Remarks on homomorphisms based on Vertex Connectivity of Weighted Directed Graphs

静岡理工科大学・総合情報学部　國持 良行

(Yoshiyuki Kunimochi)

Faculty of Comprehensive Informatics,

Shizuoka Institute of Science and Technology

abstract We give our definition of homomorphisms(called w-homomorphisms) of general weighted directed graphs and investigate the semigroups of surjective homomorphims and synthesize graphs to obtain a generator of pricipal left (or right) ideal in the semigroup. This study is motivated by reducing the redundancy in concurrent systems, for example, Petri nets which are represented by weighted bipartite graphs. Here we can more simply obtain some results in weighted directed graphs that is generalizations of Petri nets[10].

In a general weighted directed graph, weights given to edges are mesured by some quantity, for example, usually nonnegative integers. Here slightly extending the notion of weight, we adopt and fix a kind of ring R as this quantity. For weighted digraphs $(V_i, E_i, W_i)(i = 1, 2)$, a usual graph homomorphism $\phi : V_1 \to V_2$ satisfies $W_2(\phi(u), \phi(v)) = W_1(u, v)$ to preserve adjacencies of the graphs. Whereas we extend this definition slightly and our homomorphism is defined by the pair (ϕ, ρ) based on the similarity of the edge connection. (ϕ, ρ) satisfies $W_2(\phi(u), \phi(v)) = \rho(u)\rho(v)W_1(u, v)$, where $\rho : V_1 \to Q(R)$ and R is a p.i.d. ant $Q(R)$ is its quatient field.

We investigate the semigroup $\mathcal{S}$ of all surjective w-homomorphisms and develop the theory of principal ideals in $\mathcal{S}$. As an application, we show that some ordered sets of graphs based on surjective w-homomorphisms form lattice structures.

1 Preliminaries

Here we introduce an extension of homomorphisms of a usual weighted directed graph and state some properties of the semigroup of these homomorphisms.

1.1 Graphs and Morphisms

In a general weighted directed graph, weights given to edges are mesured by some quantity, for example, usually nonnegative integers. Here slightly extending the notion of weight, we adopt a kind of ring R as this quantity. More precisely we assume that $(R, +, \cdot)$ has at least two distinct elements $0, 1 \in R$ and satisfies conditions (i) to (iii):

(i) $(R, +, 0)$ is an abelian group.

(ii) $(R, \cdot, 1)$ is a monoid.

(iii) $(R, +, \cdot)$ satisfies the distributive laws.

Moreover through the manuscript we assume that R is a principal ideal domain (abbreviated as p.i.d)[9], that is, satisfies the following conditions (iv) to (vi).

(iv) $(R, \cdot, 1)$ is commutative.

(v) $ab = 0$ implies $a = 0$ or $b = 0$.

(vi) Every ideal I in R is principal, that is, $I = RaR$ for some $a \in R$.

We require the conditions (iv) and (v) that R is a domain, for defining the quotient field $Q(R) = \{r/s | r, s \in R, s \neq 0\}$ of R by $Q(R)$, which is the smallest field containing a domain R.

By (vi), for any nonempty $S = \{a_1, a_2, \ldots, a_n\} \subset R$, there exists $a \in R$ such that $a_1 R \cup a_2 R \cup \cdots \cup a_n R = aR$, which is called a greatest common divisor of S. The set of all the greatest common divisors of S is denoted by $\gcd(S)$.

DEFINITION 1.1 A *weighted directed graph* (weighted digraph, for short) is a 3-tuple (V, E, W) where

(1) V is a finite set of vertices,

(2) $E (\subset V \times V)$ is a set of edges,

(3) $W : E \to R$ is a *weight function*, where R is a p.i.d.. $\qquad\square$

According to custom, $(u, v) \in E \iff W(u, v) \neq 0$.

DEFINITION 1.2 Let $G_1 = (V_1, E_1, W_1)$ and $G_2 = (V_2, E_2, W_2)$ be weighted digraphs. Then a pair (ϕ, ρ) is called a *(weight preserving) homomorphism* (for short, *w-homomorphism*) from G_1 to G_2 if $W_i : E_i \to R$ have the same image R and the maps $\phi : V_1 \to V_2$, $\rho : V_1 \to Q(R) \setminus \{0\}$ satisfy the condition that for any $u, v \in V_1$,

$$W_2(\phi(u), \phi(v)) = \rho(u)\rho(v)W_1(u, v), \qquad (1.1)$$

where $Q(R)$ is the quotient field of R. Especially if $\rho = 1_{V_1}$, i.e., $\rho(u) = 1$ for any $u \in V_1$, then w-homomorphism is called a *strictly weight preserving homomorphism* (*s-homomorphism*, for short). $\qquad\square$

EXAMPLE 1.1 Let $G_i = (V_i, E_i, W_i)$ $(i = 1, 2)$ be the weighted digraphs depicted in Figure 1, $W_i : V_i \to Z$ the weight functions, where Z is the set of integers but we don't use its negative part. That is,

$V_1 = \{u_1, u_2, v_1, v_2\}, V_2 = \{u_3, u_4, v_3\}$.

$W_1(u_1, v_1) = 1, W_1(u_1, v_2) = 2, W_1(u_2, v_1) = 3, W_1(u_2, v_2) = 6$.

$W_2(u_3, v_3) = 3, W_2(u_4, v_3) = 9$, otherwise $W(u, v) = 0$.

The weights of any other weights are 0.

