \in M, \end{cases} \tag{2.3}$$

where $M = H^{-1/2}(\Gamma)$, and $c(\cdot,\cdot) = \langle\cdot,\cdot\rangle$. We notice that $\lambda := -\sigma(u,p)n\cdot n$ is the normal component of the traction vector.

2.2 Meshes and approximate spaces

Because Γ is smooth, there exists a covering $\{U_r\}_{r=1}^M$ of Γ such that each $\Gamma \cap U_r$ can be represented by a graph $x_d = \phi_r(x')$, where $x' = (x_1, \ldots, x_{d-1})$, after some rotation of the coordinates. Furthermore, there exists a strip neighborhood $\Gamma_\delta = \{x \in \mathbb{R}^d \,|\, \operatorname{dist}(x,\Gamma) < \delta\}$ of Γ such that the decomposition $x = \pi(x) + d(x)n(\pi(x))$, where $\pi(x) \in \Gamma$ and $d(x)$ is the signed distance function to Γ, is uniquely determined for $x \in \Gamma_\delta$ (see [6, p. 355]). We extend n from Γ to Γ_δ by $n(x) = n(\pi(x))$.

We introduce a regular family of triangulation $\{\mathcal{T}_h\}_{h\downarrow 0}$ of Ω, where $h = \max_{T\in\mathcal{T}_h} \operatorname{diam}(T)$, and put $\Omega_h = \bigcup_{T\in\mathcal{T}_h} T$ and $\Gamma_h := \partial\Omega_h$. The boundary mesh $\mathcal{S}_h$ inherited from $\mathcal{T}_h$ also becomes a regular family of triangulation of dimension $d-1$. We denote by n_h the outer unit normal assigned to Γ_h.

We assume that $\mathcal{T}_h$ is fine enough (and thus h is sufficiently small) to satisfy the following:

(1) each $S \in \mathcal{S}_h$ is contained in some local neighborhood U_r.

(2) for each r, $\Gamma_h \cap U_r$ is represented by the graph of some piecewise linear interpolation ϕ_{rh} of ϕ_r.

(3) Γ_h is contained in the strip neighborhood Γ_δ.

Under these assumptions we can show that the map $\Gamma_h \to \Gamma$; $x \mapsto \pi(x)$ is bijective, and we call it the orthogonal projection from Γ_h onto Γ.

Next we introduce finite element spaces. We consider P1/P1 or P1b/P1 approximations, to which we refer as $l = 1$ and $l = 1b$ respectively, that is,

$$V_h = \begin{cases} \{v_h \in C(\overline{\Omega})^d \mid v_h|_T \in P_1(T) \text{ for } T \in \mathcal{T}_h\} & \text{if } l = 1, \\ \{v_h \in C(\overline{\Omega})^d \mid v_h|_T \in P_1(T) \oplus B(T) \text{ for } T \in \mathcal{T}_h\} & \text{if } l = 1b, \end{cases}$$

$$Q_h = \{v_h \in C(\overline{\Omega}) \mid v_h|_T \in P_1(T) \text{ for } T \in \mathcal{T}_h\},$$

where $B(T)$ stands for the space spanned by the bubble function on T. We furthermore set $\mathring{V}_h := V_h \cap H_0^1(\Omega_h)^d$ and $\mathring{Q}_h := Q_h \cap \mathring{Q}$.

2.3 FE scheme with penalty term

Before presenting our scheme, we highlight two difficulties in finite element approximation of (2.1), which do not occur in the no-slip boundary problem (cf. [4, p. 332]). First, since the normal direction $n(x)$ does not align with an axis of the coordinates (it varies depending on x), we need a local transformation, at the element-matrix level, to enforce the Dirichlet condition $(2.1)_3$. This procedure is described e.g. in [3], but it would require additional implementation technique which is not necessary in the no-slip boundary condition.

