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REGULARIZED PETERSSON INNER PRODUCTS FOR MEROMORPHIC MODULAR FORMS

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ABSTRACT. We investigate the history of inner products within the theory of modular forms. We first give the history of the applications of Petersson's original definition for the inner product of S_{2k} and then recall Zagier's extension to a non-degenerate (but not necessarily positive-definite) inner product on all holomorphic modular forms. We then recall the history of the so-called "regularization" of the inner product to extend it to weakly holomorphic modular forms originally by Petersson and then later independently rediscovered by Harvey–Moore and Borcherds, as well as its applications to theta lifts by Borcherds, Bruinier–Funke, and many more recent authors. This has been recently extended to a well-defined inner product on all weakly holomorphic modular forms by Bringmann, Diamantis, and Ehlen. Finally, we consider inner products on meromorphic modular forms which have poles in the upper half-plane. Petersson also defined a regularization in this case by cutting out small neighborhoods around each pole occurring in the fundamental domain; Bringmann, von Pippich, and the author have recently constructed an extension of this regularization, which, when combined with the regularization of Bringmann, Diamantis, and Ehlen, yields an inner product that is well-defined and finite on all meromorphic modular forms.

1. INTRODUCTION

The Petersson inner product has a long history within the theory of automorphic forms. This expository paper serves as a brief sojourn through that history. Petersson [14] provided a well-defined and finite (see Section 2) Hermitian inner product on the space S_{2k} of weight $2k \in 2\mathbb{N}$ cusp forms on $\mathrm{SL}_2(\mathbb{Z})$ (Petersson considered his inner product on modular forms for much more general Fuchsian groups, but for simplicity of the exposition, we restrict ourselves to $\mathrm{SL}_2(\mathbb{Z})$). Roughly speaking, the idea of Petersson's inner product is to construct a function which is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ and then integrate over an arbitrary fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, where $\mathbb{H}$ is the complex upper half-plane.

For $f, g \in S_{2k}$, we denote Petersson's inner product by $\langle f, g\rangle$. The inner product has a number of applications. Firstly, the inner product is non-degenerate (and even positive-definite) on S_{2k}, yielding an orthogonal splitting; this splitting may be explicitly realized by decomposing into the (one-dimensional) simultaneous eigenspaces under the Hecke operators. Secondly, Petersson used his inner product to establish the

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well-known **Petersson coefficient formula** (see Section 2.3 and particularly Theorem 2.1). The coefficient formula gives a way to relate the coefficients of cusp forms with the inner product of the the cusp forms against certain distinguished elements called the Poincaré series. Poincaré series are generalizations of the well-known *Eisenstein series*

$$E_{2k}(z) := \sum_{M=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} (cz+d)^{-2k}, \tag{1.1}$$

where $\Gamma_\infty := \{\pm T^n : n \in \mathbb{Z}\}$ with $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Petersson's coefficient formula uses a technique called "unfolding", where the sum in (1.1) is used to extend the integral over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ to an integral over $\Gamma_\infty\backslash\mathbb{H}$. The fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$ is very simple, allowing one to explicitly compute integral by plugging in Fourier expansions. In doing so, Petersson obtains the Fourier coefficients of the modular forms by replacing the summand $(cz+d)^{-2k}$ with another appropriate function.

It is natural to ask whether one can extend the inner product to include inner products with the Eisenstein series E_{2k} defined in (1.1). Petersson's original definition suffices when one takes the inner product of E_{2k} with a cusp form, and reveals that E_{2k} is orthogonal to all cusp forms. However, the inner product diverges when trying to compute the *Petersson norm*

$$\|f\|^2 := \langle f, f\rangle \tag{1.2}$$

for $f = E_{2k}$. Zagier [21] later managed to extend the inner product to this case and proved that the Petersson inner product on holomorphic modular forms is indeed non-degenerate, but in general it is not positive-definite (in particular, the norm of E_{2k} is either positive or negative, depending on the parity of k).

We next consider the inner product on forms in the space $M_{2k}^!$ of weight $2k$ *weakly holomorphic modular forms* (i.e., meromorphic modular forms all of whose poles are contained at cusps). Unfortunately, the naive definition usually diverges, even between a cusp form and a weakly holomorphic modular form. There is however a trick which allows one to consider inner products on this space, which appears to have been first realized by Petersson [15] and then later rediscovered by Harvey–Moore [11] and Borcherds [2]. One "regularizes" the integral over $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (see Section 3). Petersson's original attempt to do so involved taking the Cauchy principal value of the integral by integrating over a part $\mathcal{F}_T$ ($T \in \mathbb{R}$) of the fundamental domain bounded away from the cusp of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ such that the limit of $\mathcal{F}_T$ as $T \to \infty$ becomes an entire fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Essentially, this is the same as choosing an ordering on the integral over the fundamental domain. Borcherds [2], Bruinier–Funke [6], and numerous other authors have used this regularized inner product to compute theta lifts between modular forms on orthogonal groups.

