数理解析研究所講究録
第2008巻 2016年 29-36

Concept of Symmetry in Closure Spaces as a Tool for Naturalization of Information

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1. Introduction

To naturalize information, i.e. to make the concept of information a subject of the study of reality independent from subjective judgment requires methodology of its study consistent with that of natural sciences. Section 2 of this paper presents historical argumentation for the fundamental role of the study of symmetries in natural sciences and in the consequence of the claim that naturalization of information requires a methodology of the study of its symmetries. Section 3 includes mathematical preliminaries necessary for the study of symmetries in an arbitrary closure space carried out in Section 4. Section 5 demonstrates the connection of the concept of information and closure spaces. This sets foundations for further studies of symmetries of information.

2. Symmetry and Scientific Methodology

Typical account of the origins of the modern theory of symmetry starts with Erlangen Program of Felix Klein published in 1872. [1] This publication was of immense influence. Klein proposed a new paradigm of mathematical study focusing not on its objects, but on their transformations. His mathematical theory of geometric symmetry was understood as an investigation of the invariance with respect to transformations of the geometric space (two-dimensional plane or higher dimensional space). Klein used this very general concept of geometric symmetry for the purpose of a classification of different types of geometries (Euclidean and non-Euclidean). The fundamental conceptual framework of Klein's Program (as it became function as a paradigm) was based on the scheme of (1) space as a collection of points → (2) algebraic structure (group) of its transformations → (3) invariants of the transformations, i.e. configurations of points which do not change as a whole, while their points are permuted by transformations. Selections of algebraic substructures (subgroups) of transformations correspond to different types and levels of invariant configurations allowing to differentiate and to compare structural properties associated with symmetry. The classical example of the mirror symmetry (symmetry with respect to the surface of the mirror) can be identified with invariance with respect to mirror reflection.

Klein's work was utilizing a new theory of groups which in the works of Arthur Cayley [2], Camille Jordan [3] and others found its identity as a part of algebra. Klein's Erlangen Program to classify geometries has been extended to many other disciplines of mathematics becoming one of most common paradigms of the mathematical research. Also, in the consequence of one of most important contributions to mathematical physics of all times published in 1918 by Emmy Noether [4] stating that every differentiable symmetry of the action of a physical system has a corresponding conservation law, the invariance with respect to transformations, i.e. symmetry with respect to these transformations, became the central subject of physics. The fact that the conservation laws for the physical magnitudes such as energy, momentum, angular momentum are associated by the theorem with transformations describing changes of reference frames, i.e. observers makes the study of symmetry a central tool for scientific methodology. Physics (and science in general) is looking for an objective description of reality, i.e. description that is invariant or covariant with changes of observers. Noether's theorem tells us that such description can be carried out with conserved magnitudes.

The year 1872 when Erlangen Program was published can be considered the starting point of the study of symmetries in terms of the group theory, but not of the scientific study of symmetries. The importance of symmetries, still understood in terms of the invariance but of the mirror reflections only was recognized long before Erlangen Program in the context of biochemistry. Louis Pasteur published in 1848 one of his most important papers explaining isomerism of tartrates, more specifically of tartaric acid by the molecular chirality. [5] He showed that the differences between optical properties of the solutions of this organic compound between samples synthesized in living organisms and samples synthesized artificially result from the fact that in artificially synthesized molecules although constructed from the same atoms as those in natural synthesis, have two geometric configurations which are symmetric with respect to the mirror reflection, but not exchangeable by spatial translations or rotations (the same way as left and right palms of human hands), while in the nature only left-handed configurations occur. Later it turned out that almost exclusively naturally synthesized amino acids (and therefore proteins) are "left-handed", and sugars are "right handed". Artificial synthesis, if not constrained by special procedures leads to equal production of the left and right handedness. There is no commonly accepted explanation of this mysterious phenomenon even today.

The structural characteristic which gives the distinction of left and right handedness was given name of chirality. Thus, our hands are chiral, while majority of simple organisms are symmetric with respect to rotations, and therefore achiral). Chirality of molecules became one of most important subjects of the 19th Century biochemistry leading to the discovery of the role of the atoms of carbon in formation of chiral molecules formulated into Le Bel – van't Hoff Rule published by these two researchers independently in 1874.

