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AN INDEFINITE SUPERLINEAR ELLIPTIC EQUATION WITH A NONLINEAR BOUNDARY CONDITION OF SUBLINEAR TYPE

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ABSTRACT. We investigate an indefinite superlinear elliptic equation coupled with a sublinear Neumann boundary condition depending on a positive parameter λ. We establish a global multiplicity result for positive solutions of this concave-convex problem in the spirit of Ambrosetti-Brezis-Cerami and obtain their asymptotic profiles as $\lambda\to 0^{+}$. Furthermore, we discuss the existence of a global subcontinuum of positive solutions bifurcating from the trivial solutions. Our arguments are based on a bifurcation analysis, a comparison principle, variational techniques, and a topological method.

1. INTRODUCTION AND STATEMENTS OF MAIN RESULTS

Let Ω be a bounded domain of $\mathbb{R}^{N}$ $(N\geq 2)$ with smooth boundary $\partial\Omega$. In this paper we consider the following nonlinear elliptic problem

$$\begin{cases} -\Delta u=a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}}=\lambda|u|^{q-2}u & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda})$$

where

- $\Delta=\sum_{j=1}^{N}\frac{\partial^{2}}{\partial x_{j}^{2}}$ is the usual Laplacian in $\mathbb{R}^{N}$,
- $\lambda>0$,
- $1<q<2<p<\infty$,
- $a\in C^{\alpha}(\overline{\Omega})$ with $\alpha\in(0,1)$,
- $\mathbf{n}$ is the unit outer normal to the boundary $\partial\Omega$.

A function $u\in X:=H^{1}(\Omega)$ is said to be a *weak solution* of (P_{λ}) if it satisfies

$$\int_{\Omega}\nabla u\nabla w-\int_{\Omega}a|u|^{p-2}uw-\lambda\int_{\partial\Omega}|u|^{q-2}uw=0, \quad \forall w\in X.$$

A weak solution u of (P_{λ}) is said to be *nontrivial and non-negative* if it satisfies $u\geq 0$ and $u\not\equiv 0$. Under the condition

$$p\leq 2^{*}=\frac{2N}{N-2} \quad \text{if } N>2, \qquad (1.1)$$

we shall prove that such solutions are strictly positive on $\overline{\Omega}$ (Proposition 2.1) and belong to $C^{2+\theta}(\overline{\Omega})$ for some $\theta\in(0,1)$ (Remark 2.2). To this end, we use the weak maximum principle [12] to deduce that any nontrivial non-negative weak solution u of (P_{λ}) is strictly positive in Ω. In addition, by making good use of a comparison principle [16, Proposition A.1], we shall prove that u is positive on the whole of $\overline{\Omega}$. Finally, a bootstrap argument will provide $u\in C^{2+\theta}(\overline{\Omega})$ for some $\theta\in(0,1)$, so that u is a (*classical*) *positive solution*. Note that

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the standard boundary point lemma (as in [14]) can not be applied directly to nontrivial non-negative weak solutions of (P_λ).

The purpose of this paper is to study existence, non-existence, and multiplicity of positive solutions of (P_λ), as well as their asymptotic properties as the parameter λ approaches 0. It is promptly seen that (P_λ) has no positive solution if $a \geq 0$. More precisely, we shall see that (P_λ) has a positive solution only if $\int_\Omega a < 0$ (cf. Proposition 2.3). This condition is known to be necessary for the existence of positive solutions of problems with Neumann boundary conditions at least since the work of Bandle-Pozio-Tesei [3]. In this paper we focus on the case where a changes sign.

In view of the condition $1 < q < 2 < p$, we note that if a changes sign then (P_λ) belongs to the class of concave-convex type problems with nonlinear boundary conditions. The main reference on concave-convex type problems is the work of Ambrosetti-Brezis-Cerami [2], which deals with

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $1 < q < 2 < p$. Under the condition (1.1) the authors proved a *global multiplicity result*, namely, the existence of some $\Lambda > 0$ such that (1.2) has at least two positive solutions for $\lambda \in (0, \Lambda)$, at least one positive solution for $\lambda = \Lambda$, and no positive solution for $\lambda > \Lambda$. In addition, they analysed the asymptotic behavior of the solutions as $\lambda \to 0^+$. Tarfulea [21] considered a similar problem with an indefinite weight and a Neumann boundary condition, namely,

$$\begin{cases} -\Delta u = \lambda |u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $a \in C(\overline{\Omega})$. He proved that $\int_\Omega a < 0$ is a necessary and sufficient condition for the existence of a positive solution of (1.3). Making use of the sub-supersolutions technique, he has also shown the existence of $\Lambda > 0$ such that problem (1.3) has at least one positive solution for $\lambda < \Lambda$ which converges to 0 in $L^\infty(\Omega)$ as $\lambda \to 0^+$, and no positive solution for $\lambda > \Lambda$. Garcia-Azorero, Peral, and Rossi [10] have considered the problem

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda |u|^{q-2}u & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

By means of a variational approach, they proved that if $1 < q < 2 < p$ and $p < 2^*$ when $N > 2$, then there exists $\Lambda_0 > 0$ such that (1.4) has infinitely many nontrivial weak solutions for $0 < \lambda < \Lambda$. Moreover, they have also proved that if $1 < q < 2$ and $p = 2^*$ when $N > 2$ then there exists $\Lambda_1 > 0$ such that (1.4) has at least two positive solutions for $\lambda < \Lambda_1$, at least one positive solution for $\lambda = \Lambda_1$, and no positive solution for $\lambda > \Lambda_1$.

When a changes sign we shall prove a global multiplicity result in the style of Ambrosetti-Brezis-Cerami result. However, in doing so we shall encounter some particular difficulties. First of all, the obtention of a first solution by the sub-supersolution method seems difficult since the existence of a strict supersolution of (P_λ) for $\lambda > 0$ small is not evident at all. As a matter of fact, in [21] the author shows that this is a rather delicate issue. Another difficulty in this case is related to the variational structure: note that unlike in problems with Dirichlet boundary conditions, the left-hand side of (P_λ) lacks coercivity, since the term $\int_\Omega |\nabla u|^2$ does not correspond to $\|u\|^2$ in X. This sort of problems has been considered in [15, 16] for other kinds of nonlinearities and we shall use a similar approach here to prove existence results for (P_λ). This approach is based on the Nehari manifold method, which is known to be useful when dealing with elliptic problems with powerlike nonlinearities and sign-changing weights. Brown and Wu [5] used this method to deal with the problem

$$\begin{cases} -\Delta u = \lambda m(x)|u|^{q-2}u + a(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where m, a are smooth functions which are positive somewhere in Ω. We refer also to Brown [4] for a combination of sublinear and linear terms and to Wu [23] for a problem with a nonlinear boundary condition.

Whenever $\int_\Omega a < 0$ we set

$$c^* = \left(\frac{|\partial\Omega|}{-\int_\Omega a} \right)^{\frac{1}{p-q}}. \tag{1.6}$$

We also set

$$\overline{\lambda} = \sup\{\lambda > 0 : (P_\lambda) \text{ has a positive solution}\}.$$

Let us recall that a positive solution u of (P_λ) is said to be *asymptotically stable* (respect. *unstable*) if $\gamma_1(\lambda, u) > 0$ (respect. < 0), where $\gamma_1(\lambda, u)$ is the smallest eigenvalue of the linearized eigenvalue problem at u, namely,

$$\begin{cases} -\Delta\phi = (p-1)a(x)u^{p-2}\phi + \gamma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda(q-1)u^{q-2}\phi + \gamma\phi & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

In addition, u is said *weakly stable* if $\gamma_1(\lambda, u) \geq 0$.

We state now our main result:

Theorem 1.1.

(1) (P_λ) *has a positive solution for* $\lambda > 0$ *sufficiently small if*

$$\int_\Omega a < 0. \tag{1.8}$$

Conversely, if (P_λ) *has a positive solution for some* $\lambda > 0$ *then* (1.8) *is satisfied.*

(2) Assume (1.8). *Then the following assertions hold:*

(a) $0 < \overline{\lambda} \leq \infty$ *and* (P_λ) *has a minimal positive solution* $\underline{u}_\lambda$ *for* $\lambda \in (0, \overline{\lambda})$, *i.e. any positive solution* u *of* (P_λ) *satisfies* $\underline{u}_\lambda \leq u$ *in* $\overline{\Omega}$. *Furthermore* $\underline{u}_\lambda$ *has the following properties:*

(i) $\lambda \mapsto \underline{u}_\lambda(x)$ *is strictly increasing in* $(0, \overline{\lambda})$.

(ii) $\underline{u}_\lambda$ *is asymptotically stable for every* $\lambda \in (0, \overline{\lambda})$.

(iii) $\lambda \mapsto \underline{u}_\lambda$ *is* C^∞ *from* $(0, \overline{\lambda})$ *to* $C^{2+\alpha}(\overline{\Omega})$.

(iv) $\underline{u}_\lambda \to 0$ *and* $\lambda^{-\frac{1}{p-q}}\underline{u}_\lambda \to c^*$ *in* $C^{2+\alpha}(\overline{\Omega})$ *as* $\lambda \to 0^+$.

