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A generaliged automorphism group and triality groups for nonassociative algebras

In memory of Professor Susumu Okubo (1930-2015)

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Abstract

In this paper, we introduce for triality groups and study examples of the group.

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(1) Triality Group

Let A be an algebra (do not need the associativity) over a field F of charachteristic not 2 with a bi-linear product denoted by juxtaposition xy for $x, y \in A$. Suppose that a triple $g = (g_1, g_2, g_3) \in (EpiA)^3$, where $EpiA$ denote the set of epimorphisms of A equipped with $g_j(x+y) = g_jx + g_jy$ and $g_j(\alpha x) = \alpha g_j(x), \alpha \in F$, satisfies a global triality relation

$$g_j(xy) = (g_{j+1}x)(g_{j+2}y) \tag{1.1}$$

for any $x, y \in A$ and for any $j = 1, 2, 3$, Here the index j is defined by modulo 3 so that

$$g_{j\pm 3} = g_j. \tag{1.2}$$

For the second triple $g' = (g'_1, g'_2, g'_3) \in (EpiA)^3$ satisfying the same triality relation, we introduce their product componetwise by

$$gg' = (g_1g'_1, g_2g'_2, g_3g'_3). \tag{1.3}$$

Then, a set consisting of all such triples forms a group which we call the triality group of A, and write

$$\mathrm{Trig}(A) = \{g = (g_1, g_2, g_3) \in (EpiA)^3 | g_j(xy) = (g_{j+1}x)(g_{j+2}y), \forall x, y \in A, \ \forall\, j = 1, 2, 3\}. \tag{1.4}$$

Here we emphasize that this group is clearly a generalization of the automorphism group defined by

$$\text{Auto}(A) = \{g \in EpiA | g(xy) = (gx)(gy), \forall x, y \in A\}. \tag{1.5}$$

We then note that Trig(A) is invariant under actions of an alternative group A_4 (or equivalently the tetrahedral group) as follows. First, let $\phi \in End(\text{Trig}(A))$ by

$$\phi : g_1 \to g_2 \to g_3 \to g_1 \tag{1.6}$$

which satisfies $\phi^3 = id$ and leaves Eq.(1.1) invariant. Tus, Trig(A) is invariant under actions of the cyclic group Z_3 generated by ϕ. We next introduce $\tau_\mu \in End(\text{Trig}A)$ for $\mu = 1, 2, 3$ by

$$\begin{aligned} &\tau_1 : g_1 \to g_1,\ g_2 \to -g_2,\ g_3 \to -g_3 \\ &\tau_2 : g_1 \to -g_1,\ g_2 \to g_2,\ g_3 \to -g_3 \\ &\tau_3 : g_1 \to -g_1,\ g_2 \to -g_2,\ g_3 \to g_3, \end{aligned} \tag{1.7}$$

which leave Eq.(1.1) invariant again. Moreover, they satisfy

$$\tau_\mu \tau_\nu = \tau_\nu \tau_\mu,\ \tau_\mu^2 = id,\ \tau_1 \tau_2 \tau_3 = id \tag{1.8}$$

for $\mu, \nu = 1, 2, 3$, so that $(id, \tau_1, \tau_2, \tau_3)$ is isomorphic to the Klein's 4-group K_4.

Further we note

$$\phi \tau_\mu \phi^{-1} = \tau_{\mu+1},\ (\text{with } \tau_4 = \tau_1). \tag{1.9}$$

Since Z_3 and K_4 generate the alternative group A_4, this shows that A_4 is an invariant group of Trig(A).

If A is invalntive with the involution map $x \to \bar{x}$ satisfying

$$\bar{\bar{x}} = x,\ \overline{xy} = \bar{y}\bar{x}, \tag{1.10}$$

we define $\bar{Q} \in End(A)$ for any $Q \in End(A)$ by

$$\overline{Qx} = \bar{Q}\bar{x}. \tag{1.11}$$

then, taking the involution of Eq.(1.10) and letting $x \leftrightarrow \bar{y}$, we find

$$\bar{g}_j(xy) = (\bar{g}_{j+2}x)(\bar{g}_{j+1}y), \tag{1.12}$$

so that $\theta \in End(\text{Trig}(A)$ defined by

$$\theta : g_1 \to \bar{g}_2,\ g_2 \to \bar{g}_1,\ g_3 \to \bar{g}_3 \tag{1.13}$$

yields also a invariant operation of Trig(A). Moreover, we obtain

$$\phi\theta\phi = \theta,\ \theta^2 = id,\ \theta\tau_1\theta^{-1} = \tau_2,\ \theta\tau_2\theta^{-1} = \tau_1,\ \theta\tau_3\theta^{-1} = \tau_3. \tag{1.14}$$