Finally, we study the inner product on meromorphic modular forms with poles in the upper half-plane. The naive inner product again diverges, and one requires a regularization. Petersson [15] defined the Cauchy principal value in this case by cutting out small neighborhoods around each pole and shrinking the volume of these neighborhoods to zero in the limit. His definition extended the inner product to many cases, but it still diverges in many cases; in particular, the Petersson inner product for non-cusp forms always diverges with Petersson's regularization. In Section 4, we discuss in detail a recent extension of Petersson's regularization by Bringmann, von

Pippich, and the author [5] which may be combined with Bringmann, Diamantis, and Ehlen's [3] regularization to yield a well-defined and finite inner product on the space $\mathcal{S}_{2k}$ of all meromorphic modular forms. One application of the new regularization is a formula relating the higher Green's functions evaluated at CM-points with the inner product between certain distinguished weight $2k$ meromorphic modular forms f_Q (Q a positive-definite integral binary quadratic form) which generalize the cusp forms $f_{k,D}$ ($D > 0$ a discriminant) which first occurred in Zagier's paper [20] and were later used by Kohnen and Zagier [12] to construct a kernel function for the Shimura [18] and Shintani [19] lifts between integral and half-integral weight modular forms.

2. Petersson inner products

2.1. Holomorphic modular forms and their generalizations. Define the weight $2k$ slash action $|_{2k}$ with a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ by

$$f|_{2k}M(z) := (cz+d)^{-2k}f(Mz),$$

where M acts on $\mathbb{H}$ via fractional linear transformations. A weight $2k$ (holomorphic) *modular form* (on $\mathrm{SL}_2(\mathbb{Z})$) is a function $f : \mathbb{H} \to \mathbb{C}$ for which the following hold.

(1) For all $M \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$f|_{2k}M = f. \tag{2.1}$$

(2) The function f is holomorphic on $\mathbb{H}$.
(3) The function f has a Fourier expansion of the type

$$f(z) = \sum_{n\geq 0} a_f(n)e^{2\pi i n z}. \tag{2.2}$$

If $a_f(0) = 0$, then we call f a *cusp form.*

More generally, if we replace condition (2) with meromorphicity (resp. holomorphicity) and condition (3) with Fourier expansions (2.2) with the weaker restriction $n \gg -\infty$, then we obtain the definition for meromorphic modular forms (resp. weakly holomorphic modular forms). Later in the paper, we will even replace condition (2) with the property that f is real analytic and annihilated by a certain differential operator Δ_{2k} called the weight $2k$ hyperbolic Laplacian (see (3.4)); in this case, the coefficients $a_f(n)$ in (2.2) are replaced with coefficients $a_f(y; n)$ which may depend on the imaginary part y of z and there is not restriction on n (i.e., $n \in \mathbb{Z}$). Doing so (replacing (2) with annihilation by Δ_{2k}) yields the definition of a special class of *non-holomorphic modular forms* known as *harmonic Maass forms.* Analogously to the change in condition (2) from holomorphic modular forms to meromorphic modular forms, for non-holomorphic modular forms we may also allow (not necessarily meromorphic) singularities in the upper half-plane or at cusps. This final class of forms are called *polar harmonic Maass forms.*

In all of the above generalizations, the one property which has remained unchanged is (2.1). This is the main essence of the definition. Of course, there are generalizations where the condition $M \in \mathrm{SL}_2(\mathbb{Z})$ is restricted to $M \in \Gamma$ for some subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ and one can slightly augment the definition of the slash operator $|_{2k}$ (for example, allowing a character) or allow $k \in \mathbb{Q}$, $k \in \mathbb{R}$ or even $k \in \mathbb{C}$, but essentially these changes do not modify (2.1). The condition (2.1) is thus aptly called weight $2k$ *modularity.*

2.2. **Definition of the inner product.** Considering the variables z and $\overline{z}$ as independent variables, note that for a weight $2k$ modular form $f(z)$, the function $\overline{f(z)}$ satisfies weight $2k$ modularity as a function of $\overline{z}$. Furthermore, writing $z = x+iy \in \mathbb{H}$, the function y^{2k} satisfies simultaneous weight $-2k$ modularity in both z and $\overline{z}$ because

$$\mathrm{Im}(Mz) = \mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \frac{\mathrm{Im}((az+b)(c\overline{z}+d))}{|cz+d|^2} = \frac{y}{|cz+d|^2},$$

where we used the fact that $ad - bc = 1$.