(b) Assume (1.1). *If* $\overline{\lambda} < \infty$ *then* (P_λ) *has a minimal positive solution* $\underline{u}_{\overline{\lambda}}$ *for* $\lambda = \overline{\lambda}$. *Moreover the solution set around* $(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ *consists of a* C^∞*-curve* $(\lambda(s), u(s)) \in \mathbb{R} \times C^{2+\alpha}(\overline{\Omega})$ *of positive solutions, which is parametrized by* $s \in (-\varepsilon, \varepsilon)$, *for some* $\varepsilon > 0$, *and satisfies* $(\lambda(0), u(0)) = (\overline{\lambda}, \underline{u}_{\overline{\lambda}})$, $\lambda'(0) = 0$, $\lambda''(0) < 0$, *and* $u(s) = \underline{u}_{\overline{\lambda}} + s\phi_1 + z(s)$, *where* ϕ_1 *is a positive eigenfunction associated to the smallest eigenvalue* $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}})$ *of* (1.7), *and* $z(0) = z'(0) = 0$. *Finally, the lower branch* $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, *is asymptotically stable, whereas the upper branch* $(\lambda(s), u(s))$, $s \in (0, \varepsilon)$, *is unstable.*

(c) Assume $p < 2^*$ if $N > 2$. Then the set of positive solutions of (P_λ) for $\lambda > 0$ around $(\lambda, u) = (0,0)$ in $\mathbb{R} \times X$ consists of $\{(\lambda, \underline{u}_\lambda)\}$.

(d) Bifurcation from zero of (P_λ) never occurs at any $\lambda > 0$, i.e. there is no sequence (λ_n, u_n) of positive solutions of (P_λ) such that $u_n \to 0$ in $C(\overline{\Omega})$ and $\lambda_n \to \lambda^* > 0$.

(e) (P_λ) has at most one weakly stable positive solution.

Remark 1.2.

(1) Under conditions (1.8) and (1.1), by the left-continuity of $\underline{u}_\lambda$ [1, Theorem 20.3], we infer that $(\lambda(s), u(s))$, $s \in (-\varepsilon, 0)$, in Theorem 1.1(2)(b) represents minimal positive solutions. In particular, the mapping $\lambda \mapsto \underline{u}_\lambda$ is continuous from $(0, \overline{\lambda}]$ into $C(\overline{\Omega})$.

(2) Under (1.1) the minimal positive solution $\underline{u}_{\overline{\lambda}}$ obtained for $\lambda = \overline{\lambda}$ satisfies in addition $\gamma_1(\overline{\lambda}, \underline{u}_{\overline{\lambda}}) = 0$.

(3) In accordance with Theorem 1.1, if $\overline{\lambda} < \infty$ then the set of bifurcating positive solutions at $(0,0)$ is represented in Figure 1.

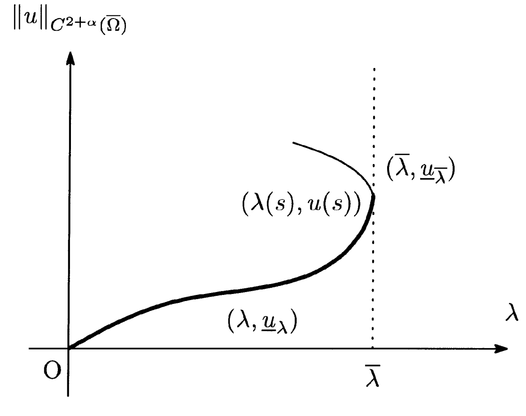

FIGURE 1. A smooth positive solution curve when $\overline{\lambda} < \infty$.

Theorem 1.3. *Assume that a changes sign and (1.8) is satisfied. Then the following assertions hold:*

(1) If $a > 0$ on $\partial\Omega$ then $\overline{\lambda} < \infty$.

(2) Assume in addition $p < \frac{2N}{N-2}$ if $N > 2$. Then (P_λ) has a second positive solution $u_{2,\lambda}$ satisfying $\underline{u}_\lambda < u_{2,\lambda}$ in $\overline{\Omega}$ for every $\lambda \in (0, \overline{\lambda})$. Moreover, $u_{2,\lambda}$ is unstable for every $\lambda \in (0, \overline{\lambda})$ and there exists $\lambda_n \to 0^+$ such that $u_{2,\lambda_n} \to u_{2,0}$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$ as $n \to \infty$, where $u_{2,0}$ is a positive solution of

$$\begin{cases} -\Delta u = a(x) u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Remark 1.4. In accordance with Theorems 1.1 and 1.3, a possible positive solutions set of (P_λ) is depicted in Figure 2.

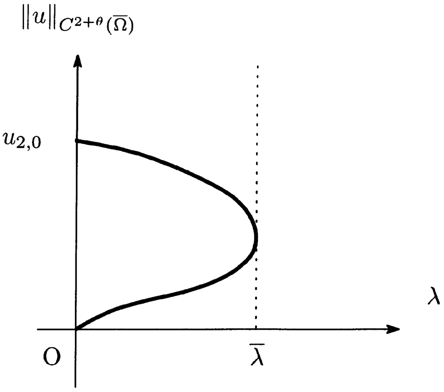

FIGURE 2. A possible bifurcation diagram for (P_λ) when $\int_\Omega a < 0$ and a changes sign.

The outline of this article is the following: in Section 2 we show that nontrivial non-negative solutions of (P_λ) are positive on $\overline{\Omega}$ and that (1.8) is a necessary condition for the existence of positive solutions of (P_λ). In Section 3 we carry out a bifurcation analysis to discuss existence of bifurcating positive solutions to the region $\lambda > 0$ at $(0, 0)$. In Section 4 we use variational techniques to discuss multiplicity of positive solutions and their asymptotic profiles as $\lambda \to 0^+$. Finally, in Section 5 we discuss existence of a unbounded subcontinuum of positive solutions of (P_λ) in $\lambda \in \mathbb{R}$. The details of the proofs of Theorems 1.1 and 1.3 appear in [18].

2. Positivity and a necessary condition

We begin this section showing the positivity on $\partial\Omega$ of nontrivial non-negative weak solutions of (P_λ). As mentioned in the Introduction, the boundary point lemma is difficult to apply directly to (P_λ) since $0 < q - 1 < 1$. However, by making good use of a comparison principle for a class of nonlinear boundary value problems of concave type, we are able to show that nontrivial non-negative weak solutions of (P_λ) with $\lambda > 0$ are positive on the whole of $\overline{\Omega}$:

Proposition 2.1. *Assume* (1.1). *Then any nontrivial non-negative weak solution of* (P_λ) *is strictly positive on* $\overline{\Omega}$.

Proof. First of all, we note that under (1.1) any nontrivial non-negative weak solution belongs to $X \cap C^\theta(\overline{\Omega})$ for some $\theta \in (0, 1)$, cf. Rossi [20, Theorem 2.2]. We consider the following boundary value problem of concave type

$$\begin{cases} -\Delta u = -a_0 u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda u^{q-1} & \text{on } \partial\Omega, \end{cases}$$

where $a^- = a^+ - a$, and $a_0 = \sup_\Omega a^-$. A nontrivial non-negative weak solution u_λ of (P_λ) for $\lambda > 0$ satisfies

$$\int_\Omega \nabla u_\lambda \nabla w + a_0 \int_\Omega u_\lambda^{p-1} w - \lambda \int_{\partial\Omega} u_\lambda^{q-1} w \geq 0,$$

for every $w \in X$ such that $w \geq 0$. On the other hand, we consider the following eigenvalue problem:

$$\begin{cases} -\Delta\phi = \sigma\phi & \text{in } \Omega, \\ \frac{\partial\phi}{\partial\mathbf{n}} = \lambda\phi & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is easy to see that for any $\lambda > 0$ this problem has a smallest eigenvalue σ_1, which is negative. So, using a positive eigenfunction ϕ_1 associated to σ_1, we deduce that if ε is sufficiently small then $\varepsilon\phi_1$ satisfies

$$\int_\Omega \nabla(\varepsilon\phi_1)\nabla w + a_0\int_\Omega (\varepsilon\phi_1)^{p-1}w - \lambda\int_{\partial\Omega}(\varepsilon\phi_1)^{q-1}w \leq 0,$$

for every $w \in X$ such that $w \geq 0$. By the comparison principle [16, Proposition A.1], we infer that $\varepsilon\phi_1 \leq u_\lambda$ on $\overline{\Omega}$. In particular, we have $0 < \varepsilon\phi_1 \leq u_\lambda$ on $\partial\Omega$. □

Remark 2.2. Thanks to the positivity property, the assumption $a \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, allows us to prove that under (1.1), any nontrivial non-negative weak solution u of (P_λ) belongs to $C^{2+\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$, by elliptic regularity. Proposition 2.1 will be needed in a combination argument of bifurcation and variational techniques, since our purpose in this paper is to discuss the existence of a classical solution of (P_λ) which is positive in the closure $\overline{\Omega}$.

We prove now that (1.8) is a necessary condition for (P_λ) to have a positive solution for some $\lambda > 0$.

Proposition 2.3. *If (P_λ) has a positive solution for some $\lambda > 0$ then (1.8) is satisfied.*

Proof. Let u be a positive solution of (P_λ). Then we have

$$\int_\Omega \nabla u\nabla w - \int_\Omega a u^{p-1}w - \lambda\int_{\partial\Omega} u^{q-1}w = 0, \quad \forall w \in X.$$

Since $u^{1-p} \in X$, we deduce that

$$\int_\Omega a = \int_\Omega \nabla u\nabla\left(u^{1-p}\right) - \lambda\int_{\partial\Omega} u^{q-1}\frac{1}{u^{p-1}} = (1-p)\int_\Omega u^{-p}|\nabla u|^2 - \lambda\int_{\partial\Omega} u^{-(p-q)} < 0,$$

as desired. □

Remark 2.4. By virtue of Proposition 2.1, under (1.1) we can prove that Proposition 2.3 holds for nontrivial non-negative weak solutions of (P_λ) .