The study of symmetry in biology, in particular of chirality in complex organisms could not have been explained in the 19th Century, but researchers published some phenomenological laws of evolution and phenotypic development of organisms, such as Bateson's Rule. Much later Bateson's son provided explanation of this rule in terms of information science. [6, 7] Similar interpretation can be given to Curie's Dissymmetry Principle. Pierre Curie made so many important contributions to physics and chemistry that this fundamental principle of great importance is rarely invoked. Its little bit outdated original formulation using the term "dissymmetry" instead of now common "asymmetry" was: A physical effect cannot have a dissymmetry absent from its efficient cause.

The real importance of these early developments could be fully appreciated a half century later when it became fully clear thanks to advances in physics (elementary particle theory) that the study of the conditions for maintaining symmetry is no more important than the study of breaking symmetry.

By the mid-20th Century the study of symmetry became a fundamental tool for mathematics, physics, chemistry and for several branches of biology. This can explain sudden explosion of interest in symmetry among philosophers. The swing of the pendulum of dominating philosophical interests between seeking an objective methodology for philosophical inquiry inspired by scientific methodology and introspective and therefore subjective phenomenal experience reached the side of the former.

The beginnings of structuralism can be traced to the works of Ferdinand de Saussure on linguistics (more specifically his lectures 1907-1911 posthumously published by his disciples in 1916 [8]). The emphasis on the structural characteristics of language and on their synchronous analysis prompted increased interest in the meaning of the concept of structure. It was a natural consequence that the tools used in science for the structural analysis in terms of symmetry found their way to psychology, anthropology and philosophy. The most clear programmatic work on structuralism by Jean Piaget published originally in 1968 is referring explicitly to the concept of the group of transformations. [9] Piaget based his theory of child development on the so called Klein's (sic!) "four group". The works of others, for instance of Claude Levi-Strauss, also employed directly the methods developed in consequence of Erlangen Program. [10]

The swing of the pendulum reversed its direction and structuralism lost his dominating position to its critique, but its importance can be seen in the name of this reversed swing as "post-structuralism". Some of this criticism is naïve. For instance, structuralism was criticized as "dry" or "too much formal". The definite record of "dryness" is held and probably always will be held by Aristotle, together with the title of the most influential philosopher of all times. More justified objection of the lack of explanation of the origin of the structures considered in the studies of Piaget, Levi-Strauss and others and the missing evolutionary or dynamic theory of structures can be blamed on these authors, but it is more a matter of misunderstanding of the mathematical tools. Physics, chemistry have powerful dynamic theories of their structures, so there is no good reason to believe that such dynamic approach is impossible in philosophy.

Symmetry can be easily identified in the studies of visual arts and music. Actually, the structural study of music initiated by Pythagoreans found its way to mediaeval philosophy via Neoplatonic authors and then to the works of the founders of modern science such as Johannes Kepler. The music of Heavens, understood literally as music produced by the motion of the planets was a mathematical model of the universe. In modern mathematics we can easily understand the reasons for the effectiveness of such models. Modern spectral analysis in physics is not very far from the decomposition of functions describing physical phenomena into harmonic components, the same way as recording of music in digital format is done.

The universal character of the study of symmetry in the spirit of Klein's Erlangen Program became commonly recognized in the second half of the 20th Century. The immense popularity of the book "Symmetry" by Hermann Weyl greatly contributed to this recognition. [11] At this time group theory in the context of symmetries became an everyday tool for all physicists and assumed a permanent place in university curricula for studies in physics, chemistry and biology. [12] The statement from an article published in Science in 1972 by a future Nobel Prize laureate in Physics Philip Warren Anderson "It is only slightly overstating the case to say that physics is the study of symmetry" was already at that time commonly accepted truth. [13]

Study of symmetry became a fundamental methodological tool. Anderson's article was not only closing the century of the development of this tool, but it also included another very important message. Anderson emphasized the role of "breaking symmetry". He demonstrated that the physical reality has a hierarchic structure of increasing complexity and that the transition from one level of complexity to the next is associated with breaking symmetry understood as a transformation of the group of symmetry to another of lower level. Thus, not only the study of symmetry, but also of the ways of its changes is important. For this purpose the concept of symmetry has to have very clear and precise formulation. Unfortunately there are many common misunderstandings.