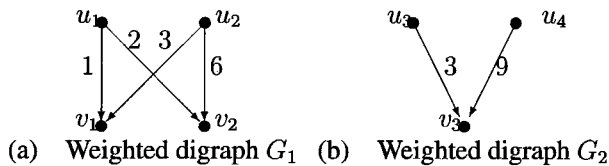(a)   Weighted digraph G_1 (b)   Weighted digraph G_2

Figure 1. Weighted Digraph G_1 and G_2 with $G_1 \sqsupseteq G_2$.

The following (ϕ_1, ρ_1) is a w-homomorphism from G_1 to G_2.

$$\phi_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u_3 & u_4 & v_3 & v_3 \end{pmatrix}, \; \rho_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 1 & 1 & 3 & 3/2 \end{pmatrix}.$$

$\qquad\square$

The w-homomorphism $(\phi,\,\rho)$ is called *injective* (resp. *surjective*) if ϕ is injective (resp. surjective). In particular, it is called a *w-isomorphism* from G_1 to G_2 if it is injective and surjective. Then G_1 is said to be *w-isomorphic* to G_2 and we write $G_1 \simeq_w G_2$. Moreover, in case of $G_1 = G_2 = G$, a w-isomorphism is called an *w-automorphism* of G. By $\mathbf{Aut}_w(G)$ we denote the set of all the w-automorphisms of G. Similarily s-isomorphism $\simeq_s$ s-automorphism and $\mathbf{Aut}_s(G)$ are defined.

1.2 Composition of the w-homomorphisms

We define the composition of the w-homomorphisms. In this manuscript, we write $\phi\psi$ for the composition $\psi \circ \phi$ of maps.

DEFINITION 1.3 Let $G_i = (V_i, E_i, W_i)\,(i = 1, 2, 3)$ be weighted digraphs, (ϕ,ρ) : $G_1 \to G_2$ and $(\psi,\sigma) : G_2 \to G_3$ be w-homomorphisms. Then the composition of these w-homomorphisms are defined by the semidirect product

$$(\phi,\rho)(\psi,\sigma) \stackrel{\text{def}}{=} (\phi,\rho) \rtimes (\psi,\sigma) = (\phi\psi, \rho \otimes (\phi\sigma)),$$

where $\rho \otimes (\phi\sigma) : V \to Q(R), u \mapsto \rho(u)\sigma(\phi(u))$. The set $Q(R)^V$ of maps from V to $Q(R)$ forms abelian group under the operation $\otimes$: $(f \otimes g)(v) = f(v)g(v)$. $\qquad\square$

Indeed, checking the validity of the definition.

$$W_3(\psi(\phi(u)), \psi(\phi(v)))$$
$$= \sigma(\phi(u))\sigma(\phi(v))W_2(\phi(u), \phi(v))$$
$$= \sigma(\phi(u))\sigma(\phi(v))\rho(u)\rho(v)W_1(u, v)$$
$$= \sigma(\phi(u))\rho(u)\sigma(\phi(v))\rho(v)W_1(u, v)$$
$$= ((\phi\sigma) \otimes \rho)(u)((\phi\sigma) \otimes \rho)(v)W_1(u, v)$$

hold.

EXAMPLE 1.2 Let $G_i = (V_i, E_i, W_i)\,(i = 2, 3)$ be weighted digraphs depicted in Figure 2. The following (ϕ_1, ρ_1) is the w-homomorphism from G_1 to G_2 in Example 1.1. (ϕ_2, ρ_2) is a w-homomorphism from G_2 to G_3.

$$\phi_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u_3 & u_4 & v_3 & v_3 \end{pmatrix}, \rho_1 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 1 & 1 & 3 & 3/2 \end{pmatrix},$$
$$\phi_2 = \begin{pmatrix} u_3 & u_4 & v_3 \\ u & u & v \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} u_3 & u_4 & v_3 \\ 5/3 & 5/9 & 1 \end{pmatrix}.$$

We have

$$\phi_1\rho_2 = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 5/3 & 5/9 & 1 & 1 \end{pmatrix}.$$

Therefore, $(\phi,\rho) = (\phi_1\phi_2, \rho_1 \otimes (\phi_1\rho_2)) = (\phi_1,\rho)(\phi_2,\rho_2)$ is the composition of them, where

$$\phi = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ u & u & v & v \end{pmatrix}, \quad \rho = \begin{pmatrix} u_1 & u_2 & v_1 & v_2 \\ 5/3 & 5/9 & 3 & 3/2 \end{pmatrix}.$$
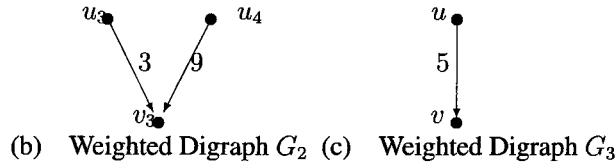
$\qquad\square$

(b) Weighted Digraph G_2 (c) Weighted Digraph G_3

Figure 2. Weighted digraphs G_2 and G_3.

Immediately, we obtain the following lemma regarding to $\otimes$.