3. A bifurcation analysis

Throughout this section, we assume (1.8). As we shall discuss bifurcation from the zero solution, the following result will be useful (see [17] for a similar proof):

Lemma 3.1. *Assume (1.1). If (λ_n, u_n) are weak solutions of (P_λ) with (λ_n) bounded then $\|u_n\| \to 0$ if and only if $\|u_n\|_{C(\overline{\Omega})} \to 0$.*

We use now a bifurcation technique to show the existence of at least one positive solution of (P_λ) for $\lambda > 0$ close to 0. To this end, we consider positive solutions of the following problem, which corresponds to (P_λ) after the change of variable $w = \lambda^{-\frac{1}{p-q}}u$:

$$\begin{cases} -\Delta w = \lambda^{\frac{p-2}{p-q}} a w^{p-1} & \text{in } \Omega, \\ \frac{\partial w}{\partial n} = \lambda^{\frac{p-2}{p-q}} w^{q-1} & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Proposition 3.2.

(1) If (3.1) *has a sequence of positive solutions* (λ_n, w_n) *such that* $\lambda_n \to 0^+$, $w_n \to c$ *in* $C(\overline{\Omega})$ *and* c *is a positive constant then* $c = c^*$, *where* c^* *is given by* (1.6).

(2) Conversely, (3.1) *has for* $|\lambda|$ *sufficiently small a secondary bifurcation branch* $(\lambda, w(\lambda))$ *of positive solutions (parametrized by* λ*) emanating from the trivial line* $\{(0, c) : c$ *is a positive constant*$\}$ *at* $(0, c^*)$ *and such that, for* $0 < \theta \le \alpha$, *the mapping* $\lambda \mapsto w(\lambda) \in C^{2+\theta}(\overline{\Omega})$ *is continuous. Moreover, the set* $\{(\lambda, w)\}$ *of positive solutions of* (3.1) *around* $(\lambda, w) = (0, c^*)$ *consists of the union*

$$\{(0, c) : c \text{ is a positive constant}, \ |c - c^*| \le \delta_1\} \cup \{(\lambda, w(\lambda)) : |\lambda| \le \delta_1\}$$

for some $\delta_1 > 0$.

Proof. The proof is similar to the one of [16, Proposition 5.3]:

(1) Let w_n be positive solutions of (3.1) with $\lambda = \lambda_n$, where $\lambda_n \to 0^+$. By the Green formula we have

$$\int_\Omega a w_n^{p-1} + \int_{\partial\Omega} w_n^{q-1} = 0.$$

Passing to the limit as $n \to \infty$, we deduce the desired conclusion.

(2) We reduce (3.1) to a bifurcation equation in $\mathbb{R}^2$ by the Lyapunov-Schmidt procedure: we use the usual orthogonal decomposition

$$L^2(\Omega) = \mathbb{R} \oplus V,$$

where $V = \{v \in L^2(\Omega) : \int_\Omega v = 0\}$ and the projection $Q : L^2(\Omega) \to V$, given by

$$v = Qu = u - \frac{1}{|\Omega|}\int_\Omega u.$$

The problem of finding a positive solution of (3.1) reduces then to the following problems:

$$\begin{cases} -\Delta v + \frac{\mu}{|\Omega|}\int_{\partial\Omega}(t+v)^{q-1} = \mu Q[a(t+v)^{p-1}] & \text{in } \Omega, \\ \frac{\partial v}{\partial \mathbf{n}} = \mu(t+v)^{q-1} & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

$$\mu\left(\int_\Omega a(t+v)^{p-1} + \int_{\partial\Omega}(t+v)^{q-1}\right) = 0, \tag{3.3}$$

where $\mu = \lambda^{\frac{p-2}{p-q}}$, $t = \frac{1}{|\Omega|}\int_\Omega w$, and $v = w - t$. To solve (3.2) in the framework of Hölder spaces, we set

$$Y = \left\{v \in C^{2+\theta}(\overline{\Omega}) : \int_\Omega v = 0\right\},$$

$$Z = \left\{(\phi, \psi) \in C^\theta(\overline{\Omega}) \times C^{1+\theta}(\partial\Omega) : \int_\Omega \phi + \int_{\partial\Omega} \psi = 0\right\}.$$

Let $c > 0$ be a constant and $U \subset \mathbb{R} \times \mathbb{R} \times Y$ be a small neighborhood of $(0, c, 0)$. We consider the nonlinear mapping $F : U \to Z$ given by

$$F(\mu, t, v) = \left(-\Delta v - \mu Q[a(t+v)^{p-1}] + \frac{\mu}{|\Omega|}\int_{\partial\Omega}(t+v)^{q-1}, \ \frac{\partial v}{\partial \mathbf{n}} - \mu(t+v)^{q-1}\right).$$

The Fréchet derivative F_v of F with respect to v at $(0, c, 0)$ is given by the formula

$$F_v(0, c, 0)v = \left(-\Delta v, \ \frac{\partial v}{\partial \mathbf{n}}\right).$$

Since $F_v(0,c,0)$ is a homeomorphism, the implicit function theorem implies that the set $F(\mu,t,v)=0$ around $(0,c,0)$ consists of a unique C^∞ function $v=v(\mu,t)$ in a neighborhood of $(\mu,t)=(0,c)$ and satisfying $v(0,c)=0$. Now, plugging $v(\mu,t)$ in (3.3), we obtain the bifurcation equation

$$\Phi(\mu,t)=\int_\Omega a(t+v(\mu,t))^{p-1}+\int_{\partial\Omega}(t+v(\mu,t))^{q-1}=0, \quad \text{for } (\mu,t)\simeq(0,c).$$

It is clear that $\Phi(0,c^*)=0$. Differentiating Φ with respect to t at $(0,c^*)$ we get

$$\begin{aligned}\Phi_t(0,c^*)&=\int_\Omega a(p-1)(c^*+v(0,c^*))^{p-2}(1+v_t(0,c^*))\\&\quad+\int_{\partial\Omega}(q-1)(c^*+v(0,c^*))^{q-2}(1+v_t(0,c^*))\\&=(p-1)(c^*)^{p-2}\int_\Omega a(1+v_t(0,c^*))+(q-1)(c^*)^{q-2}\int_{\partial\Omega}(1+v_t(0,c^*)).\end{aligned}$$

Differentiating now (3.2) with respect to t, and plugging $(\mu,t)=(0,c^*)$ therein, we have $v_t(0,c^*)=0$. Hence

$$\Phi_t(0,c^*)=(p-1)(c^*)^{p-2}\left(\int_\Omega a\right)+(q-1)(c^*)^{q-2}|\partial\Omega|=|\partial\Omega|^{\frac{p-2}{p-q}}\left(-\int_\Omega a\right)^{\frac{2-q}{p-q}}(q-p)<0.$$

By the implicit function theorem, the function $w(\lambda)=t(\mu)+v(\mu,t(\mu))$ with $\mu=\lambda^{\frac{p-2}{p-q}}$ satisfies the desired assertion.

□

By considering the transform $u(\lambda)=\lambda^{\frac{1}{p-q}}w(\lambda)$, we get the following result:

Proposition 3.3. *Let $0<\theta\le\alpha$ and $w(\lambda)$ be given by Proposition 3.2. If $\lambda>0$ is sufficiently small then $u(\lambda)=\lambda^{\frac{1}{p-q}}w(\lambda)$ is a positive solution of (P_λ) which satisfies $\lambda^{-\frac{1}{p-q}}u(\lambda)\to c^*$ in $C^{2+\theta}(\overline{\Omega})$ as $\lambda\to 0^+$. In particular, $u(\lambda)\to 0$ in $C^{2+\theta}(\overline{\Omega})$ as $\lambda\to 0^+$.*

4. Variational approach

We associate to (P_λ) the C^1 functional

$$I_\lambda(u):=\frac{1}{2}E(u)-\frac{1}{p}A(u)-\frac{\lambda}{q}B(u), \quad u\in X,$$

where

$$E(u)=\int_\Omega|\nabla u|^2, \quad A(u)=\int_\Omega a(x)|u|^p, \quad \text{and} \quad B(u)=\int_{\partial\Omega}|u|^q.$$

Let us recall that $X=H^1(\Omega)$ is equipped with the usual norm $\|u\|=\left[\int_\Omega\left(|\nabla u|^2+u^2\right)\right]^{\frac{1}{2}}$. We denote by $\rightharpoonup$ the weak convergence in X.

The following result will be used repeatedly in this section.

Lemma 4.1.

(1) *If (u_n) is a sequence such that $u_n\rightharpoonup u_0$ in X and $\limsup_n E(u_n)\le 0$ then u_0 is a constant and $u_n\longrightarrow u_0$ in X.*
(2) *Assume (1.8). If $v\neq 0$ and $A(v)\ge 0$, then v is not a constant.*

Proof.

(1) Since $u_n \rightharpoonup u_0$ in X and E is weakly lower semicontinuous, we have $E(u_0) \leq \liminf_n E(u_n)$, so that

$$0 \leq E(u_0) \leq \liminf_n E(u_n) \leq \limsup_n E(u_n) \leq 0.$$

Hence, $E(u_0) = 0$, which implies that u_0 is a constant. Assume $u_n \not\to u_0$ in X. Then $E(u_0) < \limsup_n E(u_n) \leq 0$, which is a contradiction. Therefore $u_n \to u_0$ in X.

(2) If $v_0 \neq 0$ is a constant then $0 \leq A(v_0) = |v_0|^p \int_\Omega a < 0$, a contradiction.