The most typical misunderstanding is a consequence of misinterpretation of Klein's Program. The missing part is the role of projective geometry. Klein did not consider arbitrary transformations of the plane (or set of points on which geometry is defined), but only those which are preserving this most fundamental geometric structure. This very important, but very frequently ignored aspect of the Program was clearly described in Weyl's book popularizing symmetry in the general audience: "What has all this to do with symmetry? It provides the adequate mathematical language to define it. Given a spatial configuration $\mathfrak{J}$, those automorphisms of space which leave $\mathfrak{J}$ unchanged form a group Γ, and *this group describes exactly the symmetry possessed by* $\mathfrak{J}$. Space itself has the full symmetry corresponding to the group of all automorphisms, of all similarities. The symmetry of any figure in space is described by a subgroup of that group." [11]

The methodological aspects of the study of symmetry in physics suggest that the concept of information can be naturalized, i.e. can become a part of the scientific description of reality, if we can develop methods of study of information in terms of symmetry. But to develop a theory of information symmetry we have to generalize the concept of symmetry from the closure space describing geometry to any closure space.

3. Algebraic Preliminaries

The following notation and terminological conventions will be used throughout the text:

Fin(S) is a set of all finite subsets of the set S. Greek letters such as ϕ, φ, θ, etc. will indicate functions on the elements of a given set and with the values belonging to a set. Small Latin letters such as f, g, h, etc. will indicate functions defined on the subsets of a given set and with the values which are subsets of this set. The double use of the symbol $\varphi^{-1}(A)$, as the set of values for the inverse function of φ, and as an inverse image of a set A with respect to function φ which does not have inverse, should not cause problems. The composition of functions will be written as a juxtaposition of their symbols, unless the fact of the use of a composition of functions is contrasted with constructing function images. The symbol $\cong$ indicates a bijective correspondence or isomorphism. Throughout the paper, partially ordered sets are often called posets.

Preliminaries include several propositions without proofs, some belong to the folklore of the subject and are well known, some have proofs straightforward. An introduction to the subject can be found in Birkhoff's "Lattice Theory". [14]

DEFINITION 3.1 *Let f be a function from the power set of a set S to itself which satisfies the following two conditions:*

(1) $\forall A \subseteq S: A \subseteq f(A)$,

(2) $\forall A,B \subseteq S: A \subseteq B \Rightarrow f(A) \subseteq f(B)$,

(3) $\forall A \subseteq S: ff(A) = f(f(A)) = f(A)$.

Then f is called an operator (or transitive closure operator) on S. The set of all operators on the set S is indicated by I(S). A set equipped with a closure operator will be called a closure space $<S, f>$.

The third conditions can be replaced by a condition: which is easier to use in proofs, but which in combination with other two gives exactly the same concept:

(3*) $\forall A,B \subseteq S: A \subseteq f(B) \Rightarrow f(A) \subseteq f(B)$.

The stronger form of this condition $\forall A,B \subseteq S$: $A \subseteq f(B)$ *iff* $f(A) \subseteq f(B)$ *can be used instead of all three conditions to define a transitive operator, but this fact does not have a significant practical importance.*

DEFINITION 3.2 *Let f be a closure operator on a set S. The subsets A of S satisfying the condition f(A)=A, called f-closed sets form a Moore family f-Cl, i.e. it is closed with respect to arbitrary intersections and includes the set S (which can be considered the intersection of the empty subfamily of subsets). Every Moore family* $\mathcal{M}$ *defines a transitive operator* $f(A) = \cap\{M \in \mathcal{M}: A \subseteq M\}$. *Set theoretical inclusion defines a partial order on* ***f-Cl*** *with respect to which it is a complete lattice. To this structure we will refer as the complete lattice* L_f *of f-closed (or just closed) subsets.*

Let f and g be operators on a set S. The relation defined by $f \leq g$ *if* $\forall A \subseteq S: f(A) \subseteq g(A)$ *is a partial order on I(S), with respect to which it is a complete lattice. This partial order corresponds to the inverse of the inclusion of the Moore families of closed subsets*