LEMMA 1.1 Let ϕ and ψ be arbitrary maps on V and $f, g : V \rightarrow Q(R)$. 1_V means the constant mapping defined by $1_V : V \rightarrow Q(R), v \mapsto 1$, f^{-1} means the mapping $V \rightarrow Q(R), v \mapsto 1/f(v)$. Then the following equations are true.
(1) $(\phi\psi)f = \phi(\psi f)$.
(2) $\phi(f \otimes g) = (\phi f) \otimes (\phi g)$.
(3) $\psi 1_V = 1_V$,
(4) $(\phi f) \otimes (\phi f^{-1}) = 1_V$.
(5) $(\phi f)^{-1} = \phi f^{-1}$.

Proof)   We can easily verify the equations.   □

For weighted digraphs G_1 and G_2, we write $G_1 \sqsupseteq G_2$ if there exists a surjective w-homomorphism from G_1 to G_2. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order $\sqsupseteq$ is regarded as an order up to w-isomorphism.

PROPOSITION 1.1 Let G_1, G_2, G_3 be weighted digraphs. Then,
(1) $G_1 \sqsupseteq G_1$.
(2) $G_1 \sqsupseteq G_2$ and $G_2 \sqsupseteq G_1$ $\Longleftrightarrow$ $G_1 \simeq_w G_2$.
(3) $G_1 \sqsupseteq G_2$ and $G_2 \sqsupseteq G_3$ imply $G_1 \sqsupseteq G_3$.

Proof)   We can easily verify the inequalities.   □

Remark that in Example 1.2, ϕ_1 and ϕ_2 are sujective, $\phi_1\phi_2$ is also. Therefore $G_1 \sqsupseteq G_2 \sqsupseteq G_3$ holds.

2   Ideals in the semigroup $\mathcal{S}$

In this section we define the set $\mathcal{S}$ of all surjective w-homomorphisms between two weighted digraphs and a (extra) zero element 0. Introducing the multiplication by the composition, $\mathcal{S}$ forms a semigroup,

For a surjective w-homomorphim $x : G_1 \rightarrow G_2$, G_1 is called the domain of x, denoted by $Dom(x)$, and G_2 is called the image(or range) of x, denoted by $Im(x)$. Especially $Dom(0) = Im(0) = \emptyset$. The multiplication of $x = (\phi, \rho)$ and $y = (\psi, \sigma)$ is defined by

$$x \cdot y \stackrel{\text{def}}{=} \begin{cases} (\phi\psi, (\phi\rho) \otimes \sigma) & \text{if } Im(x) = Dom(y). \\ 0 & \text{otherwise.} \end{cases}$$

2.1 Green's equivalences on the semigroup $\mathcal{S}$

Regarding to a general semigroup S without an identity, for convenience of notation, $S^1 = S \cup \{1\}$ is the monoid obtained from a semigroup S by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in S$ and $1 \cdot 1 = 1$.

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a semigroup S, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$x\mathcal{L}y \iff S^1x = S^1y,$$
$$x\mathcal{R}y \iff xS^1 = yS^1,$$
$$x\mathcal{J}y \iff S^1xS^1 = S^1yS^1,$$
$$\mathcal{H} = \mathcal{L} \cap \mathcal{R},$$
$$\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*,$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. S^1x (resp. xS^1) is called the *principal left* (resp. *right*) *ideal generated by* x and S^1xS^1 the *principal (two-sided) ideal generated by* x. Then, the following facts are generally true[7, 2].

FACT 1 *The following relations are true.*

$$(1)\, \mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L}$$
$$(2)\, \mathcal{H} \subset \mathcal{L}\ (resp.\, \mathcal{R}) \subset \mathcal{D} \subset \mathcal{J}$$

FACT 2 *An $\mathcal{H}$-class is a group if and only if it contains an idempotent e, that is $e^2 = e$.*

Now we consider the case of $S = \mathcal{S}$ in the rest of the maniscript. The following lemma is obviously true.

LEMMA 2.1 *Let $x : G_1 \to G_2$, $y : G_3 \to G_4 \in \mathcal{S}$. Then,*
(1) $x\mathcal{S}^1 \subset y\mathcal{S}^1 \implies G_1 = G_3$ *and* $G_2 \sqsubseteq G_4$.
(2) $\mathcal{S}^1x \subset \mathcal{S}^1y \implies G_3 \sqsubseteq G_1$ *and* $G_2 = G_4$.
(3) $x\mathcal{S}^1 = y\mathcal{S}^1 \implies G_1 = G_3$ *and* $G_2 \simeq_{\mathrm{w}} G_4$.
(4) $\mathcal{S}^1x = \mathcal{S}^1y \implies G_1 \simeq_{\mathrm{w}} G_3$ *and* $G_2 = G_4$. $\qquad\square$

Remark that any reverse implications above are not necessarily true.

PROPOSITION 2.1 The following conditions are equivalent.
(1) H is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}_{\mathrm{w}}(G)$ for some weighted digraph G.

Proof) (1)$\implies$(2) By Fact2, H contains an idempotent e, that is $e^2 = e$. This implies $Dom(e) = Im(e) = G$ for some weighted digraph G. By (3) and (4) of Lemma 2.1, $Dom(x) = Dom(e) = G$ and $Im(x) = Im(e) = G$ for any $x \in H$ because $x\mathcal{S}^1 = e\mathcal{S}^1$ and $\mathcal{S}^1x = \mathcal{S}^1e$ hold. Therefore each element of H is a w-automorphism of G. Conversely, for a w-automorphism x of G, $x \in H$ because $xx^{-1} = x^{-1}x = e$ and $ex = xe = x$. Hence we have $H = \mathbf{Aut}_{\mathrm{w}}(G)$.
(2)$\implies$(1) For $x, y \in H = \mathbf{Aut}_{\mathrm{w}}(G)$, there exist $z, w \in H$ such that $x = zy$ and $wx = y$. This implies $\mathcal{S}^1x = \mathcal{S}^1y$. Similarily we have $x\mathcal{S}^1 = y\mathcal{S}^1$. Therefore $x\mathcal{H}y$. Conversely, $x\mathcal{H}y$ and $x \in H$ imply $y \in H$ because y is a surjective w-homomorphism with $Dom(y) = Im(y) = G$. Hence H is an $\mathcal{H}$-class and a group. $\qquad\square$

PROPOSITION 2.2 On the semigroup $\mathcal{S}$, $\mathcal{J} = \mathcal{D}$.