□

Now, in addition to (1.1) and (1.8), we assume that a changes sign. Moreover, we assume $p < \frac{2N}{N-2}$ if $N > 2$. We shall prove the existence of two positive solutions of (P_λ) for $0 < \lambda < \overline{\lambda}$ and characterize their asymptotic profiles as $\lambda \to 0^+$. To this end, we use the Nehari manifold and the fibering maps associated to I_λ. Let us introduce some useful subsets of X:

$$E^+ = \{u \in X : E(u) > 0\},$$
$$A^\pm = \{u \in X : A(u) \gtrless 0\}, \quad A_0 = \{u \in X : A(u) = 0\}, \quad A_0^\pm = A^\pm \cup A_0,$$
$$B^+ = \{u \in X : B(u) > 0\}.$$

The Nehari manifold associated to I_λ is given by

$$N_\lambda := \{u \in X \setminus \{0\} : \langle I_\lambda'(u), u\rangle = 0\} = \{u \in X \setminus \{0\} : E(u) = A(u) + \lambda B(u)\}.$$

We shall use the splitting

$$N_\lambda = N_\lambda^+ \cup N_\lambda^- \cup N_\lambda^0,$$

where

$$N_\lambda^\pm := \{u \in N_\lambda : \langle J_\lambda'(u), u\rangle \gtrless 0\} = \left\{u \in N_\lambda : E(u) \lessgtr \lambda \frac{p-q}{p-2} B(u)\right\}$$
$$= \left\{u \in N_\lambda : E(u) \gtrless \frac{p-q}{2-q} A(u)\right\},$$

and

$$N_\lambda^0 = \{u \in N_\lambda : \langle J_\lambda'(u), u\rangle = 0\}.$$

Note that any nontrivial weak solution of (P_λ) belongs to N_λ. Furthermore, it follows from the implicit function theorem that $N_\lambda \setminus N_\lambda^0$ is a C^1 manifold and every critical point of the restriction of I_λ to this manifold is a critical point of I_λ (see for instance [6, Theorem 2.3]).

To analyse the structure of $N_\lambda^\pm$, we consider the fibering maps corresponding to I_λ for $u \neq 0$ in the following way:

$$j_u(t) := I_\lambda(tu) = \frac{t^2}{2}E(u) - \frac{t^p}{p}A(u) - \lambda \frac{t^q}{q}B(u), \quad t > 0.$$

It is easy to see that

$$j_u'(1) = 0 \lessgtr j_u''(1) \iff u \in N_\lambda^\pm,$$

and more generally,

$$j_u'(t) = 0 \lessgtr j_u''(t) \iff tu \in N_\lambda^\pm.$$

Having this characterisation in mind, we look for conditions under which j_u has a critical point. Set

$$i_u(t) := t^{-q} j_u(t) = \frac{t^{2-q}}{2}E(u) - \frac{t^{p-q}}{p}A(u) - \lambda B(u), \quad t > 0.$$

Let $u \in E^+ \cap A^+ \cap B^+$. Then i_u has a global maximum $i_u(t^*)$ at some $t^* > 0$, and moreover, t^* is unique. If $i_u(t^*) > 0$, then j_u has a global maximum which is positive and a local minimum which is negative. Moreover, these are the only critical points of j_u. We

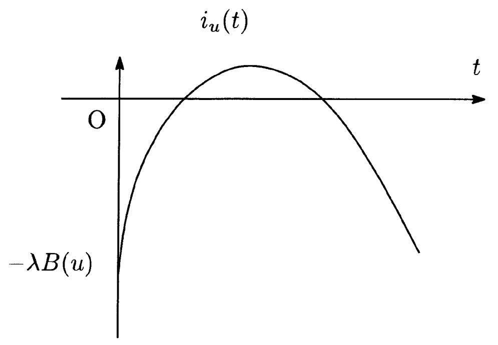

FIGURE 3. The case $i_u(t^*) > 0$.

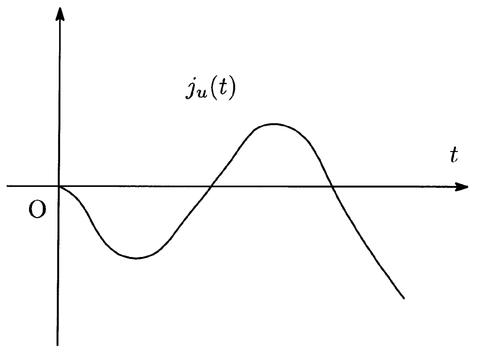

FIGURE 4. A case of j_u having a global maximum and a local minimum.

shall require a condition on λ that provides $i_u(t^*) > 0$. Note that

$$i_u'(t) = \frac{2-q}{2} t^{1-q} E(u) - \frac{p-q}{p} t^{p-q-1} A(u) = 0$$

if and only if

$$t = t^* := \left(\frac{p(2-q)E(u)}{2(p-q)A(u)} \right)^{\frac{1}{p-2}}.$$

Moreover

$$i_u(t^*) = \frac{p-2}{2(p-q)} \left(\frac{p(2-q)}{2(p-q)} \right)^{\frac{2-q}{p-2}} \frac{E(u)^{\frac{p-q}{p-2}}}{A(u)^{\frac{2-q}{p-2}}} - \frac{\lambda}{q} B(u) > 0$$

if and only if

$$0 < \lambda^{\frac{p-2}{p-q}} < C_{pq} \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}, \tag{4.1}$$

where $C_{pq} = \left(\frac{q(p-2)}{2(p-q)} \right)^{\frac{p-2}{p-q}} \left(\frac{p(2-q)}{2(p-q)} \right)^{\frac{2-q}{p-q}}$. Note that $F(u) = \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}$ satisfies $F(tu) = F(u)$ for $t > 0$, i.e. F is homogeneous of order 0.

We deduce then the following result, which provides sufficient conditions for the existence of critical points of j_u:

Proposition 4.2. *The following assertions hold:*

(1) If either $u \in E^+ \cap A_0^- \cap B^+$ or $u \in A^- \cap B^+$ then $j_u(t)$ has a negative global minimum at some $t_1 > 0$, i.e. $j_u'(t_1) = 0 < j_u''(t_1)$, and $j_u(t) > j_u(t_1)$ for $t \neq t_1$. Moreover, t_1 is the unique critical point of j_u and $j_u(t) \to \infty$ as $t \to \infty$.

(2) If $u \in E^+ \cap A^+ \cap B_0$ then $j_u(t)$ has a positive global maximum at some $t_2 > 0$, i.e. $j_u'(t_2) = 0 > j_u''(t_2)$ and $j_u(t) < j_u(t_2)$ for $t \neq t_1$. Moreover, t_2 is the unique critical point of j_u and $j_u(t) \to -\infty$ as $t \to \infty$.

(3) Assume (1.8). *If we set*

$$\lambda_0^{\frac{p-2}{p-q}} = \inf\{E(u) : u \in E^+ \cap A^+ \cap B^+,\ C_{pq}^{-1} B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}} = 1\}, \tag{4.2}$$

then $\lambda_0 > 0$. Moreover, for any $0 < \lambda < \lambda_0$ and $u \in E^+ \cap A^+ \cap B^+$ the map j_u has a negative local minimum at $t_1 > 0$ and a positive global maximum at $t_2 > t_1$. Furthermore, t_1, t_2 are the only critical points of j_u and $j_u(t) \to -\infty$ as $t \to \infty$ (see Figure 4).

Proof. Assertions (1) and (2) are straightforward from the definition of j_u. We prove now assertion (3). First, we show that $\lambda_0 > 0$. Assume $\lambda_0 = 0$, so that we can choose $u_n \in E^+ \cap A^+ \cap B^+$ satisfying

$$E(u_n) \longrightarrow 0, \quad \text{and} \quad C_{pq}^{-1} B(u_n)^{\frac{p-2}{p-q}} A(u_n)^{\frac{2-q}{p-q}} = 1.$$

If (u_n) is bounded in X then we may assume that $u_n \rightharpoonup u_0$ for some $u_0 \in X$ and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows from Lemma 4.1(1) that u_0 is a constant and $u_n \to u_0$ in X. From $u_n \in A^+$ we deduce that $u_0 \in A_0^+$. In addition, we have

$$C_{pq}^{-1} B(u_0)^{\frac{p-2}{p-q}} A(u_0)^{\frac{2-q}{p-q}} = 1,$$

so that $u_0 \not\equiv 0$. From Lemma 4.1(2) we get a contradiction.

Let us assume now that $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^p(\Omega)$. Since $E(v_n) \to 0$ and $v_n \in A^+$, we have $v_n \to v_0$ in X, v_0 is a constant, and $v_0 \in A_0^+$. In particular, $\|v_0\| = 1$, i.e. $v_0 \not\equiv 0$. Lemma 4.1 provides again a contradiction.

Finally, for any $u \in E^+ \cap A^+ \cap B^+$ we have

$$\lambda_0^{\frac{p-2}{p-q}} \leq C_{pq} \frac{E(u)}{B(u)^{\frac{p-2}{p-q}} A(u)^{\frac{2-q}{p-q}}}.$$

Thus, if $0 < \lambda < \lambda_0$ then $i_u(t^*) > 0$ from (4.1). This completes the proof of assertion (3). □

Proposition 4.3. *We have, for $0 < \lambda < \lambda_0$:*

(1) N_λ^0 is empty.
(2) $N_\lambda^\pm$ are non-empty.

Proof.

(1) From Proposition 4.2 it follows that there is no $t > 0$ such that $j_u'(t) = j_u''(t) = 0$, i.e. N_λ^0 is empty.

(2) Consider the following eigenvalue problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Under (1.8) it is known that this problem has a unique positive principal eigenvalue λ_N with a positive principal eigenfunction φ_N. From $\varphi_N > 0$ on $\partial\Omega$ and the fact that φ_N is not a constant, we deduce that $\varphi_N \in E^+ \cap A^+ \cap B^+$. Since $0 < \lambda < \lambda_0$, Proposition 4.2(3) provides the desired conclusion.