Petersson [14] then realized that, for functions f and g satisfying (2.1) (i.e., satisfying modularity) for all $M \in \mathrm{SL}_2(\mathbb{Z})$, the function

$$f(z)\overline{g(z)}y^{2k}$$

is $\mathrm{SL}_2(\mathbb{Z})$-invariant. Moreover, the metric

$$\frac{dxdy}{y^2}$$

is also $\mathrm{SL}_2(\mathbb{Z})$-invariant. Hence the integral

$$\langle f, g\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2} \tag{2.3}$$

is well-defined whenever it converges absolutely. Using bounds for cusp forms (in particular, they exponentially decay as $y \to \infty$), one can show that the integral (2.3) converges absolutely for $f, g \in S_{2k}$. This exponential decay also suffices to show convergence when taking the inner product between $f \in S_{2K}$ and the Eisenstein series E_{2k} defined in (1.1).

2.3. **Petersson coefficient formula.** The Petersson coefficient formula uses an explicit evaluation of the inner product to compute the Fourier coefficients (in the expansion (2.2)) of modular forms. To describe this result, we require the classical *Poincaré series* (see [16, 17])

$$P_{2k,m}(z) := \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \varphi_m\big|_{2k}M(z), \tag{2.4}$$

where $k \in \mathbb{N}_{\geq 2}$ and for $m \in \mathbb{Z}$

$$\varphi_m(z) := e^{2\pi i m z}.$$

These converge locally and absolutely uniformly. For $m = 0$, the Poincaré series is precisely the Eisenstein series (1.1), while for $m > 0$ we have $P_{2k,m} \in S_{2k}$ and for $m < 0$ we have $P_{2k,m} \in M_{2k}^!$.

Theorem 2.1 (Petersson coefficient formula). *If $f \in S_{2k}$ and $m \in \mathbb{N}$, then*

$$\langle f, P_{2k,m}\rangle = \frac{(2k-2)!}{(4\pi m)^{2k-1}}a_f(m).$$

Sketch of proof. Plugging in the definition (2.4) of the Poincaré series $P_{2k,m}$ and choosing a *fundamental domain* $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (a "nice" connected set of representatives $z \in \mathbb{H}$ of the orbits of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ under fractional linear transformations), we *unfold* the integral on the left-hand side by rewriting (formally, but this is valid because of the exponential decay of cusp forms towards the cusps)

$$\begin{aligned}
&\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z) \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \frac{\overline{\varphi_m(Mz)}}{(c\overline{z}+d)^{2k}} y^{2k}\frac{dxdy}{y^2}\\
&\quad= \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \int_{\mathcal{F}} f(Mz)\varphi_m(Mz)\mathrm{Im}(Mz)^{2k}\frac{dxdy}{y^2}\\
&\quad= \sum_{M\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})} \int_{M\mathcal{F}} f(z)\varphi_m(z)y^{2k}\frac{dxdy}{y^2} = \int_{\Gamma_\infty\backslash\mathbb{H}} f(z)\overline{\varphi_m(z)}y^{2k}\frac{dxdy}{y^2}. \quad (2.5)
\end{aligned}$$

Since the fundamental domain for $\Gamma_\infty\backslash\mathbb{H}$ is very simple, this unfolding argument results in the double integral

$$\int_0^\infty \int_0^1 f(z)\overline{\varphi_m(z)}y^{2k}\frac{dxdy}{y^2}.$$

The integral over x essentially picks off the mth coefficient and then explicitly computing the integral over y yields the claim. □

2.4. **Orthogonal splitting.** The inner product on S_{2k} is positive-definite. Hence, by the Gram-Schmidt process, one can construct an orthonormal basis. A particular choice of the basis elements turns out to be very natural.

There are certain operators T_n known as the *Hecke operators* and defined for each $n \in \mathbb{N}$ by (these are normalized differently in different books and papers for various purposes, but the normalization is not important for the discussion at hand)

$$f|_{2k}T_n := \sum_{M\in\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{M}_n} f|_{2k}M,$$