Proof)   Since $\mathcal{D} \subset \mathcal{J}$ holds, it is enough to show the reverse inclusion.

$$x\mathcal{J}y \iff \mathcal{S}^1 x \mathcal{S}^1 = \mathcal{S}^1 y \mathcal{S}^1$$
$$\iff \exists u, v, z, w \in \mathcal{S}^1 \, (x = uyv, y = zxw)$$

implies that $x = uzxwv, y = zuyvw$. Setting
$P = Dom(x), Q = Dom(y), R = Im(x)$ and $S = Im(y)$, $uz : P \to P$, $zu : Q \to Q$, $wv : R \to R$, $vw : S \to S$ are w-automorphisms. This implies that u, v, z, w are w-isomorphisms and $u^{-1} = z, v^{-1} = w$. Let $t = xw$. Then,

$$x = x(ww^{-1}) = (xw)w^{-1} = tw^{-1}$$
$$y = z(xw) = zt$$
$$t = (z^{-1}z)t = z^{-1}(zt) = z^{-1}y$$

Therefore $x\mathcal{S}^1 = t\mathcal{S}^1$ and $\mathcal{S}^1 t = \mathcal{S}^1 y$, that is, $x\mathcal{R}t\mathcal{L}y$. Thus $\mathcal{D} \subset \mathcal{J}$. $\square$

2.2   Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}$ we find a elements z such that $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$ (resp. $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$). $x\mathcal{S}^1 \cap y\mathcal{S}^1 = \{0\}$ (resp. $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \{0\}$) is a trivial case($z = 0$). We should only consider the non-trivial case. For a surjective map $\phi : V_1 \to V_2$, we denote the equivalence relation $\phi\phi^{-1} = \{(u, v) | v \in \phi\phi^{-1}(u)\}$ on V_1 by ker ϕ.

LEMMA 2.2 Let $G_i = (V_i, E_i.W_i)(i = 1, 2, 3)$ be weighted graphs, $x = (\phi, \rho) : G_1 \to G_3, y = (\psi, \sigma) : G_2 \to G_3$ be elements of $\mathcal{S}$. If $|\phi^{-1}(u)| \leq |\psi^{-1}(u)|$ for any $u \in V_3$, then $\mathcal{S}^1 y \subset \mathcal{S}^1 x$.

Proof)   By the assumption, we can choose some surjective morphism $\xi : V_2 \to V_1$ such that $\xi(\psi^{-1}(u)) = \phi^{-1}(u)$ for any $u \in V_3$.

$$W_1(\xi(u), \xi(v)) = \frac{\sigma(u)\sigma(v)}{\rho(\xi(u))\rho(\xi(v))} W_2(u, v).$$

So $\tau : V_2 \to Q(R)$ is defined by $\tau = \sigma \otimes (\xi\rho)^{-1}$. Then, we can verify that (ξ, τ) is a surjective morphism from G_2 to G_1 and thus $z \in \mathcal{S}^1$, $y = zx$. Therefore $\mathcal{S}^1 y \subset \mathcal{S}^1 x$. $\square$

Remark that enumerating all the surjective maps such as ξ in the proof, the number N of them is represented as

$$N = \prod_{i=1}^{k} (s_{n_i}^{m_i} \times m_i!),$$

where $V_3 = \{u_1, u_2, \ldots, u_k\}$, $m_i = |\phi^{-1}(u_i)|$, $n_i = |\psi^{-1}(u_i)|$, and $s_{n_i}^{m_i}$ is the Stirling number (of the second kind). $s_{n_i}^{m_i}(n_i \geq m_i)$ is the number of partitions of a set of n_i objects into m_i classes[1].

LEMMA 2.3 Let $G_i = (V_i, E_i.W_i)(i = 0, 1, 2)$ be weighted digraphs, $x = (\phi, \rho) : G_0 \to G_1, y = (\psi, \sigma) : G_0 \to G_2$ be elements of $\mathcal{S}$. If ker $\phi \subset$ ker ψ, then $y\mathcal{S}^1 \subset x\mathcal{S}^1$.