□

The following result provides some properties of N_λ^+:

Lemma 4.4. *Let $0 < \lambda < \lambda_0$. Then, we have the following two assertions:*

(1) N_λ^+ is bounded in X.
(2) $I_\lambda(u) < 0$ for any $u \in N_\lambda^+$ and moreover $t > 1$ if $j_u'(t) > 0$.

Proof.

(1) Assume $(u_n) \subset N_\lambda^+$ and $\|u_n\| \to \infty$. Set $v_n = \frac{u_n}{\|u_n\|}$. It follows that $\|v_n\| = 1$, so we may assume that $v_n \rightharpoonup v_0$, $B(v_n)$ is bounded, and $v_n \to v_0$ in $L^p(\Omega)$ (implying $A(v) \to A(v_0)$). Since $u_n \in N_\lambda^+$, we see that

$$E(v_n) < \lambda \frac{p-q}{p-2} B(v_n)\|u_n\|^{q-2},$$

and thus $\limsup_n E(v_n) \leq 0$. Lemma 4.1(1) yields that v_0 is a constant and $v_n \to v_0$ in X. Consequently, $\|v_0\| = 1$, and v_0 is a non-zero constant. However, since $u_n \in N_\lambda$, we see that

$$0 \leq E(u_n) = A(u_n) + \lambda B(u_n),$$

and it follows that

$$0 \leq A(v_n) + \lambda B(v_n)\|u_n\|^{q-p}.$$

Passing to the limit as $n \to \infty$, we deduce $0 \leq A(v_0)$. Lemma 4.1(2) leads us to a contradiction. Therefore N_λ^+ is bounded in X.

(2) Let $u \in N_\lambda^+$. Then

$$0 \leq E(u) < \lambda \frac{p-q}{p-2} B(u),$$

so that $B(u) > 0$. First we assume that u is not a constant. In this case $E(u) > 0$. If $A(u) > 0$ then Proposition 4.2(3) tells us that $I_\lambda(u) < 0$ and $t > 1$ if $j_u'(t) > 0$. On the other hand, if $A(u) \leq 0$ then $u \in E^+ \cap A_0^- \cap B^+$. So Proposition 4.2(1) gives the same conclusion. Assume now that u is a constant. In this case $A(u) = |u|^p \int_\Omega a < 0$, so that $u \in A^- \cap B^+$. Proposition 4.2(1) again yields the desired conclusion.

□

Next we prove that $\inf_{N_\lambda^+} I_\lambda$ is achieved by some $u_{1,\lambda} > 0$ for $\lambda \in (0, \lambda_0)$, which implies the estimate $\overline{\lambda} \geq \lambda_0$. Furthermore, we can show that $u_{1,\lambda}$ is in fact the minimal positive solution of (P_λ) for $\lambda > 0$ sufficiently small.

Proposition 4.5. *For any $0 < \lambda < \lambda_0$, there exists $u_{1,\lambda}$ such that $I_\lambda(u_{1,\lambda}) = \min_{N_\lambda^+} I_\lambda$. In particular, $u_{1,\lambda}$ is a positive solution of (P_λ).*

Proof. Let $0 < \lambda < \lambda_0$. We consider a minimizing sequence $(u_n) \subset N_\lambda^+$, i.e.

$$I_\lambda(u_n) \longrightarrow \inf_{N_\lambda^+} I_\lambda < 0.$$

Since (u_n) is bounded in X, we may assume that $u_n \rightharpoonup u_0$, $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$I_\lambda(u_0) \leq \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda(u) < 0,$$

so that $u_0 \not\equiv 0$. We claim that $u_n \to u_0$ in X. We have two possibilities:

- If u_0 is a constant, then $0 = E(u_0) \leq \lambda^{\frac{p-q}{p-2}} B(u_0)$. If $B(u_0) = 0$ then $u_0 = 0$ on $\partial\Omega$, so that $u_0 = 0$ in Ω, which yields a contradiction. Hence $B(u_0) > 0$. In this case, we have $A(u_0) = |u_0|^p \int_\Omega a < 0$, so that $u_0 \in A^- \cap B^+$. Proposition 4.2(1) implies that $t_1 u_0 \in N_\lambda^+$ and j_{u_0} has a global minimum at t_1. If $u_n \not\to u_0$ then

  $$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \tag{4.3}$$

  which is a contradiction since $t_1 u_0 \in N_\lambda^+$. Therefore $u_n \to u_0$.

- If u_0 is not a constant then $E(u_0) > 0$ and $B(u_0) > 0$. So either $u_0 \in E^+ \cap A_0^- \cap B^+$ or $u_0 \in E^+ \cap A^+ \cap B^+$. In the first case, j_{u_0} has a global minimum point t_1 and we can argue as in the previous case. In the second case, since $0 < \lambda < \lambda_0$, Proposition 4.2 yields that $t_1 u_0 \in N_\lambda^+$ for some $t_1 > 0$. Assume $u_n \not\to u_0$. If $1 < t_1$ then we have again

  $$I_\lambda(t_1 u_0) = j_{u_0}(t_1) \leq j_{u_0}(1) < \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^+} I_\lambda, \tag{4.4}$$

  If $t_1 < 1$ then $j'_{u_n}(t_1) < 0$ for every n, so that $j'_{u_0}(t_1) < \liminf j'_{u_n}(t_1) \leq 0$, which is a contradiction. Therefore $u_n \to u_0$.

Now, since $u_n \to u_0$ we have $j'_{u_0}(1) = 0 \leq j''_{u_0}(1)$. But $j''_{u_0}(1) = 0$ is impossible by Proposition 4.3(1). Thus $u_0 \in N_\lambda^+$ and $I_\lambda(u_0) = \inf_{N_\lambda^+} I_\lambda$. □

Remark 4.6. From Proposition 4.5 we derive $\overline{\lambda} \geq \lambda_0$.

Next we obtain a second nontrivial non-negative weak solution of (P_λ), which achieves $\inf_{N_\lambda^-} I_\lambda$ for $\lambda \in (0, \lambda_0)$. The following result provides some properties of N_λ^-:

Lemma 4.7. *Let $0 < \lambda < \lambda_0$. Then we have $I_\lambda(u) > 0$ for any $u \in N_\lambda^-$. Moreover $t < 1$ if $j'_u(t) > 0$.*

Proof. If $u \in N_\lambda^-$ then $A(u) > 0$ and u is not a constant from Lemma 4.1(2). It follows immediately that $E(u) > 0$. If $B(u) > 0$, then, by Proposition 4.2(3), we have that $I_\lambda(u) > 0$ and $t < 1$ if $j'_u(t) > 0$. If $B(u) = 0$, then Proposition 4.2(2) provides the same conclusion. □

Proposition 4.8. *For any $\lambda \in (0, \lambda_0)$, there exists $u_{2,\lambda}$ such that $I_\lambda(u_{2,\lambda}) = \min_{N_\lambda^-} I_\lambda$. In particular, $u_{2,\lambda}$ is a positive solution of (P_λ).*

Proof. Since $I_\lambda(u) > 0$ for $u \in N_\lambda^-$, we can choose $u_n \in N_\lambda^-$ such that

$$I_\lambda(u_n) \longrightarrow \inf_{N_\lambda^-} I_\lambda(u) \geq 0.$$

We claim that (u_n) is bounded in X. Indeed, there exists $C > 0$ such that $I_\lambda(u_n) \le C$. Since $u_n \in N_\lambda$, we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) \le C.$$

Assume $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$, so that $\|v_n\| = 1$. We may assume that $v_n \rightharpoonup v_0$, and $v_n \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Then, from

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_n) \le \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(v_n)\|u_n\|^{q-2} + \frac{C}{\|u_n\|^2},$$

we infer that $\limsup_n E(v_n) \le 0$. Lemma 4.1(1) yields that v_0 is a constant, and $v_n \to v_0$ in X, which implies $\|v_0\| = 1$. However, since $u_n \in N_\lambda^-$, we observe that

$$E(v_n)\|u_n\|^{2-p} < \frac{p-q}{2-q} A(v_n).$$

Passing to the limit $n \to \infty$, we get $0 \le A(v_0)$, which is contradictory by Lemma 4.1(2). Hence (u_n) is bounded. We may then assume that $u_n \rightharpoonup u_0$, and $u_n \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. We claim that $u_n \to u_0$ in X. Assume $u_n \not\to u_0$. Then, since $u_n \in N_\lambda^-$, we deduce

$$0 \le E(u_0) < \liminf_n E(u_n) \le \liminf_n \frac{p-q}{2-q} A(u_n) = \frac{p-q}{2-q} A(u_0).$$

This implies that u_0 is not a constant by Lemma 4.1(2), so that $E(u_0) > 0$. Since $u_0 \in E^+ \cap A^+$, Proposition 4.2 tells us that there exists $t_2 > 0$ such that $t_2 u_0 \in N_\lambda^-$. Moreover, $0 = j'_{u_0}(t_2) < \liminf_n j'_{u_n}(t_2)$, since $u_n \not\to u_0$. We deduce that $j'_{u_n}(t_2) > 0$ for n large enough. Since $u_n \in N_\lambda^-$, we have $t_2 < 1$ from Lemma 4.7. Then, we observe that

$$I_\lambda(t_2 u_0) = j_{u_0}(t_2) < \liminf_n j_{u_n}(t_2) \le \liminf_n j_{u_n}(1) = \liminf_n I_\lambda(u_n) = \inf_{N_\lambda^-} I_\lambda.$$

This is a contradiction, which implies that $u_n \to u_0$ and $I_\lambda(u_n) \to I_\lambda(u_0) = \gamma$.