Second, approximation of the space V_τ, especially the constraint $v\cdot n = 0$, is rather problematic. The naive choice $V_{h\tau} = \{v_h \in V_h \mid v_h\cdot n_h = 0 \text{ on } \Gamma_h\}$ is known to lead to a variational crime. In fact, if $d = 2$, $V_{h\tau}$ coincides with $\mathring{V}_h$, so that the finite element solution converges to that of the no-slip boundary problem and never satisfies the slip boundary condition. From theoretical point of view, $V_{h\tau} = \{v_h \in V_h \mid (v_h \cdot n)(x) = 0 \text{ at each vertex } x \in \Gamma_h\}$ will be the best choice. However, this implies that one has to remember n, i.e., the information of the exact geometry Γ, which would be inconvenient in case one is given only $\partial\Omega_h$.

In view of these situations, we would like to propose a scheme to problem (2.2) such that

- implementation is easy;
- only n_h is involved;
- optimal rate of convergence $O(h)$ is achieved.

For this purpose, we approximate the Dirichlet condition $u \cdot n = 0$ by the Robin-type one $\sigma(u,p)n \cdot n + \frac{1}{\epsilon} u \cdot n = 0$ with small $\epsilon > 0$. In the variational form, this amounts to using the whole V instead of V_τ and introducing the penalty term $\frac{1}{\epsilon} c(u \cdot n, v \cdot n)$. To avoid over-constraint, we apply a reduced-order numerical integration to the penalty term. Then the resulting finite element scheme now reads as follows: find $(u_h, p_h) \in V_h \times Q_h$ such that, for all $(v_h, q_h) \in V_h \times Q_h$,

$$\begin{cases} a_h(u_h, v_h) + b_h(v_h, p_h) + \dfrac{1}{\epsilon} c_h(u_h \cdot n_h - \tilde{g}, v_h \cdot n_h) = (\tilde{f}, v_h)_h + \langle \tilde{h}, v_{h\tau} \rangle_h, \\ \hfill b_h(u_h, q_h) = d_h(p_h, q_h). \end{cases} \tag{2.4}$$

Here, we let $\tilde{\Omega}$ be a bounded smooth domain containing $\Omega \cup \Omega_h \cup \Gamma_\delta$ and denote by $\tilde{f}$ an extension of f from Ω to $\tilde{\Omega}$. We also extend g to $\tilde{g}$ and h to $\tilde{h}$ defined in $\tilde{\Omega}$, then their traces on Γ_h can be defined. We suppose $\tilde{g} \in C(\Gamma_h)$.

We set $a_h = a_{\Omega_h}$, $b_h = b_{\Omega_h}$, $(\cdot,\cdot)_h = (\cdot,\cdot)_{\Omega_h}$, and

$$c_h(\mu_h, \eta_h) = \sum_{S \in \mathcal{S}_h} |S| \mu_h(m_S) \eta_h(m_S), \quad m_S = \begin{cases} \text{midpoint of } S \text{ if } d = 2, \\ \text{barycenter of } S \text{ if } d = 3, \end{cases}$$

$$d_h(p_h, q_h) = \gamma h^2 (\nabla p_h, \nabla q_h)_h, \qquad \gamma = \begin{cases} 1 & \text{if } l = 1, \\ 0 & \text{if } l = 1b. \end{cases}$$

We recall Korn's inequality for a_h and the $\mathring{V}_h$-$\mathring{Q}_h$ inf-sup condition for b_h, which are uniform in h (see [9, 14]):

$$C\|v_h\|^2_{H^1(\Omega_h)^d} \leq a_h(v_h, v_h), \qquad \forall v_h \in V_h, \tag{2.5}$$

$$C\|q_h\|_{L^2(\Omega_h)} - \gamma Ch\|\nabla q_h\| \leq \sup_{v_h \in \mathring{V}_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{H^1(\Omega)^d}}, \qquad \forall q_h \in \mathring{Q}_h. \tag{2.6}$$

Then we have

Theorem 2.2. *There exists a unique solution* (u_h, p_h) *of problem* (2.4).