where $\mathcal{M}_n$ denotes the set of 2×2 integral matrices with determinant n. The Hecke operators commute and are Hermitian with respect to the Petersson inner product. By the Spectral Theorem, one may therefore diagonalize to obtain simultaneous eigenfunctions under all T_n. These simultaneous eigenfunctions are known as *Hecke eigenforms.* The Hecke eigenforms $f \in S_{2k}$ are often normalized to have $a_f(1) = 1$, but another natural normalization to take is $\|f\|^2 = 1$, where the Petersson norm $\|\cdot\|^2$ was defined in (1.2). The Hecke operators satisfy what is known as *multiplicity one*, which means that the eigenspaces of simultaneous eigenfunctions under all Hecke operators are all one-dimensional (indeed, they satisfy a much stronger condition known as strong multiplicity one). Hence, for two distinct Hecke eigenforms $f, g \in S_{2k}$, there exists $n \in \mathbb{N}$ for which the eigenvalues $\lambda_f(n)$ and $\lambda_g(n)$ differ. However, since the Hecke operators are Hermitian, we have

$$\lambda_f(n)\langle f,g\rangle = \langle \lambda_f(n)f,g\rangle = \langle f|_{2k}T_n,g\rangle = \langle f,g|_{2k}T_n\rangle = \langle f,\lambda_g(n)g\rangle = \lambda_g(n)\langle f,g\rangle.$$

Since $\lambda_f(n) \neq \lambda_g(n)$, this leads to a contradiction if $\langle f,g\rangle \neq 0$. We thus conclude that f and g are orthogonal to each other. Hence the splitting of S_{2k} into eigenspaces precisely yields the orthogonal splitting, with the orthonormal basis given by the Hecke eigenforms normalized such that $\|f\|^2 = 1$.

We note that the other normalization $a_f(1) = 1$ is also natural. Under this normalization (and appropriately normalizing the Hecke operators), the coefficients $a_f(n)$ and the eigenvalues $\lambda_f(n)$ coincide. This realization "de-mystifies" the coefficients of the Hecke eigenforms and plays an important role in understanding Fourier expansions.

3. Inner products for weakly holomorphic modular forms

3.1. The regularization of Petersson, Harvey–Moore, and Borcherds and its extension. For $f, g \in M_{2k}^!$, the integral (2.3) generally diverges. Petersson established a Cauchy principal value for the integral as a partial solution to this problem. Firstly, one chooses a specific fundamental domain for $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. We choose the *standard fundamental domain* (for simplicity, we take the closed fundamental domain; this is easier to write down, but technically there are points on the boundary which are $\mathrm{SL}_2(\mathbb{Z})$-equivalent; however, since we will ultimately integrate over it and the boundary is a measure zero set, this is irrelevant for our consideration)

$$\mathcal{F} := \left\{ z \in \mathbb{H} : |z| \geq 1,\ -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}.$$

Instead of integrating over $\mathcal{F}$ in (2.3), we integrate over a *cut-off fundamental domain* whose closure does not include the cusp on the boundary of the chosen fundamental domain. In our case, the cusp is $i\infty$ and the cut-off fundamental domain is given by

$$\mathcal{F}_T := \left\{ z \in \mathbb{H} : |z| \geq 1,\ y \leq T,\ -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}.$$

For $f, g \in M_{2k}^!$, Petersson then defined the regularized inner product (see [15])

$$\langle f, g \rangle := \lim_{T \to \infty} \int_{\mathcal{F}_T} f(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2}. \tag{3.1}$$

The key to the above regularization is that it essentially gives an ordering to the integrals over x and y.

This construction was further independently rediscovered and extended by Harvey–Moore [11] and Borcherds [2] by multiplying the integrand by y^s for some $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$ and then taking the constant term of the Laurent expansion of the meromorphic continuation (in s) at $s = 0$.

One can use the regularized inner product to show that for $m < 0$ the Poincaré series $P_{2k,m}$, defined in (2.4), is orthogonal to cusp forms. This was shown by Petersson in a much more general setting in [15, Satz 4].

The regularization of Petersson/Harvey–Moore/Borcherds does not always converge, however. In particular, Petersson found a necessary and sufficient condition for his regularization (3.1)to converge (see [15, Satz 1]) and Petersson norms once again pose a problem, as they did for the Eisenstein series. This problem has been recently resolved by Bringmann, Diamantis, and Ehlen [3], who were able to extend the regularization in a way so that the inner product $\langle f, g \rangle$ is well-defined and finite for all $f, g \in M_{2k}^!$. We do not give any of the technical details here, but the reader is encouraged to look at [3, Section 3, and in particular Theorem 3.2].