Now we verify that $u_0 \ne 0$. Assume $u_0 = 0$. Then, since $u_n \in N_\lambda$, we have

$$E(v_n)\|u_n\|^{2-q} = A(v_n)\|u_n\|^{p-q} + \lambda B(v_n),$$

where $v_n = \frac{u_n}{\|u_n\|}$. We may assume again that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^q(\partial\Omega)$ and $L^p(\Omega)$. Passing to the limit as $n \to \infty$, we obtain $0 = \lambda B(v_0)$, so that $v_0 = 0$ on $\partial\Omega$. On the other hand, we observe that

$$0 < I_\lambda(u_n) = \frac{1}{2}E(u_n) - \frac{1}{p}A(u_n) - \frac{\lambda}{q}B(u_n).$$

Since $u_n \in N_\lambda$, we deduce

$$\left(\frac{1}{q} - \frac{1}{2}\right) E(v_n) \le \left(\frac{1}{q} - \frac{1}{p}\right) A(v_n)\|u_n\|^{p-2}.$$

From the assumption $u_n \to 0$ in X, it follows that $\limsup E(v_n) \le 0$. By Lemma 4.1(1) we get that v_0 is a constant, and $v_n \to v_0$ in X, so that $\|v_0\| = 1$. Since v_0 is a constant and $v_0 = 0$ on $\partial\Omega$, we have $v_0 = 0$ in Ω. This is a contradiction, as desired.

Finally, since $u_n \to u_0$ in X we have $j'_{u_0}(1) = 0 \ge j''_{u_0}(1)$. But $j''_{u_0}(1) = 0$ is impossible by Proposition 4.3(1). Thus $u_0 \in N_\lambda^-$ and $I_\lambda(u_0) = \inf_{N_\lambda^-} I_\lambda$.

□

We discuss now the asymptotic profiles of $u_{1,\lambda}, u_{2,\lambda}$ as $\lambda \to 0^+$. The following lemma is concerned with the behavior of positive solutions in N_λ^+ as $\lambda \to 0^+$:

Proposition 4.9. *If u_λ is a positive solution of (P_λ) such that $u_\lambda \in N_\lambda^+$ for $\lambda > 0$ sufficiently small then $u_\lambda \to 0$ in X as $\lambda \to 0^+$. Moreover there holds $\lambda^{-\frac{1}{p-q}} u_\lambda \to c^*$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$ as $\lambda \to 0^+$.*

Proof. First we show that u_λ remains bounded in X as $\lambda \to 0^+$. Indeed, assume that $\|u_\lambda\| \to \infty$ and set $v_\lambda = \frac{u_\lambda}{\|u_\lambda\|}$. We may then assume that for some $v_0 \in X$ we have $v_\lambda \rightharpoonup v_0$ in X, and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since $u_\lambda \in N_\lambda$, we have

$$E(v_\lambda)\|u_\lambda\|^{2-p} = A(v_\lambda) + \lambda B(v_\lambda)\|u_\lambda\|^{q-p}.$$

Passing to the limit as $\lambda \to 0^+$, we obtain $A(v_0) = 0$. From $u_\lambda \in N_\lambda^+$ we have

$$E(v_\lambda) < \lambda \frac{p-q}{p-2} B(v_\lambda)\|u_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(v_\lambda) \le 0$. By Lemma 4.1(1) we infer that v_0 is a constant and $v_\lambda \to v_0$ in X, so that $\|v_0\| = 1$, i.e. $v_0 \neq 0$. This is contradictory with Lemma 4.1(2), and therefore u_λ stays bounded in X as $\lambda \to 0^+$.

Hence we may assume that $u_\lambda \rightharpoonup u_0$ in X and $u_\lambda \to u_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_\lambda \in N_\lambda^+$, we observe that

$$E(u_\lambda) < \lambda \frac{p-q}{p-2} B(u_\lambda).$$

Passing to the limit as $\lambda \to 0^+$, we get $\limsup_\lambda E(u_\lambda) \le 0$. Lemma 4.1(2) provides that u_0 is a constant and $u_\lambda \to u_0$ in X. Since $u_\lambda \in N_\lambda$, we have

$$E(u_\lambda) = A(u_\lambda) + \lambda B(u_\lambda).$$

which implies $A(u_0) = 0$, so that $u_0 = 0$ from Lemma 4.1(2). Therefore $u_\lambda \to 0$ in X as $\lambda \to 0^+$.

Now we obtain the asymptotic profile of u_λ as $\lambda \to 0^+$. Let $w_\lambda = \lambda^{-\frac{1}{p-q}} u_\lambda$. We claim that w_λ remains bounded in X as $\lambda \to 0^+$. Indeed, since $u_\lambda \in N_\lambda^+$, we have

$$E(w_\lambda) < \frac{p-q}{p-2} \lambda^{\frac{p-2}{p-q}} B(w_\lambda).$$

Let us assume that $\|w_\lambda\| \to \infty$ and set $\psi_\lambda = \frac{w_\lambda}{\|w_\lambda\|}$. We may assume that $\psi_\lambda \rightharpoonup \psi_0$ and $\psi_\lambda \to \psi_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that

$$E(\psi_\lambda) < \frac{p-q}{p-2} \lambda^{\frac{p-2}{p-q}} B(\psi_\lambda)\|w_\lambda\|^{q-2},$$

so that $\limsup_\lambda E(\psi_\lambda) \le 0$. By Lemma 4.1(1) we infer that ψ_0 is a constant and $\psi_\lambda \to \psi_0$ in X. On the other hand, from $u_\lambda \in N_\lambda$ it follows that

$$0 \le A(u_\lambda) + \lambda B(u_\lambda),$$

so that

$$-B(\psi_\lambda)\|w_\lambda\|^{q-p} \le A(\psi_\lambda).$$

Taking the limit as $\lambda \to 0^+$ we get $0 \le A(\psi_0)$, which contradicts Lemma 4.1(2). Hence w_λ stays bounded in X as $\lambda \to 0^+$ and we may assume that $w_\lambda \rightharpoonup w_0$ in X and $w_\lambda \to w_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. It follows that $\limsup_\lambda E(w_\lambda) \le 0$, and by Lemma 4.1(1) we get that w_0 is a constant and $w_\lambda \to w_0$ in X.

It remains to show that $w_0 = c^*$. We note that w_λ satisfies

$$\int_\Omega \nabla w_\lambda \nabla w - \lambda^{\frac{p-2}{p-q}} \int_\Omega a w_\lambda^{p-1} w - \lambda^{\frac{p-2}{p-q}} \int_{\partial\Omega} w_\lambda^{q-1} w = 0, \quad \forall w \in X, \tag{4.5}$$

since u_λ is a weak solution of (P_λ). Taking $w = 1$, we see that

$$\int_\Omega a w_\lambda^{p-1} + \int_{\partial\Omega} w_\lambda^{q-1} = 0.$$

Passing to the limit as $\lambda \to 0^+$, we see that either $w_0 = 0$ or $w_0 = c^*$. However, taking $w = \frac{1}{w_\lambda^{q-1}}$ in (4.5) we obtain

$$0 > -(q-1)\int_\Omega w_\lambda^{-q}|\nabla w_\lambda|^2 = \lambda^{\frac{p-2}{p-q}}\left(\int_\Omega a w_\lambda^{p-q} + |\partial\Omega|\right),$$

so that

$$|\partial\Omega| < -\int_\Omega a w_\lambda^{p-q}.$$

It is clear then that $w_0 \neq 0$, i.e. $w_0 = c^*$, and consequently we obtain $\lambda^{-\frac{1}{p-q}} u_\lambda \to c^*$ in X. By a standard bootstrap argument, we get the desired conclusion. □

We turn now to the asymptotic behavior of $u_{2,\lambda}$ as $\lambda \to 0^+$. We shall prove initially that solutions in N_λ^- are bounded away from zero as $\lambda \to 0^+$:

Lemma 4.10. *If u_λ is a positive solution of (P_λ) such that $u_\lambda \in N_\lambda^-$ for $\lambda > 0$ sufficiently small then $\|u_\lambda\| \geq C$ for some constant $C > 0$ as $\lambda \to 0^+$.*

Proof. Assume by contradiction that (u_n) is a sequence of positive solutions of (P_{λ_n}) with $\lambda_n \to 0^+$, $u_n \in N_{\lambda_n}^-$ and $\|u_n\| \to 0$. Then, since $u_n \in N_{\lambda_n}^-$, we deduce

$$E(v_n) < \frac{p-q}{2-q} A(v_n)\|u_n\|^{p-2},$$

where $v_n = \frac{u_n}{\|u_n\|}$. We may assume that $v_n \rightharpoonup v_0$ in X and $v_n \to v_0$ in $L^p(\Omega)$. It follows that $\limsup E(v_n) \leq 0$. By Lemma 4.1(1) we get that v_0 is a constant and $v_n \to v_0$ in X, so that $\|v_0\| = 1$. On the other hand, we see that $A(v_n) > 0$, since $u_n \in N_{\lambda_n}^-$. We obtain then $0 \leq A(v_0)$, which is a contradiction with Lemma 4.1(2). □

We prove now that $u_{2,\lambda}$ is bounded in X as $\lambda \to 0^+$:

Lemma 4.11. *There exists a constant $C > 0$ such that $C^{-1} \leq \|u_{2,\lambda}\| \leq C$ as $\lambda \to 0^+$.*