Proof) Let u, v be arbitrary elements of V_1, respectively. By the assumption, $\bar{u}, \bar{v} \in V_2$ are uniquely determined and let

$$\phi^{-1}(u) = \{u_1, u_2, \dots, u_n\} \subset \psi^{-1}(\bar{u}),$$
$$\phi^{-1}(v) = \{v_1, v_2, \dots, v_m\} \subset \psi^{-1}(\bar{v}),$$

Then we can easily check that

$$W_1(u, v) = W_1(\phi(u_i), \phi(v_j)) = \rho(u_i)\rho(v_j)W_0(u_i, v_j),$$
$$W_2(\bar{u}, \bar{v}) = W_2(\psi(u_i), \psi(v_j)) = \sigma(u_i)\sigma(v_j)W_0(u_i, v_j),$$

for any $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. The right hand sides of the equations above are constants not depending on i and j. So

$$\xi : V_1 \to V_2, \ u \mapsto \bar{u}, \text{ where } \phi^{-1}(u) \subset \psi^{-1}(\bar{u}), \text{ and}$$
$$\nu : V_1 \to Q(R), \ u \mapsto \sigma(u_i)\rho^{-1}(u_i), \text{ where } \phi(u_i) = u$$

are well-defined. Therefore we have $z = (\xi, \nu) \in \mathcal{S}$ and thus $y = xz$, that is, $y\mathcal{S}^1 \subset x\mathcal{S}^1$. $\square$

PROPOSITION 2.3 (Intersection of Principal Left Ideals) Let $G_i = (V_i, E_i, W_i)(i = 1, 2, 3)$ be weighted digraphs, $x = (\phi_1, \rho_1) : G_1 \to G_3$, $y = (\phi_2, \rho_2) : G_2 \to G_3$ be elements of $\mathcal{S}$, $V_3 = \{u_1, u_2, \dots, u_N\}$. Let

$$n_i = \max\{|\phi_1^{-1}(u_i)|, |\phi_2^{-1}(u_i)|\} \text{ for each } i = 1, 2, \dots, N.$$

Taking sets $U_1, U_2, \dots, U_N$ with their sizes $|U_i| = n_i (i = 1, 2, \dots, N)$, we construct a weighted digraph $G = (V, E, W)$, where $V = \bigcup_{1 \le i \le N} U_i$ and for any $u, v \in V$,

$$W(u, v) = W_3(u_i, u_j) \quad \text{if } u \in U_i, v \in U_j,$$

Then, $z = (\phi, 1_{\otimes_V}) : G \to G_3$, where $\phi : U_i \ni u \mapsto u_i$ and $1_{\otimes_V} : V \to Q(R), v \mapsto 1 \in Q(R)$, is a surjective morphism. Moreover, $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$.

Proof) By Lemma 2.2 and the construction of G, $z = ax = by$ for some $a, b \in \mathcal{S}^1$. Therefore $z \in \mathcal{S}^1 x \cap \mathcal{S}^1 y$.

Conversely we show that $w = (\psi, \sigma) \in \mathcal{S}^1 x \cap \mathcal{S}^1 y$ implies $w \in \mathcal{S}^1 z$. We can write $w = a'x = b'y$ for some $a', b' \in \mathcal{S}^1$. Let $u_i \in V_3$. In our construction, $|\phi^{-1}(u_i)| = \max\{|\phi_1^{-1}(u_i)|, |\phi_2^{-1}(u_i)|\}$. Since $w = a'x = b'y$ holds, we have $|\phi_1^{-1}(u_i)| \le |\psi^{-1}(u_i)|$ and $|\phi_2^{-1}(u_i)| \le |\psi^{-1}(u_i)|$ and thus $|\phi^{-1}(u_i)| \le |\psi^{-1}(u_i)|$. By Lemma 2.2, we conclude $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$. $\square$

COROLLARY 2.1 (Diamond Property I) Let G_1, G_2, G_3 be weighted digraphs with $G_i \sqsupseteq G_3 (i = 1, 2)$. Then there exists a unique least weighted digraph G up to w-isomorphism such that $G_i \sqsupseteq G (i = 1, 2)$. $\square$

We consider the intersection of two principal right ideals. The case of principal right ideals is rather difficult compared to that of principal left ideals.

$(ker\phi \cup ker\,\psi)^*$ is the smallest equivalence relation on V which includes both $ker\,\phi$ and $ker\,\psi$, that is, it is the reflexive and transitive closure of $ker\phi \cup ker\,\psi$.

PROPOSITION 2.4 (Intersection of Principal Right Ideals) Let $G_i = (V_i, E_i.W_i)(i = 0, 1, 2)$ be weighted digraphs, $x = (\phi_1, \rho_1) : G_0 \to G_1$, $y = (\phi_2, \rho_2) : G_0 \to G_2$ be elements of $\mathcal{S}$. Let $C_1, C_2, \ldots, C_N$ be all the $(\ker \phi_1 \cup \ker \phi_2)^*$-classes in V_0.

$\rho : V_0 \to Q(R)$ is defined by if u is 0-isolated then $\rho(u) = 1$ and otherwise

$$\rho(u) = 1/\gcd(\{W_0(u, v), W_0(v, u) \,|\, v \in V_0\})$$

where $n = |V_0|$ and $V_0 = \{v_1, v_2, \ldots, v_n\}$.
(1) The weighted graph $G_3 = (V_3, E_3.W_3)$ can be constructed in the following way:

$$V_3 = \{C_1, C_2, \ldots, C_N\},$$

For each $i, j \in \{1, 2, \ldots, N\}$,

$$W_3(C_i, C_j) = \rho(u)\rho(v)W_0(u, v) \text{for any } u \in C_i, v \in C_j,$$

are well-defined.

(2) Let $z = (\phi, \rho) : G_0 \to G_3$, where ϕ is the canonical surjection from V_0 onto V_3. Then, z is a surjective morphism and $x\mathcal{S}^1 \cap y\mathcal{S}^1 = z\mathcal{S}^1$.