sketch of proof. We can prove the V_h-Q_h inf-sup condition

$$C\|q_h\|_{L^2(\Omega_h)} - \gamma Ch\|\nabla q_h\| \leq \sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_{H^1(\Omega_h)^d}}, \qquad \forall q_h \in Q_h.$$

Then we have the coupled inf-sup condition

$$C(h)(\|v_h\|_{H^1(\Omega_h)^d} + \|q_h\|_{L^2(\Omega_h)}) \leq \sup_{(u_h,p_h)\in V_h\times Q_h} \frac{B_h(u_h, p_h; v_h, q_h)}{\|u_h\|_{H^1(\Omega_h)^d} + \|p_h\|_{L^2(\Omega_h)}}$$

for all $(v_h, q_h) \in V_h \times Q_h$, where $B_h(u_h, p_h; v_h, q_h) := a_h(u_h, v_h) + b_h(v_h, p_h) + \frac{1}{\epsilon}c_h(u_h \cdot n_h, v_h \cdot n_h) - b_h(u_h, q_h) + d_h(p_h, q_h)$. The solvability of (2.4) is a consequence of the generalized Lax-Milgram theorem. □

By introducing the auxiliary variable $\lambda_h := (u_h \cdot n_h - \tilde{g})/\epsilon \in L^2(\Gamma_h)$, one can rewrite (2.4) as

$$\begin{cases} a_h(u_h, v_h) + b_h(v_h, p_h) + c_h(v_h \cdot n_h, \lambda_h) = (\tilde{f}, v_h)_h + \langle \tilde{h}, v_h \rangle_h, & \forall v_h \in V_h, \\ b_h(u_h, q_h) = d_h(p_h, q_h), & \forall q_h \in Q_h, \\ c_h(u_h \cdot n_h - \tilde{g}, \mu_h) = \epsilon c_h(\lambda_h, \mu_h), & \forall \mu_h \in M_h, \end{cases} \tag{2.7}$$

where $M_h = \{\mu_h \in L^2(\Gamma_h) \mid \mu|_S \in P_1(S) \text{ for } S \in \mathcal{S}_h\}$ is a discontinuous P1 space.

2.4 Estimation of consistency error

From now on, we consider the case $g = h = \tilde{g} = \tilde{h} = 0$ for simplicity. Since $\Omega_h \neq \Omega$ and the approximation is nonconforming, we cannot expect to have the so called Galerkin orthogonality relation. However, we still have the following estimate.

Proposition 2.1. *Let* (u, p, λ) *and* (u_h, p_h, λ_h) *be solutions of* (2.3) *and* (2.7) *respectively. We assume* $f \in L^3(\Omega)^d$ *and* $(u, p, \lambda) \in H^2(\Omega)^d \times H^1(\Omega) \times W^{1,\infty}(\Gamma)$. *Then we have, for all* $v_h \in V_h$,

$$|a_h(\tilde{u} - u_h, v_h) + b_h(v_h, \tilde{p} - p_h) + c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h)| \leq C(\tilde{u}, \tilde{p}, \tilde{\lambda}) h \|v_h\|_{H^1(\Omega_h)^d}, \tag{2.8}$$

where $\tilde{u}$ *and* $\tilde{p}$ *are (any suitable) extensions of* u *and* p *to* $\tilde{\Omega}$, *and* $\tilde{\lambda} = \lambda \circ \pi$.

To prove this we need auxiliary lemmas (see [9, 14] for the proof) concerning estimates on the boundary skin $\Omega \triangle \Omega_h = (\Omega \setminus \Omega_h) \cup (\Omega_h \setminus \Omega)$.

Lemma 2.1. *There exists an extension operator* $P_h \in \mathcal{L}(H^1(\Omega_h)^d, H_0^1(\tilde{\Omega})^d)$ *such that*

$$\|P_h v_h\|_{H^1(\tilde{\Omega})^d} \leq C \|v_h\|_{H^1(\Omega_h)^d}, \qquad \forall v_h \in V_h,$$
$$\|P_h v_h\|_{H^1(\Omega \triangle \Omega_h)^d} \leq C h^{1/2} \|v_h\|_{H^1(\Omega_h)^d}, \qquad \forall v_h \in V_h.$$

Lemma 2.2. *For all* $q_h \in Q_h$ *we have* $\|q_h\|_{L^2(\Omega_h \setminus \Omega)} \leq C h^{1/2} \|q_h\|_{L^2(\Omega_h)}$.