Then, A_4 togehter with θ give the S_4-symmetry for Trig(A) with identification of

$$\phi = (1, 2, 3),\ \tau_1 = (2, 3)(1, 4)\ \tau_2 = (3.1)(2.4), \tau_3 = (1, 2)(3, 4), \theta = (1, 2) \tag{1.15}$$

in terms of the transpositions of the S_4-group.

Before going into further discussion, we note that Trig(A) for any A always contains a Klein's 4-group K_4 Let $Id \in Epi(A)$ be defined by $(Id)x = x$, for $x \in A$, and set

$$\tilde{\tau}_0 = (Id, Id, Id),\ \tilde{\tau}_1 = (Id. -Id, -Id),\ \tilde{\tau}_2 = (-Id, Id, -Id),\ \tilde{\tau}_3 = (-Id, -Id, Id). \quad (1.16)$$

Then, these 4 element of $(Epi(A)^3)$ satisfy Eq.(1.8) in view of Eq.(1.3), so that they give another Klein's 4-group. Moreover, they satisfy Eq.(1.1). Similarly, we have Trig(F) = K_4, for the simplest case of $A = F$.

In contrast to the global triality relation Eq.(1.1), we may also consider the local triality relation

$$t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y) \quad (1.17)$$

for $t_j \in End(A)$ with $t_{j\pm3} = t_j$. Analogously to Eq.(1.4), we introduce

$$s{\circ}Lrt(A) = \{t = (t_1, t_2, t_3) \in (End\ A)^3 | t_j(xy) = (t_{j+1}x)y + x(t_{j+2}y), \forall x, y \in A, \forall i = 1, 2, 3\}. \quad (1.18)$$

Then, it defines a Lie algebra now with component-wise commutation relation (]K-O,15]). Here, $s \circ Lrt(A)$ stands for symmetric Lie-related triple, which has been reformed to as stri(A) in [O.05] instead. If we set

$$t_j^{'} = \sum_{k=1}^{3} \alpha_{j-k} t_k, \ (j = 1, 2, 3) \quad (1.19)$$

for any $\alpha_j \in F$ satisfying $\alpha_{j\pm3} = \alpha_j$, then it is easy to verify that we have also

$$t^{'} = (t_1^{'}, t_2^{'}, t_3^{'}) \in s \circ Lrt(A). \quad (1.20)$$

If the exponential map $t_j \to \xi_j$ is given by

$$\xi_j = \exp(t_i) = \sum_{n=0}^{\infty} \frac{1}{n!}(t_j)^n, \quad (1.21)$$

is well defined, then we have shown in [K-O.15] that the vality of

$$\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y) \quad (1.22)$$

provided with respect to $t = (t_1, t_2, t_3) \in s \circ Lrt(A)$ and vice-versa.

Note that the existence ot the exponantial map requires the underlying field F to be at least of zero charachteristic.

We next introduce multiplication operators $L(x), R(x) \in End\ A$ by

$$L(x)y = xy\ R(x)y = yx \quad (1.23)$$

as usual. Then, Eq.(1.17) yields

$$t_j L(x) = L(x)t_{j+2} + L(t_{j+1}x), \quad (1.24a)$$

$$t_j R(y) = R(y)t_{j+1} + R(t_{j+2}y) \quad (1.24b)$$

while Eq.(1.1) gives

$$g_j L(x) = L(g_{j+1}x)g_{j+2}, \quad (1.25a)$$

$$g_jR(y) = R(g_{j+2}y)g_{j+1} \tag{1.25b}$$

any $g = (g_1, g_2, g_3) \in \mathrm{Trig}(A)$.