Proof) Let $i, j \in \{1, 2, \ldots, N\}$. We shall show that for any $u, u' \in C_i$ and $v, v' \in C_j$,

$$\rho(u)\rho(v)W_0(u, v) = \rho(u')\rho(v')W_0(u', v'), \tag{2.1}$$

Before proving the equation (2.1), under the condition that $\phi_k(u) = \phi_k(u')$ and $\phi_k(v) = \phi_k(v')$ hold for $k = 1, 2$, we show the equation (2.1). First,

$$\begin{aligned}
\rho_k(u)\rho_k(v)W_0(u, v) &= W_k(\phi_k(u), \phi_k(v)) \\
&= W_k(\phi_k(u'), \phi_k(v')) = \rho_k(u')\rho_k(v')W_0(u', v')
\end{aligned} \tag{2.2}$$

holds and especially considering the case of $v = v'$, we have

$$\begin{aligned}
\rho_k(u)W_0(u, v) &= \rho_k(u')W_0(u', v), \text{ and similarly} \\
\rho_k(u)W_0(v, u) &= \rho_k(u')W_0(v, u').
\end{aligned} \tag{2.3}$$

Next the following equation (2.4) holds.

$$\begin{aligned}
&\text{neither } u \text{ nor } v \text{ is 0-isolated} \Longrightarrow \\
&\rho(u)\rho(v)\rho_k(u')\rho_k(v') = \rho(u')\rho(v')\rho_k(u)\rho_k(v).
\end{aligned} \tag{2.4}$$

Indeed since u and u' are not 0-isolated, the greatest common divisors give the following equations.

$$\begin{aligned}
&\rho(u)\rho_k(u') \\
&= \rho(u')\rho(u)\rho_k(u')\rho^{-1}(u') \\
&= \rho(u')\rho(u)\rho_k(u') \gcd(\{W_0(u', v), W_0(v, u') \,|\, v \in V_0\}) \\
&= \rho(u')\rho(u)\rho_k(u) \gcd(\{W_0(u, v), W_0(v, u) \,|\, v \in V_0\}) \; \because (2.3) \\
&= \rho(u')\rho(u)\rho_k(u)\rho^{-1}(u) \\
&= \rho(u')\rho_k(u)\{\rho(u)\rho^{-1}(u)\} \\
&= \rho(u')\rho_k(u)
\end{aligned}$$

Similarily we have $\rho(v)\rho_k(v') = \rho(v')\rho_k(v)$. These imply that the equation (2.4) holds. The equation (2.2)implies that $W_0(u,v) = 0 \iff W_0(u',v') = 0$. Since it is trivial in case of $W_0(u,v) = 0$, we may assume that $W_0(u,v) \neq 0$ and thus u is not 0-isolated.

$$
\begin{aligned}
&\rho(u)\rho(v)W_0(u,v) \\
&= \rho(u)\rho(v)\rho_k(u)^{-1}\rho_k(v)^{-1}\rho_k(u)\rho_k(v)W_0(u,v) \\
&= \rho(u)\rho(v)\rho_k(u)^{-1}\rho_k(v)^{-1}\rho_k(u')\rho_k(v')W_0(u',v') \quad \because (2.2) \\
&= \rho(u')\rho(v')\rho_k(u)^{-1}\rho_k(v)^{-1}\rho_k(u)\rho_k(v)W_0(u',v') \quad \because (2.4) \\
&= \rho(u')\rho(v')W_0(u',v')
\end{aligned}
$$

If $\phi_k(u) = \phi_k(u')$ and $\phi_k(v) = \phi_k(v')$ hold for $k = 1,2$, we have shown the equation (2.1) and return to the proof of the equation (2.1) in case of $u, u' \in C_i$ and $v, v' \in C_j$.

Since $u, u' \in C_i$ and $v, v' \in C_j$, there exist sequences

$$
\begin{aligned}
&s_0 = u, s_1, \ldots, s_\ell = u', \\
&\quad \text{with}(s_{k-1}, s_k) \in \ker\phi_1 \cup \ker\phi_2 (0 < k \leq \ell), \\
&t_0 = v, t_1, \ldots, t_m = v', \\
&\quad \text{with}(t_{k-1}, t_k) \in \ker\phi_1 \cup \ker\phi_2 (0 < k \leq m).
\end{aligned}
$$

Then,

$$
\begin{aligned}
&\rho(s_0)\rho(t_0)W_0(s_0, t_0) = \rho(s_1)\rho(t_0)W_0(s_1, t_0) = \ldots \\
&= \rho(s_\ell)\rho(t_0)W_0(s_\ell, t_0) = \rho(s_\ell)\rho(t_1)W_0(s_\ell, t_1) = \ldots \\
&= \rho(s_\ell)\rho(t_m)W_0(s_\ell, t_m)
\end{aligned}
$$

Therefore the equation (2.1) and thus W_3 are well-defined.