DEFINITION 3.3 *Let f be a closure operator on a set S, g a closure operator on set T, and* φ *be a function from S to T. The function* φ *is (f,g)-continuous if* $\forall A \subseteq S$: $\varphi f(A) \subseteq g\varphi(A)$. *We will write continuous, if no confusion is likely.*

PROPOSITION 3.1 *Continuity of the function* φ *as defined above is equivalent to each of the following statements:*

(1) $\forall A \subseteq S: f(A) \subseteq \varphi^{-1} g\varphi(A)$,

(2) $\forall B \subseteq T$: $f\varphi^{-1}(B) \subseteq \varphi^{-1} g(B)$,

(3) $\forall B \subseteq T$: $\varphi f\varphi^{-1}(B) \subseteq g(B)$.

(4) $\forall B \in g\text{-}Cl$: $\varphi^{-1}(B) \in f\text{-}Cl$.

DEFINITION 3.4 *Let f be a closure operator on a set S, g a closure operator on set T, and φ be a function from S to T. The function φ is (f,g)-isomorphism if it is bijective and $\forall A \subseteq S$: $\varphi f(A) = g\varphi(A)$. We will write isomorphism, if no confusion is likely. If S=T, we will call φ an (f,g)-automorphism, or smply automorphism.*

PROPOSITION 3.2 *The conditions for a function φ to be an isomorphism, as defined above, are equivalent to either one below:*

(1) φ has an inverse φ^{-1}, and both are continuous,

(2) There exists a function ψ from T to S such that $\varphi\psi = id_T$ and $\psi\varphi = id_S$ and both φ and ψ are continuous.

PROPOSITION 3.3 *Let f be a closure operator on a set S, g a closure operator on set T, and φ be a function from S to T. Then, every (f,g)-isomorphism φ generates a lattice isomorphism φ^* between the complete lattices of closed subsets L_f and L_g defined by $\forall A \in L_f$: $\varphi^*(A) = \varphi(A) \in L_g$. Also, if a function $\varphi: S \rightarrow T$ is bijective and is generating a lattice isomorphism φ^* between lattices L_f and L_g., then φ is an (f,g)-isomorphism.*

COROLLARY 3.4 *Every f-authomorphism φ of <S,f> generates a unique lattice automorphism of L_f. However, more than one f-authomorphism φ of <S,f> can correspond to the same lattice automorphism of L_f.*

PROPOSITION 3.4 *The set of all f-automorphisms of <S,f> forms a group Aut<S,f> under the function composition. This group is isomorphic to $Aut(L_f)$ of lattice automorphisms of L_f.*

We will refer to the concept of an (antisotone) Galois connection between two posets.

DEFINITION 3.5 *Let $<P,\leq>$ and $<Q,\leq>$ be posets and φ and ψ be anti-isotone (order inverting) functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$. Then the functions define a Galois connection between the posets if: $\forall x \in P$: $x \leq \psi\varphi(x)$ and $\forall y \in Q$: $y \leq \varphi\psi(y)$.*

Galois connection can be defined in an equivalent way as a pair of functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ such that $\forall x \in P\, \forall y \in Q$: $y \leq \varphi(x)$ iff $x \leq \psi(y)$.

PROPOSITION 3.5 *If a pair of functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ defines a Galois connection, then the functions $\psi\varphi: P \rightarrow P$ and $\varphi\psi: Q \rightarrow Q$ are closure operators, i.e. they satisfy the conditions 1)-3) of Definition 3.1 generalized from the inclusion $\subseteq$ to the partial order $\leq$. Moreover, the functions $\varphi: P \rightarrow Q$ and $\psi: Q \rightarrow P$ define order anti-isomorphism (order reversing functions preserving all infima and suprema) between the complete lattices of closed elements in the posets P and Q.*

PROPOSITION 3.6 *Given an anti-isotone function $\varphi: P \rightarrow Q$. If the function $\varphi: P \rightarrow Q$ defines together with $\psi: Q \rightarrow P$ a Galois connection, then the function ψ is unique. However, there are anti-isotone functions which do not form a Galois connection with any function.*