Lemma 2.3. *(i) For all* $f \in L^3(\tilde{\Omega})^d$, *we have* $\|f\|_{L^2(\Omega \triangle \Omega_h)^d} \leq C h^{1/3} \|f\|_{L^3(\tilde{\Omega})^d}$.
(i) For all $v \in H^2(\tilde{\Omega})^d$, *we have* $\|v\|_{H^1(\Omega \triangle \Omega_h)^d} \leq C h^{2/3} \|v\|_{H^2(\tilde{\Omega})^d}$.
(ii) For all $q \in H^1(\tilde{\Omega})$, *we have* $\|q\|_{L^2(\Omega \triangle \Omega_h)} \leq C h^{2/3} \|q\|_{H^1(\tilde{\Omega})}$.

Proof. These follows from $|\Omega \triangle \Omega_h| \leq C h^2$. □

Lemma 2.4. *For all* $\eta \in H^1(\tilde{\Omega})$, *we have*
(i) $\|\eta \circ \pi\|_{L^2(\Gamma_h)} \leq C \|\eta\|_{L^2(\Gamma)}$.
(ii) $|\int_\Gamma \eta \, ds - \int_{\Gamma_h} \eta \circ \pi \, ds| \leq C h^2 \|\eta\|_{L^2(\Gamma)}$.
(iii) $\|\eta - \eta \circ \pi\|_{L^2(\Gamma_h)} \leq C h \|\eta\|_{H^1(\tilde{\Omega})}$.

Lemma 2.5. *Under the assumptions of Proposition 2.1, we obtain*

$$|c(v \cdot n, \lambda) - c_h(v \cdot n_h, \tilde{\lambda})| \leq C(\lambda) h \|v\|, \qquad \forall v \in H^1(\tilde{\Omega})^d.$$

Proof. First we note that

$$\begin{aligned}
&|c(v \cdot n, \lambda) - \langle v \cdot n_h, \tilde{\lambda} \rangle_h| \\
&\leq \left| \int_\Gamma v \cdot n \lambda \, ds - \int_{\Gamma_h} v(\pi(x)) \cdot n(\pi(x)) \lambda(\pi(x)) \, ds \right| \\
&\quad + \left| \int_{\Gamma_h} v(\pi(x)) \cdot n(\pi(x)) \lambda(\pi(x)) \, ds - \int_{\Gamma_h} v(x) \cdot n(\pi(x)) \lambda(\pi(x)) \, ds \right| \\
&\quad + \left| \int_{\Gamma_h} v(x) \cdot n(\pi(x)) \lambda(\pi(x)) \, ds - \int_{\Gamma_h} v(x) \cdot n_h(x) \lambda(\pi(x)) \, ds \right| \\
&\leq C h \|v\|_{H^1(\Omega_h)^d} \|\lambda\|_{L^2(\Gamma)}.
\end{aligned}$$

In fact, one can apply Lemma 2.4(ii)(iii) to bound the first two terms on the right-hand side. The last term is treated by $\|n - n_h\|_{L^\infty(\Gamma_h)} \le Ch$. Finally, because the mid-point formulas are exact for linear functions, one obtains

$$\begin{aligned}|\langle v \cdot n_h, \tilde{\lambda}\rangle_h - c_h(v \cdot n_h, \tilde{\lambda})| &= \left| \sum_{S \in \mathcal{S}_h} \int_{\Gamma_h} v \cdot n_h(\tilde{\lambda} - \tilde{\lambda}(m_s))\, ds \right| \\ &\le Ch\|v\|_{L^1(\Gamma_h)}\|\lambda\|_{W^{1,\infty}(\Gamma)}.\end{aligned}$$

