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A common stabilization of diagrams of a knot

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An oriented *knot* is an oriented circle smoothly embedded in the space $\mathbb{R}^3$. We consider oriented knots up to ambient isotopy in $\mathbb{R}^3$. We do not distinguish a knot and its ambient isotopy class so long as no confusion occurs. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a natural projection defined by $\pi(x, y, z) = (x, y)$. Let K be an oriented knot in $\mathbb{R}^3$. Suppose that the multiple points of the restriction of π to K are only finitely many transverse double points. Then the image $\pi(K)$ together with over/under information at each double point is called a *knot diagram* of K. A double point of a knot diagram is called a *crossing point*. We do not distinguish a knot diagram and its ambient isotopy class in $\mathbb{R}^2$ so long as no confusion occurs.

The *Reidemeister moves* are local moves on knot diagram illustrated in Figure 1. Let n be a positive integer. A sequence of knot diagrams $D_1, \cdots, D_n$ on $\mathbb{R}^2$ is said to be a *Reidemeister sequence* if D_{i+1} is obtained from D_i by an application of one of the Reidemeister moves for each i with $1 \leq i \leq n-1$. It is well-known that two knot diagrams of the same knot are transformed into each other by a finite number of applications of Reidemeister moves. Namely for any two diagrams D and E of a knot K there is a Reidemeister sequence $D_1, \cdots, D_n$ with $D = D_1$ and $D_n = E$. Then we say that $D_1, \cdots, D_n$ is a Reidemeister sequence from D to E. We denote the number of crossings of a knot diagram D by $c(D)$.

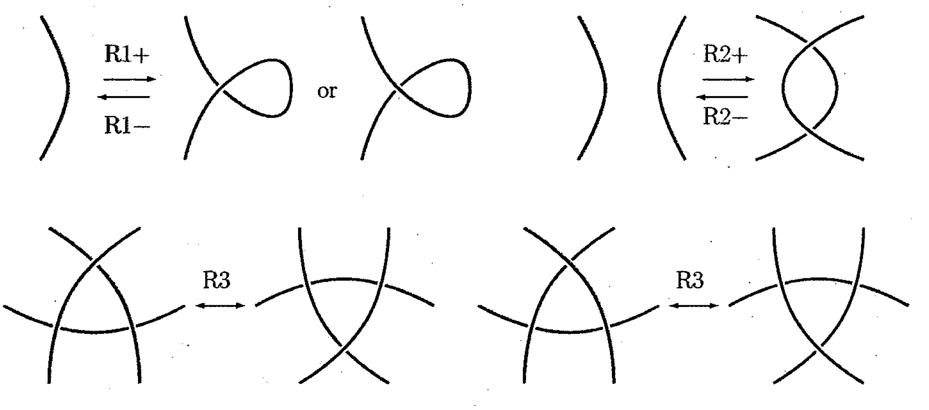

FIGURE 1. Reidemeister moves

It is also well-known that there are a knot K and two diagrams D and E of K such that for any Reidemeister sequence $D_1, \cdots, D_n$ from D to E, there exists $i \in \{2, \cdots, n-1\}$ such that $c(D_i) > \max\{c(D), c(E)\}$. For example, let K_0 be a knot that bounds a disk in $\mathbb{R}^3$, D Goeritz's unknot illustrated in Figure 2 and E a unit circle on the plane. Goeritz's unknot is a knot diagram of K_0 [2]. Note that it has no loops and triangles, and each 2-gon of it has alternating crossings. Therefore we can only apply R1+ or R2+ to it among other Reidemeister moves. Therefore if $D_1, \cdots, D_n$ is a Reidemeister sequence from D to E, then $c(D_2) = c(D_1) + 1$ or $c(D_2) = c(D_1) + 2$ and therefore $c(D_2) > c(D_1) = c(D) = \max\{c(D), c(E)\}$.

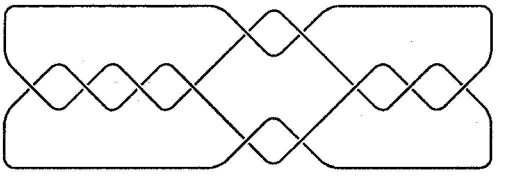

FIGURE 2. Goeritz's unknot

However it is known that there is a function $f(x)$ such that for any diagram D of the knot K_0, there exists a Reidemeister sequence $D_1, \cdots, D_n$ from D to a unit circle E with $\max\{c(D_i) | i \in \{1, \cdots, n\}\} \leq f(c(D))$. See [3] and [4].

A knot diagram E is said to be a *stabilization* (resp. *strong stabilization*) of a knot diagram D if there exists a Reidemeister sequence $D_1, \cdots, D_n$ with $n \geq 1$ from D to E such that $c(D_1) \leq \cdots \leq c(D_n)$ (resp. $c(D_1) < \cdots < c(D_n)$). By definition D is a strong stabilization of D itself. Note that Goeritz's unknot is not a stabilization of a unit circle. Let $D_1, \cdots, D_m$ be knot diagrams. A knot diagram D is said to be a *common stabilization* (resp. *common strong stabilization*) of $D_1, \cdots, D_m$ if D is a stabilization (resp. strong stabilization) of D_i for each $i \in \{1, \cdots, m\}$.