(2)   Let $k \in \{1,2\}$. By the statement (1) above, the following maps are well-defined.

$$
\begin{aligned}
\phi_k' &: V_k \to V_3, v \mapsto \phi(u) & \text{where } \phi_k(u) = v, \\
\rho_k' &: V_k \to Q(R), v \mapsto \rho(u)\rho_k(u)^{-1} & \text{where } \phi_k(u) = v.
\end{aligned}
$$

For any $v, t \in V_k$, there exists $u, s \in V_0$ such that $\phi_k(u) = v$ and $\phi_k(s) = t$, and thus we have

$$
\begin{aligned}
&W_3(\phi_k'(v), \phi_k'(t)) = W_3(\phi(u), \phi(s)) = \rho(u)\rho(s)W_0(u,s) \\
&= \rho(u)\rho(s)\rho_k(u)^{-1}\rho_k(s)^{-1}W_k(\phi_k(u), \phi_k(s)) \\
&= \rho_k'(v)\rho_k'(t)W_k(v,t).
\end{aligned}
$$

Therefore $x' = (\phi_1', \rho_1') : G_1 \to G_3$ and $y' = (\phi_2', \rho_2') : G_2 \to G_3$ are w-homomorphisms. We can easily show that $\phi_k'(k = 1,2)$ are surjective, that is, $z = xx' = yy'$ $(x', y' \in \mathcal{S})$. Therefore $z\mathcal{S}^1 \subset x\mathcal{S}^1 \cap y\mathcal{S}^1$.

Conversely, we show that for any $w \subset x\mathcal{S}^1 \cap y\mathcal{S}^1$ there exists $z' \in \mathcal{S}^1$ such that $w = zz'$.

If we can write $w = xa = yb$, $a = (\psi_1, \sigma_1), b = (\psi_2, \sigma_2) \in \mathcal{S}$, then $w = (\psi, \sigma) = (\phi_1\psi_1, \rho_1 \otimes \phi_1\sigma_1) = (\phi_2\psi_2, \rho_2 \otimes \phi_2\sigma_2)$. Let $Im(w) = G_4 = (V_4, E_4, W_4)$

Let $u, u' \in C_i$. Since a sequence $s_0 = u, s_1, \ldots, s_\ell = u'$ such that for $0 \leq j < \ell$ $\phi_1(s_j) = \phi_1(s_{j+1})$ or $\phi_2(s_j) = \phi_2(s_{j+1})$ exists, $\psi(s_j) = \psi(s_{j+1})$ holds. This implies that there exists $v \in V_4$ such that $C_i \subset \psi^{-1}(v)$. By Lemma 2.3, $w\mathcal{S}^1 \subset z\mathcal{S}^1$. Therefore, $x\mathcal{S}^1 \cap y\mathcal{S}^1 \subset z\mathcal{S}^1$. $\square$

COROLLARY 2.2 (Diamond Property II) Let G_i $(i = 0,1,2)$ be weighted digraphs with $G_0 \sqsupseteq G_i$ $(i = 1,2)$. Then, there exists a unique maximum weighted digraph G up to isomorphism such that $G_0 \sqsupseteq G_i \sqsupseteq G$ $(i = 1,2)$. $\square$

We define the notion of irreducible forms of a weighted digraph with respect to $\sqsupseteq$.

DEFINITION 2.1 (Irreducible) A weighted digraph G is called a $\sqsupseteq$-*irreducible* if $G \sqsupseteq G'$ implies $G \simeq G'$ for any weighted digraph G'. Then G is called an $\sqsupseteq$-irreducible form. $\square$

COROLLARY 2.3 Let G, G' and G'' be weighted digraphs with $G \sqsupseteq G'$ and $G \sqsupseteq G''$. Then one has: If G' and G'' are $\sqsupseteq$-irreducible, then $G' \simeq_w G''$.

Proof) Trivial by Corollary 2.2 and the definition of $\sqsupseteq$-irreducibility. $\square$

2.3 Lattice structures of $\simeq_w$-classes of weighted digraphs

As an application of the theory of principal ideals developed in the previous section, we deal with lattice structures of equivalence classes ($\simeq_w$-classes) of digraphs divided by the w-isomorphism relation $\simeq_w$. By $[G]$ we denote the $\simeq_w$-class of a graph G. The set of all $\simeq_w$-class is an ordered set because $\sqsupseteq$ is well-defined and LEMMA 1.1 holds.

Let G_{irr} be an $\sqsupseteq$-irreducible form and $L(G_{irr}) = \{[G] \mid G \sqsupseteq G_{irr}\}$ through this section. By COROLLALY 2.3, the class $[G_{irr}]$ is the least element of $L(G_{irr})$ because any other $\simeq_w$-class in $L(G_{irr})$ cannot contain an $\sqsupseteq$-irreducible form.

PROPOSITION 2.5 (conditional LUB and GLB) The following claims hold.

(1) Let $[G_1], [G_2], [G_3]$ be $\simeq_w$-classes with $[G_i] \sqsupseteq [G_3]$ $(i = 1, 2)$. There exists the minimum $[G]$ such that $G \sqsupseteq [G_i] \sqsupseteq [G_3]$ $(i = 1, 2)$, denoted by $\mathrm{lub}([G_1], [G_2]; [G_3])$.

(2) Let $[G_0], [G_1], [G_2]$ be $\simeq_w$-classes with $[G_0] \sqsupseteq [G_i]$ $(i = 1, 2)$. There exists the maximum $[G]$ such that $[G_0] \sqsupseteq [G_i] \sqsupseteq [G]$ $(i = 1, 2)$, denoted by $\mathrm{glb}([G_0]; [G_1], [G_2])$.

Proof) Immediate from COROLLALY 2.1 and COROLLALY 2.2. $\square$

PROPOSITION 2.6 The following claims hold.

(1) Let $[G_1], [G_2], [G_3], [G'_3]$ be $\simeq_w$-classes with $[G_i] \sqsupseteq [G_3]$ and $[G_i] \sqsupseteq [G'_3]$ $(i = 1, 2)$. If $[G_3] \sqsupseteq [G'_3]$, then $\mathrm{lub}([G_1], [G_2]; [G_3]) \sqsupseteq \mathrm{lub}([G_1], [G_2]; [G'_3])$.