This completes the proof. □

proof of Proposition 2.1. We add the following three identities:

$$\begin{aligned}a_h(\tilde{u} - u_h, v_h) &= a(u, P_h v_h) - a_h(u_h, v_h) + a_{\Omega_h \setminus \Omega}(\tilde{u}, v_h) - a_{\Omega \setminus \Omega_h}(u, P_h v_h), \\ b_h(v_h, \tilde{p} - p_h) &= b(P_h v_h, p) - b_h(v_h, p_h) + b_{\Omega_h \setminus \Omega}(v_h, \tilde{p}) - b_{\Omega \setminus \Omega_h}(P_h v_h, p), \\ c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h) &= c(P_h v_h \cdot n, \tilde{\lambda}) - c_h(v_h \cdot n_h, \lambda_h) + c_h(v_h \cdot n_h, \tilde{\lambda}) - c(P_h v \cdot n, \tilde{\lambda}).\end{aligned}$$

Then, from $(2.3)_1$ and $(2.7)_1$ we deduce that

$$\begin{aligned}&a_h(\tilde{u} - u_h, v_h) + b_h(v_h, \tilde{p} - p_h) + c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h) \\ &= (\tilde{f}, P_h v_h)_{\Omega \triangle \Omega_h} + a_{\Omega_h \setminus \Omega}(\tilde{u}, v_h) - a_{\Omega \setminus \Omega_h}(u, P_h v_h) \\ &\qquad + b_{\Omega_h \setminus \Omega}(v_h, \tilde{p}) - b_{\Omega \setminus \Omega_h}(P_h v_h, p), \\ &\qquad + c_h(v_h \cdot n_h, \tilde{\lambda}) - c(P_h v \cdot n, \tilde{\lambda}).\end{aligned}$$

Now Lemmas 2.1, 2.2, 2.3 and 2.5 conclude (2.8). □

2.5 Error estimate

We change the additive constant of $\tilde{p}$ in such a way that $\int_{\Omega_h}(\tilde{p} - p_h)\, dx = 0$. Let $I_h\tilde{u}$ and $\Pi_h\tilde{p}$ be a Lagrange interpolation of $\tilde{u}$ and $L^2(\Omega_h)$-projection of $\tilde{p}$ respectively.

Lemma 2.6. *Let $S \in \mathcal{S}_h$ be arbitrary.*
(i) When $d = 2$, we have $|n(m_S) - n_h(m_S)| \le Ch^2$.
(ii) When $d = 3$, if $u \in W^{2,\infty}(\tilde{\Omega})$ satisfies $\operatorname{div} u = 0$ in $\tilde{\Omega}$ and $u \cdot n = 0$ on Γ, then $|I_h\tilde{u} \cdot n_h(m_S)| \le Ch^2\|u\|_{W^{2,\infty}(\tilde{\Omega})}$.

Proof. (i) This follows from a Taylor expansion of ϕ.

(ii) For simplicity we assume that Ω is convex. Then, for each $S \in S_h$, the plane containing S, denoted by P_S, divide Ω into exactly two parts. We denote the one which contains $\pi(S)$ by G_S. One sees that $\partial G_S = (\Gamma \cap G_S) \cup$

$(P_S \cap G_S) =: \tilde{S} \cup S^*$. By the assumption, $\int_{S^*} u \cdot n_h \, ds = -\int_{\tilde{S}} u \cdot n \, ds = 0$. Since the barycenter formula is exact for linear functions, it follows that

$$\begin{aligned} I_h u \cdot n_h(m_S) &= \frac{1}{|S|} \int_S I_h u \cdot n_h \, ds \\ &= \frac{1}{|S|} \int_{S^*} (I_h u - u) \cdot n_h \, ds + \frac{1}{|S|} \int_{S^* \setminus S} I_h u \cdot n_h \, ds. \end{aligned}$$