PEOPOSITION 3.7 *If posets $<P,\leq>$ and $<Q,\leq>$ are complete lattices, then for every anti-isotone function $\varphi: P \rightarrow Q$, there exists (by Prop. 3.6 unique) function $\psi: Q \rightarrow P$, such that they form a Galois connection. The function $\psi: Q \rightarrow P$ is defined by: $\forall y \in Q$: $\psi(y) = \vee\{x \in P: y \leq \varphi(x)\}$, where is the lowest upper bound of the set, which must exist in a complete lattice.*

REMARK 3.8 *We were using only the fact that the poset $<P,\leq>$ is a complete lattice.*

4. Concept of Symmetry in General Closure Spaces

In the abstract formulation of geometry on the plane in the terms of closure spaces the only closed subsets are entire plane, empty subset, points and straight lines. Geometric configurations are collections of points or lines. However, the concepts of closure spaces do not give us any tools for analysis of such configurations beyond the intersections of lines producing points and pairs of points defining lines. Our goal is to provide the tools for the analysis of such configurations not only for abstract geometries, but for arbitrary closure spaces. The approach presented below was informed by the analogy with geometric symmetries in the choice of group theory as a foundation. Since we will use only rudimentary facts about group actions on a set, there will be no need for extensive explanation of the concepts of this theory.

We will use in the presentation of the approach to the study of symmetry of configurations a selected closure space $\langle S, f\rangle$ with the group G = Aut$\langle S, f\rangle$ of its f-automorphisms. A configuration in this space will be an arbitrary, but not empty set $\Im$ of f-closed subsets of S. It is a natural question how the complete lattice of subgroups of the group G is related to symmetries of configurations, i.e. to symmetries of subsets of the complete lattice L_f of closed subsets in $\langle S, f\rangle$.

We will start from a simple observation related to the generalization of one of examples in Birkhoff's "Lattice Theory" [14]. Its proof is so elementary that it is left as an exercise.

LEMMA 4.1 *Let H be a subset of a group G acting on a set S, such that the identity ε_G of G belongs to H. Define the family $\Im_H$ of subsets of S by $\forall A\subseteq S$: $A\in\Im_H$ iff $\forall x\in A\ \forall\varphi\in H$: $\varphi(x)\in A$. Then $\Im_H$ is a complete lattice with respect to the order of inclusion of sets.*

To avoid coming to too fast conclusion we have to notice that we are not interested in stabilizers of sets of elements of the closure space $\langle S, f\rangle$, but of the families of closed subsets. Therefore we have to apply this lemma to the families of sets of closed subsets of $\langle S, f\rangle$. We will use the notation introduced in the previous section and the concepts defined and explained there.

PROPOSITION 4.2 *Let H be a subgroup of the group G = Aut(L_f). Define the family $\mathfrak{S}_H$ of subsets of L_f by $\forall\mathfrak{K}\subseteq L_f$: $\mathfrak{K}\in\mathfrak{S}_H$ iff $\forall A\in\mathfrak{K}\ \forall\varphi\in H$: $\varphi^*(A)\in\mathfrak{K}$. Then $\mathfrak{S}_H$ is a complete lattice with respect to the order of inclusion of sets.*

PROPOSITION 4.3 *Function Φ: $H\rightarrow\mathfrak{S}_H$ defined in Prop. 4.2 is anti-isotone function between two posets, one of them (the lattice of subgroups of a group G) is a complete lattice.*

Proof: Let K be a subgroup of H. Then $\forall\mathfrak{K}\subseteq L_f$: $\mathfrak{K}\in\mathfrak{S}_H$ iff $\forall A\in\mathfrak{K}\ \forall\varphi\in H$: $\varphi^*(A)\in\mathfrak{K}$. But $\forall\varphi\in K$: $\varphi\in H$, therefore $\mathfrak{K}\in\mathfrak{S}_K$.

Now we can define a Galois connection. By Proposition 3.7 and Remark 3.8 we know that there exists a Galois connection between the poset of complete lattices $\mathfrak{S}_H$ and the complete lattice of subgroups of G = Aut(L_f) $\cong$ Aut $\langle S,f\rangle$.