(2) Let $[G_0], [G'_0], [G_1], [G_2]$ be $\simeq_w$-classes with $[G_0] \sqsupseteq [G_i]$ and $[G'_0] \sqsupseteq [G_i]$ $(i = 1, 2)$. If $[G_0] \sqsupseteq [G'_0]$, then $\mathrm{glb}([G_0]; [G_1], [G_2]) \sqsupseteq \mathrm{glb}([G'_0]; [G_1], [G_2])$.

Proof) (1) Put $[G] = \mathrm{lub}([G_1], [G_2]; [G_3])$, $G' = \mathrm{lub}([G_1], [G_2]; [G'_3])$. By Proposition 2.3, there exist surjective w-homomorphisms $z : G \to G_3$, $z' : G' \to G'_3$ and $u : G_3 \to G'_3$ such that $\mathcal{S}^1 x \cap \mathcal{S}^1 y = \mathcal{S}^1 z$ and $\mathcal{S}^1 xu \cap \mathcal{S}^1 yu = \mathcal{S}^1 z'$. Since $zu \in \mathcal{S}^1 xu$ and $zu \in \mathcal{S}^1 yu$ hold, $zu \in \mathcal{S}^1 z'$ and thus $zu = vz'$ for some $v : G \to G'$ and $v \in \mathcal{S}^1$.

(2) By the left-right duality of (1). $\square$

COROLLARY 2.4 Let $[G_1], [G_2]$ be elements in $L(G_{irr})$. There exists the unique least (resp. greatest) $\simeq_w$ class $[G_U]$ (resp. $[G_L]$) such that $[G_U] \sqsupseteq [G_i]$ $(i = 1, 2)$ (resp. $[G_i] \sqsupseteq [G_L]$ $(i = 1, 2)$), denoted by $\mathrm{lub}(G_1, G_2)$ (resp. $\mathrm{glb}(G_1, G_2)$).

Proof) By PROPOSITION 2.6, $[G_U] = \mathrm{lub}([G_1], [G_2]; [G_{irr}])$ is least. Again, $[G_L] = \mathrm{glb}([G_U]; [G_1], [G_2])$ is greatest. $\square$

From this proposition we get the following theorem.

THEOREM 2.1 The ordered set $(L(G_{irr}), \sqsupseteq)$ forms a lattice with the least element $[G_{irr}]$.

3 Conclusion

In this paper we introduced our graph homomorphisms based on similarity of vertecies. Some algebraic properties related to them were investigeted. We first considered Green's relations and ideals in the semigroup S of all surjecvtive w-homomorphisms between two weighted digraphs, to which is adjoined the extra zero 0. In the semigroup S, the intersection of principal left (resp. right) ideals is also a principal left (resp. right) ideal. This implies two kinds of diamond properties with respect to the pre-order induced by surjective homomorphisms. It is technically interesting to construct such two kinds of synthesis of weighted digraphs. Moreover we apply this results to the lattice structure of $\sim_w$-classes of digraphs.

The following problems remains open, for example, whether the intersection of two principal (two-sided) ideals is also a principal ideal in S,

In addition to these problems, we develop the application of algebraic theories to automorphim groups of weighted digraphs and apply our w-homomorphism to formal languages and codes and to fundamental and common problems related to weighted digraphs.

References

[1] C. Berge. *Principles of Combinatorics*, volume 72 of *Mathematics in Science and Engineering: A Series of Monographs and Textbooks*. Academic Press, 1971.

[2] J. Berstel and D. Perrin. *Theory of Codes*. Academic Press , INC., Orlando, Florida, 1985.

[3] C. Borgs, J. Chayes, L. Lovász, V. Sós, and K. Vesztergombi. Counting graph homomorphisms. In *Topics in discrete mathematics*, pages 315–371. Springer, 2006.

[4] M. Freedman, L. Lovász, and A. Schrijver. Reflection positivity, rank connectivity, and homomorphism of graphs. *Journal of the American Mathematical Society*, 20(1):37–51, 2007.

[5] C. Godsil and G. Royle. *Algebraic Graph Theory*, volume 207 of *Graduate texts in mathematics*. Springer-Verlag New York, Inc, 2001.

[6] P. Hell and J. Nesetril. *Graphs and homomorphisms*. Oxford University Press, 2004.

[7] J. Howie. *Fundamentals of Semigroup Theory*. Oxford University Press, INC., New York, 1995.

[8] M. Ito and Y.Kunimochi. Some petri nets languages and codes. *Lecture Notes in Computer Science*, 2295:69–80, 2002.

[9] N. Jacobson. *Basic Algebra. Vol. 1*. Freeman, 1974.

[10] Y. Kunimochi. Algebraic properties of petri net morphisms based on place connectivity. In P.Dömösi and S. Iván, editors, *Proceedings of Automata and Formal Languages. AFL2011*, pages 270–284, 2011.

[11] Y. Kunimochi, T. Inomata, and G. Tanaka. Automorphism groups of transformation nets (in japanese). *IEICE Trans. Fundamentals*, J79-A,(9):1633–1637, Sep. 1996.

[12] G. Lallement. *Semigroups and Combinatorial Applications*. Pure and applied Mathematics. A Wiley-Interscience Publication, 1979.