From Eqs.(1.25), we find

$$g_jL(x)R(y)g_j^{-1} = L(g_{j+1}x)R(g_{j+1}y) \tag{1.26a}$$

$$g_jR(y)L(x)g_j^{-1} = R(g_{j+2}y)L(g_{j+2}x). \tag{1.26b}$$

We next introduce the notion of a regular triality algebra.

Def.1.1

Let $d_j(x, y) \in End(A)$ for $x, y \in A$ and for $j = 1, 2, 3$ be to satisfy

(i)

$$d_1(x, y) = R(y)L(x) - R(x)L(y), \tag{1.27a}$$

$$d_2(x, y) = L(y)R(x) - L(x)R(y). \tag{1.27b}$$

(ii) The explicit form for $d_3(x, y)$ is unspecified except for

$$d_3(y, x) = -d_3(x, y) \tag{1.27c}$$

(iii) $(d_[(x, y), d_2(x, y), d_3(x, y)) \in s \circ Lrt(A)$, i.e., they satisfy

$$d_j(x, y)(uv) = (d_{j+1}(x, y)u)v + u(d_{j+2}(x, y)v) \tag{1.28}$$

for any $x, y, u, v \in A$ and for any $j = 1, 2, 3$. here the index over j is defined modulo 3 as before.

We call the algebra A satisfying these condition to be a regular triality algebra. ([K-O.15])

Def.1.2

Condition (B): We have $AA = A$.

Condition (C): If some $b \in A$ satisfies either $L(b) = 0$, or $R(b) = 0$, then $b = 0$.
We can now prove.

Proposition 1.3

Let A be a regular triality algebra satisfying either the condition (B) or (C). We then obtain the followings:

For any $t = (t_1, t_2, t_3) \in s \circ Lrt(A)$, we jave

$$[t_j, d_k(x, y)] = d_k(t_{j-k}x, y) + d_k(x, t_{j-k}y). \tag{1.29a}$$

Especially, if we choose $t_j = d_j(u, v)$, it yields also

$$[d_j(u, v), d_k(x, y)] = d_k(d_{j-k}(u, v)x, y) + d_k(x, d_{j-k}(u, v)y). \tag{1.29b}$$

(ii) For any $g = (g_1, g_2, g_3) \in \mathrm{Trig}(A)$, we have

$$g_jd_k(x, y)g_j^{-1} = d_k(g_{j-k}x, y) + d_k(x, g_{j-k}y) \tag{1.30}$$

for any $j,k=1,2,3$ and for any $u,v,x,y \in A$,.

Proof

Since Eqs.(1.29) have been already proved in [K-O,15], we will give only a proof of Eq.(1.30) below. In view of Eq.(1.26) and (1.27), we see that Eq.(1.30) holds valid for any $j=1,2,3$ and for $k=1,2$. Therefore, it suffices to prove of the case of $k=3$. To this end, we set

$$D_{j,k} :\equiv g_j d_k(x,y) g_j^{-1} - d_k(s_{j-k}x, y) - d_k(x, g_{j-k}y) \tag{1.31}$$

for a fixed $x,y \in A$. Then, as we have noted, we have $D_{j.1} = D_{j.2} = 0$ identically. Moreover, we will show that it satisfies also

$$D_{j,k}(uv) = (D_{j+1,k+1}u)v + u(D_{j+2,k+2}v). \tag{1.32}$$

We first calculate

$$g_j d_k(x,y) g_j^{-1}(uv) = g_j d_k(x,y)\{(g_{j+1}^{-1}u)(g_{j+2}^{-1}v)\} =$$

$$g_j(\{d_{k+1}(x,y)(g_{j+1}^{-1}u)\}(g_{j+2}^{-1}v)) + g_j((g_{j+1}^{-1}u)(d_{k+2}(x,y)g_{j+2}^{-1}v))$$
$$= g_{j+1}(d_{k+1}(x,y)(g_{j+1}^{-1}u)g_{j+2}(g_{j+2}^{-1}v)) + (g_{j+1}g_{j+1}^{-1}u)(g_{j+2}d_{k+2}(x,y)g_{j+2}^{-1}v)$$
$$= g_{j+1}d_{k+1}(x,y)g_{j+1}^{-1}u)v + u(g_{j+2}d_{k+2}(x,y)g_{j+2}^{-1}v\}.$$