The first term is bounded by an interpolation estimate. For the second term, noting that $|S| = Ch_S^2$, $|S^* \setminus S| = Ch_S^3$ and that $\|n - n_h\|_{L^\infty(S)} \le Ch_S$, one obtains the desired bound. □

Theorem 2.3. *Let (u, p) and (u_h, p_h) be solutions of (2.2) and (2.4), respectively, for $g = h = 0$. We assume $f \in L^3(\Omega)^d$ and $(u, p) \in W^{2,\infty}(\Omega)^d \times W^{1,\infty}(\Omega)$. Then we obtain*

$$\|\tilde{u} - u_h\|_{H^1(\Omega_h)^d} + \|\tilde{p} - p_h\|_{L^2(\Omega_h)} \le C(\tilde{u}, \tilde{p})(h + \sqrt{\epsilon} + \frac{h^2}{\sqrt{\epsilon}}),$$

where $(\tilde{u}, \tilde{p})$ is any $W^{2,\infty} \times W^{1,\infty}$ extension of (u, p) to $\tilde{\Omega}$. In particular, if $\epsilon = O(h^2)$, then the error is of $O(h)$.

Proof. Let $v_h = I_h \tilde{u}$. It is obvious that $\|\tilde{u} - u_h\|_{H^1(\Omega_h)^d} \le C(\tilde{u})h + \|v_h - u_h\|_{H^1(\Omega_h)^d}$. By (2.5) one has

$$\begin{aligned} &C\|v_h - u_h\|^2_{H^1(\Omega_h)^d} \le a_h(v_h - u_h, v_h - u_h) \\ &= a_h(v_h - \tilde{u}, v_h - u_h) \\ &\quad + a_h(\tilde{u} - u_h, v_h - u_h) + b_h(v_h - u_h, \tilde{p} - p_h) + c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h) \\ &\quad - b_h(v_h - u_h, \tilde{p} - p_h) \\ &\quad - c_h((v_h - u_h) \cdot n_h, \tilde{\lambda} - \lambda_h). \end{aligned} \tag{2.9}$$

Let us majorize each term on the right-hand side. The first line is easily estimated by $C(\tilde{u})h\|v_h - u_h\|_{H^1(\Omega_h)^d}$. Proposition 2.1 tells us that the second line is bounded by $C(\tilde{u}, \tilde{p})h\|v_h - u_h\|_{H^1(\Omega_h)^d}$.

For the third line, it follows that

$$\begin{aligned} &- b_h(v_h - u_h, \tilde{p} - p_h) + d_h(p_h - q_h, p_h - q_h) \\ &= b_h(\tilde{u} - v_h, \tilde{p} - p_h) + b_h(u_h, \tilde{p} - p_h) + d_h(p_h, p_h - q_h) - d_h(q_h, p_h - q_h) \\ &= b_h(\tilde{u} - v_h, \tilde{p} - p_h) + b_h(u_h, \tilde{p} - q_h) - d_h(q_h, p_h - q_h) \\ &= b_h(\tilde{u} - v_h, \tilde{p} - p_h) + b_h(u_h - \tilde{u}, \tilde{p} - q_h) - \gamma h(\nabla q_h, h\nabla(p_h - q_h)), \end{aligned} \tag{2.10}$$

where we have used $\operatorname{div}\tilde{u} = 0$ and $(2.4)_2$. We combine Proposition 2.1, test functions being in $\mathring{V}_h$, with (2.6) to obtain

$$\|\tilde{p}-p_h\|_{L^2(\Omega_h)} \le C(\tilde{u},\tilde{p})h+C(\tilde{u},\tilde{p})\|v_h-u_h\|_{H^1(\Omega_h)^d}+\gamma Ch\|\nabla(q_h-p_h)\|_{L^2(\Omega_h)} \tag{2.11}$$

We conclude from (2.10) and (2.11) that

$$-b_h(v_h - u_h, \tilde{p} - p_h) \le C(\tilde{u},\tilde{p})(h^2 + h\|v_h - u_h\|_{H^1(\Omega_h)^d}). \tag{2.12}$$

