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On the ground states of quantum electrodynamics with cutoffs

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This article is a short review of a ground state of a model of quantum electrodynamics in [13]. We investigate a system of a quantized Dirac field coupled to a quantized radiation filed in the Coulomb gauge. The classical Lagrangian density is given by

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\gamma^{\mu}D_{\mu} - M)\psi,$$

where $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$, $D_{\mu} = \partial_{\mu} - eA_{\mu}$ and $\overline{\psi} = \psi^{\dagger}\gamma^{0}$. We define the Hilbert space for the system by $\mathcal{F}_{\text{QED}} = \mathcal{F}_{\text{Dirac}} \otimes \mathcal{F}_{\text{rad}}$ where $\mathcal{F}_{\text{Dirac}}$ is a fermion Fock space over $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and $\mathcal{F}_{\text{rad}}$ is a boson Fock space over $L^2(\mathbb{R}^3 \times \{1,2\})$. The total Hamiltonian for the system is defined by

$$\begin{aligned} H_{\text{QED}} = H_{\text{D}} \otimes 1\!\!1 + 1\!\!1 \otimes H_{\text{rad}} + \kappa_{\text{I}} \sum_{j=1}^{3} \int_{\mathbf{R}^3} \chi_{\text{I}}(\mathbf{x})(\psi^{\dagger}(\mathbf{x})\alpha^{j}\psi(\mathbf{x}) \otimes A_j(\mathbf{x}))d\mathbf{x} \\ + \kappa_{\text{II}} \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{\chi_{\text{II}}(\mathbf{x})\chi_{\text{II}}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \left(\psi^{\dagger}(\mathbf{x})\psi(\mathbf{x})\psi^{\dagger}(\mathbf{y})\psi(\mathbf{y}) \otimes 1\!\!1\right) d\mathbf{x}\,d\mathbf{y}, \end{aligned}$$

where

$$\begin{aligned} H_{\text{D}} &= \sum_{s=\pm 1/2} \int_{\mathbf{R}^3} \omega_M(\mathbf{p}) \left(b_s^{\dagger}(\mathbf{p})b_s(\mathbf{p}) + d_s^{\dagger}(\mathbf{p})d_s(\mathbf{p})\right) d\mathbf{p}, \\ H_{\text{rad}} &= \sum_{r=1,2} \omega(\mathbf{k})a_r^{\dagger}(\mathbf{k})a_r(\mathbf{k})d\mathbf{k}, \end{aligned}$$

with $\omega_M(\mathbf{p}) = \sqrt{\mathbf{p}^2 + M^2}, M > 0$, and $\omega(\mathbf{k}) = |\mathbf{k}|$. The momentum expansions of the fields operators are

$$\begin{aligned} \psi_l(\mathbf{x}) &= \sum_{s=\pm 1/2} \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbf{R}^3} \chi_{\text{D}}(\mathbf{p}) \left(u_s^l(\mathbf{p})b_s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + v_s^l(-\mathbf{p})d_s^{\dagger}(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right) d\mathbf{p}, \\ A_j(\mathbf{x}) &= \sum_{r=1,2} \frac{1}{\sqrt{(2\pi)^3}} \int_{\mathbf{R}^3} \frac{\chi_{\text{rad}}(\mathbf{k})e_r^j(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \left(a_r(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + a_r^{\dagger}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right) d\mathbf{k}, \end{aligned}$$

respectively.

The canonical anti-commutation relations are

$$\begin{aligned}
&\{b_s(\mathbf{p}), b^\dagger_{s'}(\mathbf{p}')\} = \{d_s(\mathbf{p}), d^\dagger_{s'}(\mathbf{p}')\} = \delta_{s,s'}\delta(\mathbf{p}-\mathbf{p}'),\\
&\{b_s(\mathbf{p}), b_{s'}(\mathbf{p}')\} = \{d_s(\mathbf{p}), d_{s'}(\mathbf{p}')\} = 0,\\
&\{b^\dagger_s(\mathbf{p}), b^\dagger_{s'}(\mathbf{p}')\} = \{d^\dagger_s(\mathbf{p}), d^\dagger_{s'}(\mathbf{p}')\} = \{b_s(\mathbf{p}), d^\dagger_{s'}(\mathbf{p}')\} = 0,
\end{aligned}$$

where $\{X,Y\} = XY + YX$. The canonical commutation relations are

$$\begin{aligned}
&[a_r(\mathbf{k}), a^\dagger_{r'}(\mathbf{k}')] = \delta_{r,r'}\delta(\mathbf{k}-\mathbf{k}'),\\
&[a_r(\mathbf{k}), a_{r'}(\mathbf{k}')] = [a^\dagger_r(\mathbf{k}), a^\dagger_{r'}(\mathbf{k}')] = 0,
\end{aligned}$$

where $[X,Y] = XY - YX$.