Similarly, we find

$$\{d_k(g_{j-k}x,y) + d_k(x, g_{j-k}y)\}(uv) =$$
$$\{d_{k+1}(g_{j-k}x,y)u\}v + u\{d_{k+2}(g_{j-k}x,y)v\} + \{d_{k+1}(x,g_{j-k})u\}v + u\{d_{k+2}(x,g_{j-k}y)v\},$$

and these prove the validity of Eq.(1.31).

if we choose $k=3$ in Eq.(1.31), we obtain

$$D_{j,3}(uv) = (D_{j+1,1}u)v + v(D_{j+2,2}v) = 0$$

which gives $D_{j,3}=0$, provided that the condition (B) holds. On the other side, the choice of $k=1$ or $k=2$ in Eq.(1.32) yields

$$0 = u(D_{j+2,3}v) = (D_{j+1,3}u)v$$

for any $j=1,2,3$ and for any $u,v \in A$. Therefore, it also gives $D_{j,3}=0$ under the condition (C) .□

We are now in position that we can construct a invariant sub-group of Trig(A) as follow:

Let A be a regular triality algebra satisfying either condition (B) or (C), and set

$$L_0 = span < d_j(x,y), \forall x,y \in A, \forall j = 1,2,3 > . \tag{1.33}$$

Then L_0 is a Lie algebra by Eq.(1.29b). Moreover, it is a ideal of the larger Lie algebra $s \circ Lrt(A)$ by Eq.(1.29a). For any basis $e_1, e_2 \cdots, e_N$ of A with $N = Dim\ A$, and for any $\alpha_{j,\mu,\nu} \in F$ ($j=1,2,3,\ \mu,\nu=1,2,\cdots,N$). We set

$$D_j = \sum_{k=1}^{3}\sum_{\mu,\nu=1}^{N} \alpha_{j-k,\mu,\nu} d_k(e_\mu, e_\nu) \in L_0 \tag{1.34}$$

for $j = 1,2,3$. Then, $D = (D_1, D_2, D_3)$ is a member of $s \circ Lrt(A)$ by Eq.(1.29). Thereofore its exponential map

$$\xi_j = exp\ D_j \ (j = 1,2,3) \tag{1.35}$$

satisfies $\xi_j(xy) = (\xi_{j+1}x)(\xi_{j+2}y)$, i.e., $(\xi_1, \xi_2, \xi_3) \in \mathrm{Trig}(A)$, provided that the exponential map is well-defined. Moreover, for any $g = (g_1, g_2, g_3) \in \mathrm{Trig}(A)$. We calculate

$$g_j D_k g_j^{-1} = \sum_{l=1}^{3} \sum_{\mu,\nu=1}^{N} \alpha_{k=l,\mu,\nu} \{ d_l(g_{j-l}e_\mu, e_\nu) + d_j(e_\mu, g_{j-l}e_\nu) \} \in L_0 \tag{1.36}$$

by Eq.(1.30), so that

$$g_j \xi_k g_j^{-1} = g_j(exp\ D_k) g_j^{-1} = exp(g_j D_k g_j^{-1}) \in exp\ L_0. \tag{1.37}$$

Thereofre, the group $G_0 =< \xi_1, \xi_2, \xi_3 >$ is a invariant sub-group of $\mathrm{Trig}(A)$.

Remark 1.4

If a regular triality algebra satisfies Eq.(1.23b) as well as

$$d_3(x,y)z + d_3(y,z)x + d_3(z,x)y = 0, \tag{1.38}$$

then A is known as a pre-normal triality algebra([K-O.15]). Moreover, if we have

$$Q(x,y,z) = d_1(z,xy) + d_2(y,zx) + d_3(x,yz) = 0 \tag{1.39}$$

in addition, A is called a normal triality algebra in [K-O.15] also.

Further, suppose that A is involutive with the involution map $x \to \bar{x}$ as in Eq.(1.10) and intoroduce the second bi-linear product in the same vector space of A by

$$x * y = \overline{xy} = \bar{y}\bar{x}. \tag{1.40}$$

Then, the resulting algebra A^* is also involutive. If A^* is unital, then A^* is the structurable algebra of Allison ([A.78]), and both conditions (B) and (C) are automatically satisfied.