For the fourth line, it follows that

$$\begin{aligned} &- c_h((v_h - u_h)\cdot n_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda} - \lambda_h, \tilde{\lambda} - \lambda_h) \\ = &- c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h) + c_h(u_h \cdot n_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda} - \lambda_h, \tilde{\lambda} - \lambda_h) \\ = &- c_h(v_h \cdot n_h, \tilde{\lambda} - \lambda_h) + \epsilon c_h(\tilde{\lambda}, \tilde{\lambda} - \lambda_h), \end{aligned}$$

where we have used $(2.7)_3$. Applying Lemma 2.6 we get

$$-c_h((v_h - u_h)\cdot n_h, \tilde{\lambda} - \lambda_h) \le C(\tilde{u},\tilde{\lambda})(h^4/\epsilon + \epsilon). \tag{2.13}$$

The desired estimate now follows from (2.9), (2.12) and (2.13). □

2.6 Numerical results

• *Example 1: two-dimensional test.*
Let Ω be the unit disk $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$. We employ the exact solution $u = (-y(x^2+y^2), x(x^2+y^2))^T$, $p = 8xy$, and try to see numerical solutions computed by our scheme (2.4) reproduce the exact one. The numerical test is implemented with the software `FreeFEM++`.

The result is reported in Table 2.1, where we compare the convergence behavior of our scheme with that for the Dirichlet boundary problem (N means the division number of the circle). The P1/P1 element with $\gamma = 0.1$ is used, and the penalty parameter is chosen as $\epsilon = 0.1h^2$.. We see that the two results are comparable and infer that our scheme for a 2D slip boundary condition is equipped with a good accuracy.

• *Example 2: three-dimensional test.*
This time Ω is the unit ball $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$, and we take the exact solution $u = (10x^2yz(y-z), 10y^2zx(z-x), 10z^2xy(x-y))$, $p = 10xyz(x+y+z)$. The numerical test is implemented with the software `FEniCS`. We again use the P1/P1 element with $\gamma = 0.1$ and $\epsilon = 0.1h^2$.

The result is shown in Table 2.2. Although it is not so good as in the Dirichlet boundary problem, the rate of convergence is $O(h)$ and is consistent with Theorem 2.3.

Remark 2.1. For the solution of system of linear equations, we employ UMFPACK, a direct solver for sparse matrices when $d = 2$, and GMRES when $d = 3$. With the use of GMRES or BICG-stab when $d = 2$ for smaller ϵ, we have experienced non-convergence.

Table 2.1: Convergence behavior of the H^1-error for velocity in Example 1

N	h	$\|u - u_h^{\mathrm{Dir}}\|_{H^1(\Omega_h)^d}$	rate	$\|u - u_h^{\mathrm{Slip}}\|_{H^1(\Omega_h)^d}$	rate
32	0.316	0.485	—	0.493	—
64	0.165	0.240	1.09	0.239	1.12
128	0.078	0.118	0.94	0.118	0.94
256	0.045	0.058	1.30	0.058	1.29
512	0.023	0.029	1.02	0.029	1.01

Table 2.2: Convergence behavior of the H^1-error for velocity in Example 2

N	h	$\|u - u_h^{\mathrm{Dir}}\|_{H^1(\Omega_h)^d}$	rate	$\|u - u_h^{\mathrm{Slip}}\|_{H^1(\Omega_h)^d}$	rate
8	0.240	0.454	—	0.860	—
16	0.117	0.228	0.96	0.310	1.42
32	0.062	0.114	1.09	0.166	0.98