Assume the following conditions:

(A.1 ; Ultraviolet cutoffs for Dirac field)

$$\int_{\mathbf{R}^3} |\chi_{\mathrm{D}}(\mathbf{p})|^2 d\mathbf{p} < \infty.$$

(A.2 : Ultraviolet cutoffs for radiation field)

$$\int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{rad}}(\mathbf{k})|^2}{\omega(\mathbf{k})^l} d\mathbf{k} < \infty, \quad l = 1,2.$$

(A.3 : Spatial cutoffs)

$$\int_{\mathbf{R}^3} |\chi_{\mathrm{I}}(\mathbf{x})| d\mathbf{x} < \infty, \qquad \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|\chi_{\mathrm{II}}(\mathbf{x})\chi_{\mathrm{II}}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} d\mathbf{x}d\mathbf{y} < \infty.$$

Then H_{QED} is a self-adjoin on $\mathcal{D}(H_0)$. We are interested in the existence of the ground state of H_{QED}. Let H be a self-adjoint operator on a Hilbert space. We say that H has a ground state if the bottom of the spectrum of H is eigenvalue, i.e., $E_0(H) = \inf\sigma(H) \in \sigma_{\mathrm{p}}(H)$. It is seen that $H_0(H) = H_{\mathrm{D}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{rad}}$ has a ground state. Since the mass of the photon is zero, $E_0(H_0)$ is embedded in a continuous spectrum.

Dimassi-Guillot [6] and Brouxbaroux-Dimassi-Giollot [3] consider a QED model with generalized perturbations, and proved the existence of the ground state for sufficiently small values of coupling constants. In [11], the existence of the ground state H_{QED} was proven for sufficiently small values of coupling constants. The main purpose in the paper [13] is to prove the existence of the ground state H_{QED} for all values of coupling constants.

(A.4 : Momentum regularization of Dirac field)

$$\int_{\mathbf{R}^3} |\partial_{p^\nu}\chi_{\mathrm{D}}(\mathbf{p})|^2 d\mathbf{p} < \infty, \quad \int_{\mathbf{R}^3} |\chi_{\mathrm{D}}(\mathbf{p})\partial_{p^\nu} u_s^l(\mathbf{p})|^2 d\mathbf{p} < \infty, \quad \int_{\mathbf{R}^3} |\chi_{\mathrm{D}}(\mathbf{p})\partial_{p^\nu} v_s^l(-\mathbf{p})|^2 d\mathbf{p} < \infty.$$

(A.5 : Momentum regularization of radiation field)

$$\int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{rad}}(\mathbf{k})|^2}{|\mathbf{k}|^5} d\mathbf{k} < \infty, \quad \int_{\mathbf{R}^3} \frac{|\partial_{k^\nu}\chi_{\mathrm{rad}}(\mathbf{k})|^2}{|\mathbf{k}|^3} d\mathbf{k} < \infty, \quad \int_{\mathbf{R}^3} \frac{|\chi_{\mathrm{rad}}(\mathbf{k})\partial_{k^\nu} e_r^j(\mathbf{k})|^2}{|\mathbf{k}|^3} d\mathbf{k} < \infty.$$

(A.6 :Spatial localization)

$$\int_{\mathbf{R}^3} |\mathbf{x}||\chi_{\mathrm{I}}(\mathbf{x})| d\mathbf{x} < \infty, \quad \int_{\mathbf{R}^3\times\mathbf{R}^3} \frac{|\chi_{\mathrm{II}}(\mathbf{x})\chi_{\mathrm{II}}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|} |\mathbf{x}| d\mathbf{x} d\mathbf{y} < \infty.$$

The main theorem in [13] is as follows:

Theorem ([13];Theorem 2.1)
Assume **(A.1)-(A.6)**. Then H_{QED} has a ground state. In particular, its multiplicity is finite.

[Remaining Problems]

(i) Self-adjointness without cutoffs
It is seen that under momentum cutoffs **(A.1),(A.2)** and spatial cutoffs **(A.3)**, H_{QED} is self-adjoint. For the Nelson model, which describes the non-relativistic particles coupled to a scalar field, it was proven that by subtracting momentum divergence terms from the Hamiltonian, there is an unique self-adjoint Hamiltonian [10].

(ii) Infrared divergence
The condition $\int_{\mathbf{R}^3} \frac{\chi_{\mathrm{rad}}(\mathbf{k})|^2}{|\mathbf{k}|^5} d\mathbf{k} < \infty$ in **(A.5)** is stronger than the standard infrared regularity condition. Non-relativistic QED model [2, 7] and spin-boson model [8] have the ground state without infrared cutoffs. On the other hand, the non-existence of the ground state for massless Nelson-model (see e.g.,[5]) and the Generalized spin-boson model [1] were investigated.

(iii) Multiplicity
We see that multiplicity of the ground state of H_{QED} is finite for all values of coupling constants. In [9], the multiplicity of the ground state for various quantum filed models was investigated for sufficiently small values of coupling constants.

(iv) Asymptotic Completeness
In [11], the existence of asymptotic field for the Dirac field and the radiation fields was proven, however its asymptotic completeness has not been shown (see e.g. [4])

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