If we set

$$D(x,y) = d_0(x,y) + d_1(x,y) + d_2(x,y),$$

then for any element x, y of a structurable algebra A^*, this $D(x,y)$ satisfies the validity of a generalized structurable algebra([Ka 92]). That is, it holds

$$D(x,y) \text{ is a derivation, and } D(x,yz) + D(y,zx) * D(z,xy) = 0.$$

Finally, Eq.(1.1) is rewritten as a modified global triality relation of

$$\bar{g}_j(x * y) = (g_{j+1}x) * (g_{j+2}y). \tag{1.41}$$

Reviewing these results, we can construct an invariant sub-group of $Trig(A)$ from any regular triality algebra A satisfying condition (B) or (C), in particular from any structurable algebra A^*.

However, for some cases, we can more directly construct some invariant sub-group of Trig (A), as it happenes for the case of A being a symmetric composition algebra. The case will be explored in the next section. If we further restrict ourselves to the automorphism group, then we can also construct some explict automorphism of the Cayley algebra. which we will study in other paper, and also other case of Trig(A) will be given in future.

In the end of this section, we note that the normal triality algebra is useful concept for a construction of Lie algebras (to see, our paper [J.Alg., 416 (2014) 58-83]).

(2) Symmetric Composition Algebra

This section is devoted for constructions of Trig(A) for symmetric composition algebras. Let A be an algebra over a field F of characteristic not 2 with a symmetric bi-linear non-degenerate form $< \cdot | \cdot >$. Suppose that it satisfies

$$(xy)x = x(yx) =< x|x > y \tag{2.1}$$

for any $x, y \in A$. Then A is called a symmetric composition algebra, since it also satisfies the composition law

$$< xy|xy >=< x|x >< y|y > \text{ and } < xy|z >=< x|yz > . \tag{2.2}$$

Conversely, the validity of Eq.(2.2) implies that of Eq.(2.1) ([O,95]).

Linearizing Eq.(2.1), it gives

$$(xy)z + (zy)x = x(yz) + z(yx) = 2 < x|z > y \tag{2.3}$$

for any $x, y, z \in A$. Moreover if we replace z by yz in the first relation of Eq.(2.3) and note Eq.(2.2), it yields

$$(xy)(yz) = 2 < x|yz > y - < y|y > zx. \tag{2.4}$$

Also, for any $g = (g_1, g_2, g_3) \in$ Trig(A), we have

$$< g_j x|g_j y >=< x|y > \quad (j = 1, 2, 3) \tag{2.5}$$

by the following reason. Applying g_j to both sides of Eq.(2.1), it gives

$$< x|x > g_j y = g_j\{x(yx)\} = (g_{j+1}x)g_{j+2}(yx) =$$

$$(g_{j+1}x)\{(g_j y)(g_{j+1}x)\} =< g_{j+1}x|g_{j+1}x > g_j y$$

so that we have $< x|z >=< g_{j+1}x|g_{j+1}x >$. Linearizing the relation, and letting $j \to j-1$, we obtain then Eq.(2.5). Similarly, any $t_j \in s \circ Lrt(A)$ satisfies

$$< t_j x|y >= - < x|t_j y > . \ (j = 1, 2, 3) \tag{2.6}$$

Any symmetric composition algebra is known (see [O-O,81],[E,97], [K-M-R-T,98]) to be either a para-Hurwitz algebra or 8-dimensional pseudo-octonion algebra where the para-Hurwitz algebra is the conjugate algebra of Hurwitz algebra with the para-unit e satisfying $ex = xe = \bar{x} = 2 < e|x > e - x$. Moreover, it is a normal triality algebra ([O,05]). Further, conditions (B)and (C) of section 1 are automatically satisfied by this algebra. Since $< \circ|\circ >$ is assumed to be non-degenerate, there exists $x \in A$ satisfying $< x|x > \neq 0$. Now Eq.(2.1) implies $< x|x > y \in AA$ for any element $y \in A$, so that we have $AA = A$. If $b \in A$ satisfies $bA = 0$. then Eq.(2.1) also leads to $< x|x > b = 0$ and here $b = 0$. Therefore both conditions (B) and (C) are automatically satisfied. Especially, any symmetric composition algebra is a regular triality algebra satisfying both conditions (B) and (C) (see Remark 1.4). Thus, we can construct some invariant sub-group of Trig(A) as has been demonstrated in the previous sections. However, for special case of $Dim\ A = 1$, we can construct entire Trig(A) as follows.