Proof. By Lemma 4.10 we know that $\|u_{2,\lambda}\| \geq C^{-1}$ for some $C > 0$ as $\lambda \to 0^+$. We show now that $u_{2,\lambda}$ is bounded in X as $\lambda \to 0^+$. First, we show that there exists a constant $C_1 > 0$ such that $I_\lambda(u_{2,\lambda}) \leq C_1$ for every $\lambda \in (0, \lambda_0)$. To this end, we consider the following eigenvalue problem with the Dirichlet boundary condition.

$$\begin{cases} -\Delta\varphi = \lambda a(x)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

We denote by φ_D a positive eigenfunction associated with the positive principal eigenvalue λ_D. Multiplying (4.6) by φ_D^{p-1} we see that $\varphi_D \in A^+$. Thus $\varphi_D \in E^+ \cap A^+ \cap B_0$ and

$$j_{\varphi_D}(t) = \frac{t^2}{2}E(\varphi_D) - \frac{t^p}{p}A(\varphi_D),$$

so that j_{φ_D} has a global maximum at some $t_2 > 0$, which implies $t_2\varphi_D \in N_\lambda^-$. Moreover, neither j_{φ_D} nor $t_2\varphi_D$ depend on $\lambda \in (0, \lambda_0)$. Let $C_1 = j_{\varphi_D}(t_2) = I_\lambda(t_2\varphi_D) > 0$. Since $t_2\varphi_D \in N_\lambda^-$, we deduce that $I_\lambda(u_{2,\lambda}) \leq C_1$.

Assume now that $\|u_{2,\lambda}\| \to \infty$ as $\lambda \to 0^+$ and set $v_\lambda = \frac{u_{2,\lambda}}{\|u_{2,\lambda}\|}$. We may assume that $v_\lambda \rightharpoonup v_0$ and $v_\lambda \to v_0$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Since

$$0 \le E(u_{2,\lambda}) < \frac{p-q}{2-q} A(u_{2,\lambda}),$$

it follows that $A(v_\lambda) > 0$. Passing to the limit as $\lambda \to 0^+$, we get $A(v_0) \ge 0$. However, we will see that the condition $I_\lambda(u_{2,\lambda}) \le C_1$ leads us to a contradiction. Indeed, since $u_{2,\lambda} \in N_\lambda$, we deduce

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_{2,\lambda}) - \left(\frac{1}{q} - \frac{1}{p}\right) \lambda B(u_{2,\lambda}) = I_\lambda(u_{2,\lambda}) \le C_1.$$

Hence

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(v_\lambda) \le \left(\frac{1}{q} - \frac{1}{p}\right) \lambda B(v_\lambda) \|u_{2,\lambda}\|^{q-2} + C_1 \|u_{2,\lambda}\|^{-2}.$$

Letting $\lambda \to 0^+$ we obtain $\limsup_\lambda E(v_\lambda) \le 0$, and by Lemma 4.1 we infer that v_0 is a constant and $v_\lambda \to v_0$ in X. In particular, $\|v_0\| = 1$, which contradicts Lemma 4.1(2). The proof is now complete. □

We establish now (up to a subsequence) the precise limiting behavior of $u_{2,\lambda}$:

Proposition 4.12. *There exists a sequence $\lambda_n \to 0^+$ such that $u_{2,\lambda_n} \to u_{2,0}$ in $C^{2+\theta}(\overline{\Omega})$ for any $\theta \in (0, \alpha)$, where $u_{2,0}$ is a positive solution of* (1.9).

Proof. Since $u_{2,\lambda}$ stays bounded in X as $\lambda \to 0^+$, up to a subsequence, we have $u_{2,\lambda} \rightharpoonup u_{2,0}$, and $u_{2,\lambda} \to u_{2,0}$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$ as $\lambda \to 0^+$. Since $u_{2,\lambda}$ is a weak solution of (P_λ), we have

$$\int_\Omega \nabla u_{2,\lambda} \nabla w - \int_\Omega a u_{2,\lambda}^{p-1} w - \lambda \int_{\partial\Omega} u_{2,\lambda}^{q-1} w = 0, \quad \forall w \in X.$$

Letting $\lambda \to 0^+$, we get

$$\int_\Omega \nabla u_{2,0} \nabla w - \int_\Omega a u_{2,0}^{p-1} w = 0, \quad \forall w \in X,$$

i.e. $u_{2,0}$ is a non-negative weak solution of (1.9). If $u_{2,0} \equiv 0$ then, from

$$E(u_{2,\lambda}) < \frac{p-q}{2-q} A(u_{2,\lambda}) \quad \text{and} \quad A(u_{2,0}) = 0,$$

we deduce that $\limsup_\lambda E(u_{2,\lambda}) \le 0$. By Lemma 4.1(1) we infer that u_0 is a constant and $u_{2,\lambda} \to u_{2,0} = 0$ in X, which contradicts Lemma 4.11.

Finally, since $u_{2,0} \in C^{2+\alpha}(\overline{\Omega})$, and $u_{2,0} > 0$ in $\overline{\Omega}$ by the weak maximum principle and the boundary point lemma, we infer that $u_{2,0}$ is a positive solution of (1.9). By a standard bootstrap argument, we obtain the desired conclusion. □

We shall consider now the Palais-Smale condition for I_λ. Let us recall that I_λ satisfies the Palais-Smale condition if any sequence such that $(I_\lambda(u_n))$ is bounded and $I'_\lambda(u_n) \to 0$ in X' has a convergent subsequence.

Proposition 4.13. *I_λ satisfies the Palais-Smale condition for any $\lambda > 0$..*

Proof. Let (u_n) be a Palais-Smale sequence for I_λ. Then

$$(I_\lambda(u_n)) \text{ is bounded} \quad \text{and} \quad I'_\lambda(u_n)\phi = o(1)\|\phi\| \quad \forall \phi \in X.$$

In particular, we have

$$\left(\frac{1}{2} - \frac{1}{p}\right) E(u_n) - \lambda \left(\frac{1}{q} - \frac{1}{p}\right) B(u_n) = I_\lambda(u_n) - \frac{1}{p} I'_\lambda(u_n) u_n \le c + o(1)\|u_n\| \qquad (4.7)$$

for some constant c. Assume that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. Then we may assume that $v_n \rightharpoonup v$ in X and $v_n \to v$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. From

$$\int_\Omega \nabla u_n \nabla \phi - a(x)|u_n|^{p-2}u_n\phi - \lambda \int_{\partial\Omega} |u_n|^{q-2}u_n\phi = o(1)\|\phi\|, \quad \forall \phi \in X \tag{4.8}$$

we get, dividing it by $\|u_n\|^{p-1}$,

$$\int_\Omega a(x)|v_n|^{p-2}v_n\phi \to 0 \quad \forall \phi \in X$$

so that

$$\int_\Omega a(x)|v|^{p-2}v\phi = 0 \quad \forall \phi \in X.$$

This equality implies that $a|v|^{p-2}v = 0$ a.e. in Ω. Hence $av \equiv 0$. Taking now $\phi = v$ in (4.8), we obtain

$$\int_\Omega \nabla v_n \nabla v - \lambda \|u_n\|^{q-2} \int_{\partial\Omega} |v_n|^{q-2}v_n v \to 0.$$

Thus

$$\int_\Omega \nabla v_n \nabla v \to 0$$

and since $v_n \rightharpoonup v$ in X, we get $\int_\Omega |\nabla v|^2 = 0$. So v must be a constant. From $av \equiv 0$, we deduce that $v \equiv 0$. Finally, from (4.7), dividing it by $\|u_n\|^2$ we obtain $E(v_n) \to 0$. Therefore, by Lemma 4.1, we have $v_n \to 0$ in X, which contradicts $\|v_n\| = 1$.

So (u_n) must be bounded, and up to a subsequence, $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$ and $L^q(\partial\Omega)$. Taking $\phi = u_n - u$ in (4.8) we get

$$\int_\Omega |\nabla u_n|^2 \to \int_\Omega |\nabla u|^2$$

and consequently $\|u_n\|^2 \to \|u\|^2$. By the uniform convexity of X, we infer that $u_n \to u$ in X. □

We prove now a multiplicity result for positive solutions of (P_λ) for $\lambda \in (0, \overline{\lambda})$. First of all, by Proposition 4.5 or Proposition 4.8, we know that $\overline{\lambda} \geq \lambda_0 > 0$. We proceed now as in [9] to obtain a solution by the variational form of the sub-supersolution method. A version of this method for a problem with Neumann boundary conditions has been proved in [11, Theorem 3]. We shall use a slightly different version of this result, namely:

Theorem 4.14. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions such that for every $R > 0$ there exists $M = M(R) > 0$ satisfying $|f(x,s)| \leq M$ if $(x,s) \in \Omega \times [-R, R]$ and $|g(x,s)| \leq M$ if $(x,s) \in \partial\Omega \times [-R, R]$. If $\underline{u}, \overline{u} \in H^1(\Omega) \cap L^\infty(\Omega) \cap L^\infty(\partial\Omega)$ are a weak subsolution and supersolution of (P_λ), respectively, and $\underline{u} \leq \overline{u}$ a.e. in Ω then (P_λ) has a solution u satisfying*

$$I_\lambda(u) = \min\{I_\lambda(v) : \ v \in H^1(\Omega), \ \underline{u} \leq v \leq \overline{u} \text{ a.e. in } \Omega\}.$$

The proof of this result can be carried out following the proof of [11, Theorem 3]. As a matter of fact, the functional I_λ is not coercive but still bounded from below on the set

$$M := \{u \in H^1(\Omega) : \ \underline{u} \leq u \leq \overline{u} \ \text{ a.e. in } \Omega\}.$$