3 Generalized Robin boundary condition[3]

We consider

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \alpha u - \beta \Delta_\Gamma u = h & \text{on } \Gamma, \end{cases} \tag{3.1}$$

where $\alpha, \beta > 0$ are constants and Δ_Γ denotes the Laplace-Beltrami operator. This can be regarded as a simplified problem of the Stokes case:

$$\begin{cases} -\nu\Delta u + \nabla p = f, \quad \operatorname{div} u = 0 & \text{in } \Omega, \\ \sigma(u,p)n + \alpha u - \beta \operatorname{div}_\Gamma \Pi_\Gamma(u) + \beta\kappa\Pi_\Gamma(u)n = h & \text{on } \Gamma, \end{cases}$$

which describes a stationary version of a reduced-order model for a fluid-structure interaction problem [2]. Here, $\operatorname{div}_\Gamma$ is the surface divergence operator; $\kappa = \operatorname{div}_\Gamma n$ is the mean curvature of Γ; $\Pi_\Gamma(u) = \lambda \operatorname{div}_\Gamma u\, I + \mu(\nabla_\Gamma u +$

[3]This study is based on a joint work with C.M. Colciago (EPFL), L. Dedè (EPFL) and A. Quarteroni (EPFL and MOX).

$\nabla_\Gamma u^T$) denotes the membrane stress tensor, where $\lambda, \mu > 0$ are Lamé constants and ∇_Γ means the surface gradient.

Let us present an idea which could be used to solve (3.1). Let $A : \phi \mapsto \frac{\partial u}{\partial n}$ be a DtN operator, where ϕ is a given function on Γ and u solves $-\Delta u = f$ in Ω, $u = \phi$ on Γ. Next we define $B : \psi \mapsto u$, where ψ is given on Γ and u solves $\alpha u - \Delta_\Gamma u = h - \psi$ on Γ. Then problem (3.1) is rewritten as a fixed-point problem $BA\phi = \phi$, which could be solved by iterative methods. This strategy separates the equations in Ω and on Γ which constitute the transmission problem (3.1).

We, however, would like to propose another method which seems more direct and simpler. By the integration-by-parts formulas in Ω and on Γ, problem (3.1) admits the following weak formulation:

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Gamma (\alpha uv + \beta \nabla_\Gamma u \cdot \nabla_\Gamma v) \, ds = \int_\Omega fv \, dx + \int_\Gamma hv \, ds, \quad (3.2)$$

for all test functions v. For this weak form to be well-defined, one finds that $V = \{v \in H^1(\Omega) \mid v|_\Gamma \in H^1(\Gamma)\}$ is a suitable function space to work with. In fact, because the bilinear form defined by the left-hand side is coercive on V, we can immediately adapt the celebrated Lax-Milgram theorem to (3.2) to show its well-posedness. Moreover, we can establish, with V-related spaces, regularity and convergence of the finite element method, namely,

Theorem 3.1. *(i) Let $f \in H^1(\Omega)'$ and $h \in H^1(\Gamma)'$. Then there exists a unique weak solution u of (3.1).*
(ii) For $m \geq 2$, if $\Gamma \in C^{m-1,1}$, $f \in H^{m-2}(\Omega)$, $h \in H^{m-2}(\Gamma)$, then $u \in H^m(\Omega)$ and $u|_\Gamma \in H^m(\Gamma)$.
(iii) Let u_h be a P_k finite element solution to (3.2), where the subscript h means the mesh size. Then we have $\|u - u_h\|_V \leq Ch^{\min\{k,m-1\}}(\|u\|_{H^m(\Omega)} + \|u\|_{H^m(\Gamma)})$.

For the details, see our preprint [8].

Our idea is to combine the equation $(3.1)_1$ and the transmission condition $(3.1)_2$ into one equation (3.2), without separating them. Then everything is treated linearly and there is no need for iterative methods to solve the problem. The idea could be applied to a more general boundary value problem such as

$$\begin{cases} -\mathcal{L}_\Omega u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n_{\mathcal{L}_\Omega}} - \alpha \mathcal{L}_\Gamma u = h & \text{on } \Gamma, \end{cases}$$

where $\mathcal{L}_\Omega$ and $\mathcal{L}_\Gamma$ denotes elliptic operators defined in Ω and Γ respectively, $\alpha > 0$, and $\frac{\partial u}{\partial n_{\mathcal{L}_\Omega}}$ means the conormal derivative.

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