Example 2.1. Case of $Dim\ A = 1$

We write $A = Fe$, where $e \in A$ satisfies $ee = e$ with $< e|e >= 1$. Then for any $g = (g_1, g_2, g_3) \in \text{Trig}(A)$, we can write

$$g_j(e) = \alpha_j e, \quad (j = 1, 2, 3)$$

for some $\alpha_j = \alpha_{j+1}\alpha_{j+2}$ while Eq.(2.5) yields $(\alpha_j)^2 = 1$. Therefore, there gives

$$\alpha_1^2 = \alpha_2^2 = \alpha_3^2 = 1 \text{ and } \alpha_1\alpha_2\alpha_3 = 1$$

so that Trig(Fe) is isomorphic to the Klein's 4-group K_4. That is,

$$\text{Trig }(Fe) \cong < (Id, Id, Id), (Id, -Id, -Id), (-Id, Id, -Id), (-Id, -Id, Id) >_{span}.$$

Note that we have

$$\text{Auto}(Fe) =< Id >$$

being trivial.

For the case of $Dim\ A = 2$, we will discuss in future.

This paper is an announcement of our recent works, and the details will be considered in future.

3 Other Examples of Triality Groups

Here, in this section, we will give some examples of the triality group for algebras other than the symmetric composition algebra.

Example 3.1 (Matrix algebra)

Let $M(n, F)$ be a set consisting all $n \times n$ matrices over the field F. For any $x, y \in M(n, F)$, the matrix product which we designate as $x * y$ is associative and we write

$$x * (y * z) = (x * y) * z := x * y * z. \tag{3.1}$$

Moreover, for the transpose matrix ${}^t x$ of any $x \in M(n, F)$, we set

$$\bar{x} =^t x. \tag{3.2}$$

Then $x \to \bar{x}$ is a involution map of the resulting algebra A^*. Then, A^* is a unital involutive associative algebra. Especially, it is structurable ([A-F,93]). Note that the $n \times n$ unit matrix e is , here, the unit element of A^*. We introducea subset of A^* by

$$A_0^* = \{x|\bar{x} * x = x * \bar{x} = e, \ x \in A^*\}. \tag{3.3}$$

For any three $a_j \in A_0^*$, $(j = 1,2,3)$, we introduce $\sigma_j(a) \in \mathrm{End}A^*$ by

$$\sigma_j(a)x := a_j * x * \bar{a}_{j+1}, \ (j = 1,2,3) \tag{3.4}$$

where the indices over j are defined modulo 3, i.e., $a_{j\pm3} = a_j$. It is easy to see the validity of

$$\sigma_j(a)\sigma_j(\bar{a}) = 1, \tag{3.5a}$$

$$\overline{\sigma_j(a)}x = a_{j+1} * x * \bar{a}_j, \tag{3.5b}$$

$$\overline{\sigma_j(a)}(x * y) = (\sigma_{j+1}(a)x) * (\sigma_{j+2}(a)y). \tag{3.5c}$$

Introducing the conjugate algebra A of A^* with the bi-linear product xy by

$$xy = \overline{x * y} = \bar{y} * \bar{x}. \tag{3.6}$$

then, these are rewritten as

$$\sigma_j(a)x = a_{j+1}(a_j x) = (x\bar{a}_{j+1})\bar{a}_j \tag{3.7a}$$

$$\sigma_j(a)(xy) = \{(\sigma_{j+1}(a)x)\}\{(\sigma_{j+2}(a)y)\}. \tag{3.7b}$$

We note that A is no longer associative but para-associative with the para-associative law of

$$\bar{z}(xy) = (yz)\bar{x}. \tag{3.8}$$

From Eqs.(3.5a) and (3.7b), we find

$$\sigma(a) = (\sigma_1(a), \sigma_2(a), \sigma_3(a)) \in \mathrm{Trig}(A). \tag{3.9}$$

Furthermore, Eq.(3.7a) gives

$$\sigma_j(a) = L(a_{j+1})L(a_j) = R(\bar{a}_j)R(\bar{a}_{j+1}), \tag{3.10}$$

which has the same structures as the relations for the symmetric composition algebra.