PROPOSITION 4.4 *The following two functions form a Galois connection:*

Φ: $H\rightarrow\mathfrak{S}_H$ defined by $\forall\mathfrak{K}\subseteq L_f$: $\mathfrak{K}\in\mathfrak{S}_H$ iff $\forall A\in\mathfrak{K}\ \forall\varphi\in H$: $\varphi^*(A)\in\mathfrak{K}$ and

Ψ: $\mathfrak{S}\rightarrow H$ defined by $\vee\{K$ subgroup of G: $\mathfrak{S}\subseteq\mathfrak{S}_K\} = \{\varphi\in G$: $\varphi(\mathfrak{S})\subseteq\mathfrak{S}\}$.The last equality is a consequence of the fact that $\{\varphi\in G$: $\varphi(\mathfrak{S})\subseteq\mathfrak{S}\}$ is a subgroup of G.

5. Symmetry and Information

In order to combine both aspects of information and to place this concept in the context of non-trivial philosophical conceptual framework, the present author introduced his definition of information in terms of the one - many categorical opposition with a very long and rich philosophical tradition. [15] Thus, information is defined as a resolution of the one-many opposition, or in other words as that, which makes one out of many. There are two ways in which many can be made one, either by the selection of one out of many, or by binding the many into a whole by some structure. The former is a selective manifestation of information and the latter is a structural manifestation. They are different manifestations of the same concept of information, not different types, as one is always accompanied by the other, although the multiplicity (many) can be different in each case.

This dualism between coexisting manifestations was explained by the author in his earlier expositions of the definition using a simple example of the collection of the keys to rooms in a hotel. It is easy to agree that the use of keys is based on their informational content, but information is involved in this use in two different ways, through the selection of the right key, or through the geometric description of its shape. We can have numbers of the rooms attached to keys which allow a selection of the appropriate key out of many other placed on the shelf. However, we can also consider the shape of key's feather made of mechanically distinguishable elements or even of molecules. In the latter case, geometric structure of the key is carrying information. The two manifestations of information make one out of very different multiplicities, but they are closely interrelated.

The definition of information presented above, which generalizes many earlier attempts and which due to its very high level of abstraction can be applied to practically all instances of the use of the term information, can be used to develop a mathematical formalism for information. It is not a surprise, that the formalism is using very general framework of algebra. [16]

The concept of information requires a variety (many), which can be understood as an arbitrary set S (called a carrier of information). Information system is this set S equipped with the family of subsets $\Im$ satisfying conditions: entire S is in $\Im$, and together with every subfamily of $\Im$, its intersection belongs to $\Im$, i.e. $\Im$ is a Moore family. Of course, this means that we have a closure operator f defined on S. The Moore family $\Im$ of subsets is simply the family f-Cl of all closed subsets, i.e. subsets A of S such that A= f(A). The family of closed subsets $\Im$ = f-Cl is equipped with the structure of a complete lattice L_f by the set theoretical inclusion. L_f can play a role of the generalization of logic for not necessarily linguistic information systems, although it does not have to be a Boolean algebra. In many cases it maintains all fundamental characteristics of a logical system. [17]

Information itself is a distinction of a subset $\Im_0$ of $\Im$, such that it is closed with respect to (pair-wise) intersection and is dually-hereditary, i.e. with each subset belonging to $\Im_0$, all subsets of S including it belong to $\Im_0$ (i.e. $\Im_0$ is a filter in L_f).

The Moore family $\Im$ can represent a variety of structures of a particular type (e.g. geometric, topological, algebraic, logical, etc.) defined on the subsets of S. This corresponds to the structural manifestation of information. Filter $\Im_0$ in turn, in many mathematical theories associated with localization, can be used as a tool for identification, i.e. selection of an element within the family $\Im$, and under some conditions in the set S. For instance, in the context of Shannon's selective information based on a probability distribution of the choice of an element in S, $\Im_0$ consists of elements in S which have probability measure 1, while $\Im$ is simply the set of all subsets of S.

The tools developed in the preceding section allow us to characterize $\Im_0$ in terms of its symmetry.

6. Conclusion

The approach presented above can be used for study of symmetry in the context of arbitrary closure spaces. Its presentation is merely an outline, which has to be elaborated in further work. In particular, the matter of special interest is its applications to already existing domains where symmetries were studied extensively, in geometry, topology and algebra.

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