Let us pick $0 < \mu < \overline{\lambda}$ and prove that (P_μ) has two positive solutions. From the definition of $\overline{\lambda}$ we can take $\mu' \in (\mu, \overline{\lambda}]$ such that $(P_{\mu'})$ has a positive solution $u_{\mu'}$. Now, we make good use of the positive eigenfunction ϕ_1 associated to the smallest eigenvalue σ_1 of (2.1) to build up a suitable positive weak subsolution. We consider the smallest eigenvalue $\hat{\sigma}_1 := \sigma_1(\mu) < 0$ of (2.1) and the corresponding positive eigenfunction $\hat{\phi}_1 = \phi_1(\mu)$. Then $\varepsilon\hat{\phi}_1$ is a strict weak subsolution of (P_μ) if $\varepsilon > 0$ is sufficiently small. Moreover, we can choose $\varepsilon > 0$ such that $\varepsilon\hat{\phi}_1 \leq u_{\mu'}$. By Theorem 4.14 with $\underline{u} = \varepsilon\hat{\phi}_1$ and $\overline{u} = u_{\mu'}$, we obtain a solution u_0 of (P_μ) such that

$$I_\mu(u_0) = \min\{I_\mu(v) : \ v \in H^1(\Omega), \ \varepsilon\hat{\phi}_1 \leq v \leq u_{\mu'} \text{ a.e. in } \Omega\}.$$

In particular, $u_0 > 0$ in $\overline{\Omega}$. Moreover, by the strong maximum principle and the boundary point lemma we have $\varepsilon\hat{\phi}_1 < u_0 < u_{\mu'}$ on $\overline{\Omega}$. It follows that u_0 is a local minimizer of I_μ with respect to the $C^1(\overline{\Omega})$ topology. We may then argue as in [10, Lemma 6.4] to deduce that u_0 is a local minimizer of I_μ with respect to the $H^1(\Omega)$ topology. Now we use an argument from [9]: let $\delta > 0$ such that u_0 minimizes I_μ in $B(u_0, \delta)$ and $0 \notin B(u_0, \delta)$. If u_0 is not a strict minimizer then there exists $v_0 \in B(u_0, \delta)$, $v_0 \not\equiv 0$ such that $I_\mu(v_0) = I_\mu(u_0)$, in which case v_0 is also a local minimizer of I_μ, and consequently a solution of (P_μ). Now, if u_0 is a strict minimizer then, by [8, Theorem 5.10], we infer that for $r > 0$ sufficiently small we have

$$I_\mu(u_0) < \inf\{I_\mu(u) : \ u \in H^1(\Omega), \|u - u_0\| = r\},$$

so that I_μ has the mountain-pass geometry (note that if $w \in A^+$ then $I_\mu(tw) \to -\infty$ as $t \to \infty$). Finally, by Proposition 4.13, I_μ satisfies the Palais-Smale condition, and since I_μ is even the mountain-pass theorem provides a second positive solution of (P_μ).

5. Unbounded subcontinuum

In this section we assume (1.8) and that a changes sign. Moreover, we assume $p < \frac{2N}{N-2}$ if $N > 2$. According to a bifurcation argument developed in [17, 19] we discuss the existence of a global subcontinuum of positive solutions bifurcating from the trivial line $\{(\lambda, 0)\}$. Note that in view of the condition $q < 2$ the nonlinearity in (P_λ) is not differentiable at $u = 0$, so that we can not apply the standard local bifurcation theory [7] directly. To overcome this difficulty we investigate the existence of a global subcontinuum of positive solutions for a regularized version of (P_λ). The regularized problem is formulated as

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = \lambda|u + \epsilon|^{q-2}u & \text{on } \partial\Omega, \end{cases} \tag{$Q_{\lambda,\epsilon}$}$$

where $\epsilon > 0$. Indeed, the mapping $t \mapsto |t + \epsilon|^{q-2}t$ is smooth at $t = 0$. We remark that $(Q_{\lambda,0}) = (P_\lambda)$, which means that (P_λ) is the limiting case of $(Q_{\lambda,\epsilon})$ as $\epsilon \to 0^+$. To study the existence of bifurcation points on the trivial line $\{(\lambda, 0)\}$ for $(Q_{\lambda,\epsilon})$, we consider the linearized eigenvalue problem at $u = 0$

$$\begin{cases} -\Delta \phi = \sigma\phi & \text{in } \Omega, \\ \frac{\partial \phi}{\partial \mathbf{n}} = \lambda\epsilon^{q-2}\phi & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

This problem has a unique principal eigenvalue σ_1, which is simple. Moreover we see that $\sigma_1 > 0$ for $\lambda < 0$, $\sigma_1 = 0$ for $\lambda = 0$, and $\sigma_1 < 0$ for $\lambda > 0$. If we denote by ϕ_1 a corresponding positive eigenfunction to σ_1 then ϕ_1 is a positive constant when $\lambda = 0$.

Now we can prove the following result for $(Q_{\lambda,\epsilon})$:

Proposition 5.1. *Let $p < \frac{2N}{N-2}$ if $N > 2$, and $\epsilon > 0$. Assume* (1.8) *and that a changes sign. Then the following assertions hold:*

(1) If u_n is a positive solution of $(Q_{\lambda,\epsilon})$ for $\lambda = \lambda_n$ such that $\lambda_n \to \lambda^*$ for some $\lambda^* \in \mathbb{R}$ and $u_n \to 0$ in $C(\overline{\Omega})$ then $\lambda^* = 0$.

(2) There exists $\Lambda_\epsilon > 0$ such that $(Q_{\lambda,\epsilon})$ has no positive solutions for $\lambda \geq \Lambda_\epsilon$.

(3) The set of positive solutions of $(Q_{\lambda,\epsilon})$ around $(\lambda, u) = (0, 0)$ consists of a curve $(\lambda, u) = (\lambda(s), s(1+w(s)))$ parametrized by $s \in (0, \delta_0)$, for some $\delta_0 > 0$. In addition, $\lambda(\cdot) : [0, \delta_0) \to \mathbb{R}$ and $w(\cdot) : [0, \delta_0) \to Z = \{u \in C^{2+\alpha}(\overline{\Omega}) : \int_\Omega u = 0\}$ are continuous and satisfy $\lambda(0) = 0$, $\lambda(s) > 0$ for $s > 0$, and $w(0) = 0$. Thus bifurcation of positive solutions of $(Q_{\lambda,\epsilon})$ at $(0,0)$ to the region $\lambda > 0$ does occur.

(4) $(Q_{\lambda,\epsilon})$ has no positive solutions for $\lambda = 0$ within a neighborhood of $u = 0$ in $C(\overline{\Omega})$.

(5) The curve $(\lambda(s), s(1 + w(s)))$, $s \in [0, \delta_0)$, can be extended as a positive solution subcontinuum of $(Q_{\lambda,\epsilon})$, denoted by $\mathcal{C}_\epsilon$, so that it is unbounded in $(-\infty, \Lambda_\epsilon) \times C(\overline{\Omega})$.

Remarks on further results with $(Q_{\lambda,\epsilon})$ for $\epsilon \geq 0$ are given as follows.

Remark 5.2.

(1) Assume that an *a priori* upper bound for positive solutions for $(Q_{\lambda,\epsilon})$ exists for every $\epsilon > 0$, i.e. for any $\mu > 0$ there exists a constant $C_\epsilon > 0$ such that for any positive solution u of $(Q_{\lambda,\epsilon})$ with $|\lambda| \leq \mu$ we have

$$\|u\|_{C(\overline{\Omega})} \leq C_\epsilon, \tag{5.2}$$

Then assertions (1), (2) and (4) of Proposition 5.1 ensure that $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_\epsilon\} = (-\infty, \overline{\lambda}_\epsilon]$ for some $\overline{\lambda}_\epsilon \in (0, \Lambda_\epsilon]$. The inequality (5.2) is still an open question. We refer to [10] for *a priori* upper bounds for positive solutions of (1.4).

(2) Assertions (1), (2) and (4) in Proposition 5.1 are valid for (P_λ). Assume that (5.2) holds for $\epsilon = 0$, and moreover, C_ϵ is provided uniformly for $\epsilon \geq 0$. Then, by the topological analysis proposed by Whyburn [22, Theorem 9.1], we can deduce from Proposition 5.1 that (P_λ) has a unbounded subcontinuum $\mathcal{C}_0$ of positive solutions, bifurcating to the region $\lambda > 0$ at $(0,0)$ and satisfying $\{\lambda \in \mathbb{R} : (\lambda, u) \in \mathcal{C}_0\} = (-\infty, \overline{\lambda}]$ as described in Figure 5. This is achieved by considering the limiting behavior of $\mathcal{C}_\epsilon$ as $\epsilon \to 0^+$.

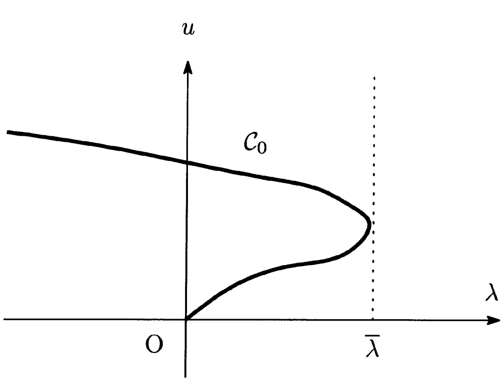

FIGURE 5. A unbounded subcontinuum of positive solutions of (P_λ) when the uniform *a priori* upper bound (5.2) with respect to $\epsilon \geq 0$ is assumed.

The proofs for the results mentioned in this section are to appear somewhere else.

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