For the corresponding to local triality case, let

$$p = (p_1, p_2, p_3) \in (A^*)^3, \ \bar{p}_j = -p_j \ (j = 1,2,3) \tag{3.11}$$

and define $d_j(p) \in \mathrm{End}\, A^*$ by

$$d_j(p)x = p_j * x - x * p_{j+1}. \tag{3.12}$$

We then have

$$\overline{d_j(p)}x = x * \bar{p}_j - \bar{p}_{j+1} * x = p_{j+1} * x - x * p_j \tag{3.13a}$$

$$\overline{d_j(p)}(x * y) = (d_{j+1}(p)x) * y + x * (d_{j+2}(p)y), \tag{3.13b}$$

and hence

$$d_j(p)(xy) = (d_{j+1}(p)x)y + x(d_{J+2}(p)y). \tag{3.14}$$

These imply that $d(p) = (d_1(p), d_2(p), d_3(p)) \in s \circ Lrt(A)$.

This implies a version of matrix algebras for well-known " the principle of triality". However, we will not go into details

In final of this section, we give some more simple examples for the triality group.

Example 3.2 Let **C** be the complex number with usual product $x * y$. If we define for $a = (\alpha, \beta, \gamma) \in \mathbf{C}^3$, and $|\alpha| = |\beta| = |\gamma| = 1$,

$$\sigma_1(a)x = \alpha * x * \beta^{-1}$$

$$\sigma_2(a)x = \beta * x * \gamma^{-1}$$

$$\sigma_3(a)x = \gamma * x * \alpha^{-1}$$

where $\sigma_{j\pm 3} = \sigma_j (j = 0, 1, 2)$, then we have

$$\sigma_j(a)(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y)$$

with respect to new product xy defined by $xy = \overline{x * y}$,
where $\bar{x}$ denotes the conjugation of x.

If we introduce for $a = (\alpha.\beta, \gamma) \in (Im\ \mathbf{C})^3$

$$d_1(a)x = \alpha * x - x * \beta, d_2(a)x = \beta * x - x * \gamma, d_3(a)x = \gamma * x - x * \alpha,$$

then we get

$$d_j(a)(xy) = (d_{j+1}(a)x)y + x(d_{j+2}(a)y).$$

Example 3.3 Let **H** be the quaternion number satisfying $i^2 = j^2 = k^2 = -1, i * j = -j * i = k$ with basis $\{1, i, j, k\}$. If we define for $a = (i, j, k) \in (Im\ \mathbf{H})^3$,

$$\sigma_1(a)x = i * x * j^{-1}$$

$$\sigma_2(a)x = j * x * k^{-1}$$

$$\sigma_3(a)x = k * x * i^{-1}$$

where the product $x * y$ is usual product of **H**,
then we have

$$\sigma_j(xy) = (\sigma_{j+1}(a)x)(\sigma_{j+2}(a)y)$$

w.r.t. new product $xy = \overline{x * y}$
where $\bar{x}$ denotes the involutive conjugation of $x \in \mathbf{H}$.

If we introduce $d_1(a)x = i * x - x * j, d_2(a)x = j * x - x * k, d_3(a)x = k * x - x * i$, then we obtain

$$d_j(a)(xy) = (d_{j+1}(a)x)y + x(d_{j+2}(a)y$$

w.r.t new product xy defined by $xy = \overline{x * y}$.

Finally, hereby we would like to note final words of Prof.S.Okubo;

Jisei no ku

"To be or not to be ? The quantum dream of the Sohrodinger Cat.
Farewell! Farewell!!-forever. Departure time now to the black hole.
Never to return, farewell, sayonara."

This paper is a cowork with Prof. S.Okubo in the end of his life.
This note is an announcement and the details will be worked in future.

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