Theorem 1. (Alexander Coward 2006 [1]) *Let K be a knot and D and E diagrams of K. Then there is a Reidemeister sequence from D to E such that the sequence is composed of a sequence of applications of R1+, followed by a sequence of applications of R2+, followed by a sequence of applications of R3, followed by a sequence of applications of R2−.*

Corollary 2. *Let K be a knot and D and E diagrams of K. Then there exists a diagram F of K such that F is a stabilization of D and F is a strong stabilization of E.*

Corollary 3. *Let K be a knot and $D_1, \cdots, D_m$ diagrams of K. Then there exists a diagram D of K such that D is a common stabilization of $D_1, \cdots, D_m$.*

Example 4. Let D and E be knot diagrams illustrated in Figure 3. Note that D is a diagram of the knot $3_1\#3_1\#3_1^*\#3_1^*$ and E is a diagram of the knot $3_1\#3_1^*\#3_1\#3_1^*$ where 3_1 denotes the right-handed trefoil knot, 3_1^* denotes the left-handed trefoil knot and $J\#K$ denotes the connected sum of two knots J and K. Since connected sum operation is commutative we have $3_1\#3_1\#3_1^*\#3_1^* = 3_1\#3_1^*\#3_1\#3_1^*$. Therefore D and E are diagrams of the same knot. By pulling one of 3_1^* tight and sliding it along one of 3_1 we have a Reidemeister sequence from D to E through diagrams at most 14 crossings. However the corresponding sequence of crossing numbers cannot be divided into a weakly increasing sequence and a subsequent weakly decreasing sequence. Let F be a knot diagram illustrated in Figure 3. Note that E is obtained from F by 12 times applications of R2− and D is obtained from F by 54 times applications of R3 followed by 12 times applications of R2−. Therefore F is a strong stabilization of E and F is a stabilization of D.

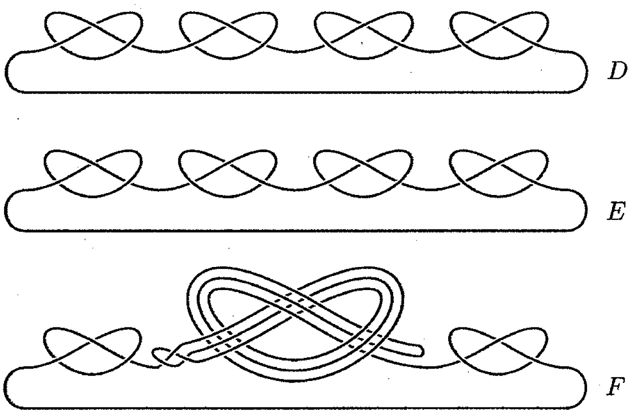

FIGURE 3. F is a common stabilization of D and E

We note that Corollary 2 is best possible. Namely we have the following results. A knot diagram D is said to be (R1, R2)-*reduced* if D has no loops and each 2-gon of D has alternating crossings. Then the following theorem is a paraphrase of a result in [8, Theorem 3.2] where not only knot diagrams but also link diagrams are considered. We note that a closely related result is shown in [5, Theorem 2.2 (3)]. See also [6] and [7].

Theorem 5. *Let K be a knot and D and E* (R1, R2)*-reduced diagrams of K. Suppose that D and E have a common strong stabilization. Then D and E are ambient isotopic on $\mathbb{R}^2$ as oriented knot diagrams, or both D and E are simple closed curves with opposite orientations.*

Corollary 6. *For any knot K, there are diagrams D and E of K that have no common strong stabilizations.*

Proof. It is clear that K has at least one (R1, R2)-reduced diagram D. Let E be a diagram-connected sum of D and a Goeritz's unknot. Then E is also a (R1, R2)-reduced diagram of K. Since D and E are not ambient isotopic on $\mathbb{R}^2$, Theorem 5 implies that they have no common strong stabilizations. □

Two knot diagrams D and E are said to be R1-R2-*equivalent* if there exists a Reidemeister sequence $D_1, \cdots, D_n$ with $D = D_1$ and $D_n = E$ such that D_{i+1} is obtained from D_i by an application of one of R1+, R1−, R2+ and R2− for each i with $1 \leq i \leq n-1$. The following is an immediate consequence of Theorem 5.

Theorem 7. *Let K be a knot and D and E diagrams of K. Let D' and E' be* (R1, R2)*-reduced diagrams obtained from D and E respectively by applications of R1− and R2−. Then the following conditions are equivalent.*

(1) *Two diagrams D and E are R1-R2-equivalent.*

(2) *Two diagrams D' and E' are ambient isotopic on $\mathbb{R}^2$ as oriented knot diagrams, or both D' and E' are simple closed curves with opposite orientations.*

Proof. Suppose that D' and E' are ambient isotopic on $\mathbb{R}^2$ as oriented knot diagrams. Then D and E are R1-R2-equivalent. Suppose that both D' and E' are simple closed curves with opposite orientations. It is easy to see that D' and E' are R1-R2-equivalent. Therefore D and E are R1-R2-equivalent. Thus we have shown that the condition (2) implies the condition (1).

Suppose that D and E are R1-R2-equivalent. Let $D_1, \cdots, D_n$ be a Reidemeister sequence with $D = D_1$ and $D_n = E$ such that D_{i+1} is obtained from D_i by an application of one of R1+, R1−, R2+ and R2− for each i with $1 \leq i \leq n-1$. Let D'_i be an (R1, R2)-reduced diagram obtained from D_i by applications of R1− and R2− for each i with $2 \leq i \leq n-1$. Let $D'_1 = D'$ and $D'_n = E'$. Then D_i or D_{i+1} is a common strong stabilization of D'_i and D'_{i+1} for each i with $1 \leq i \leq n-1$. Then by Theorem 5 D'_i and D'_{i+1} are ambient isotopic on $\mathbb{R}^2$ as oriented knot diagrams, or both D'_i and D'_{i+1} are simple closed curves with opposite orientations for each i with $1 \leq i \leq n-1$. Therefore $D' = D'_1$ and $E' = D'_n$ are ambient isotopic on $\mathbb{R}^2$ as oriented knot diagrams, or both D' and E' are simple closed curves with opposite orientations. Thus we have shown that the condition (1) implies the condition (2). □

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