3.2. Theta lifts. The inner product has been used by many authors (for example, in [2] and [6]) to obtain theta lifts from modular forms of one type to modular forms of another type. To give a rough idea, one defines a two-variable theta function $\Theta(z, \tau)$ which is modular in both variables (one calls this function the *theta kernel*), but which satisfies a different kind of modularity in each variable (for example, suppose that it satisfies weight $2k$ modularity as a function of z and weight $k + 1/2$ modularity as a function of τ). Taking the inner product in one variable against another function f satisfying the same type of modularity then yields a new function in the other variable satisfying the other type of modularity. In other words, in the example above, if f satisfies weight $2k$ modularity, then

$$\Phi(f)(\tau) := \langle \Theta(\cdot, \tau), f \rangle$$

satisfies weight $k + 1/2$ modularity. This yields a *theta lift* Φ from weight $2k$ modular forms to weight $k + 1/2$ modular forms. The example illustrated above is Shintani's construction [19] of his lift from integral weight to half-integral weight modular forms and the lift in the opposite direction can be shown to be one of Shimura's lifts [18] from half-integral weight to integral weight (see [13] and [12] for two alternative options for the theta kernel). Note: although we do not define half-integral weight modular forms here, one may simply think of these as generalizations of modular forms where the slash operator is slightly augmented to resolve the issue that the square root is multi-valued and then modularity is again defined by (2.1).

Lifts from "simpler" spaces with special properties often yield strange or exceptional modular forms which can be used to understand or narrow down conjectures that are often precisely false on the image or pre-image of such lifts. For example, the Shimura lift generally sends cusp forms to cusp forms, but there is an exceptional class of forms known as unary theta functions in weight 3/2 which are cusp forms but whose image under the Shimura lift is an Eisenstein series. These unary theta functions are also counter-examples to the *Ramanujan–Petersson conjecture*, which states that the coefficients of weight $\kappa \in \frac{1}{2}\mathbb{Z}$ cusp forms f satisfy

$$|a_f(n)| \ll_{f,\varepsilon} n^{\frac{\kappa-1}{2}+\varepsilon}.$$

The coefficients of the unary theta functions grow like $n^{1/2}$, contradicting the conjecture in this wide of generality. However, for integral weight cusp forms $f \in S_{2k}$, the conjecture is a celebrated result of Deligne [7] and it is conjectured that the Ramanujan–Petersson conjecture holds in half-integral weight as long as f is orthogonal to unary theta functions.

3.3. **Computation of the inner product by the Brunier–Funke pairing.** For $f, g \in M_{2k}^!$, we next describe a way to compute the inner product between these two forms. There is a natural function G satisfying weight $2 - 2k$ associated with g. The inner product between f and g is then given by a pairing between the function G and f given by

$$\{f, G\} := \sum_{n \in \mathbb{Z}} a_f(-n) a_G^+(n), \tag{3.2}$$

where $a_G^+(n)$ is the nth coefficient of the holomorphic part of the Fourier expansion (which has the same shape as (2.2)). In particular, we have

$$\langle f, g \rangle = \{f, G\}. \tag{3.3}$$

The pairing is useful for computing inner products because only finitely many terms in (3.2) are non-zero.

Roughly speaking, the pairing is shown by using Stokes Theorem to evaluate the integral instead of the unfolding method described in Section 2.3. When applying

Stokes Theorem, a pre-image G of g under the operator $\xi_{2-2k} := 2iy^{2-2k}\overline{\frac{\partial}{\partial \overline{z}}}$ naturally appears. Since g is weakly holomorphic, we have

$$\Delta_{2-2k}(G) = -\xi_{2k}(g) = 0,$$

where

$$\Delta_{2-2k} := -\xi_{2k} \circ \xi_{2-2k} \tag{3.4}$$

is the weight $2-2k$ hyperbolic Laplacian. Therefore, the pre-image G is what is known as a weight $2-2k$ *harmonic Maass form* (i.e., it satisfies weight $2-2k$ modularity, it is annihilated by Δ_{2-2k}, and it grows at most linear exponentially towards the cusps).

The pairing was first introduced by Bruinier and Funke in [6]. Its connection to inner products defined as regularized integrals was then realized in a number of cases by many authors and one may interpret the recent results in [3] as giving an analytic interpretation via a regularized integral for the Bruinier–Funke pairing in the general case for any arbitrary $f, g \in M_{2k}^!$.

4. Inner products for meromorphic modular forms

We would now like to define an inner product on arbitrary meromorphic modular forms $f, g \in \mathcal{S}_{2k}$. However, an arbitrary meromorphic modular form $f \in \mathcal{S}_{2k}$ may be decomposed into two pieces, one of which only has poles at the cusps (i.e., it is in $M_{2k}^!$) and one of which only has poles in the upper half-plane (vanishing towards all cusps); we call forms of the second type weight $2k$ *meromorphic cusp forms* and denote the subspace of such forms by $\mathbb{S}_{2k}$. It thus essentially suffices to consider inner products between forms $f, g \in \mathbb{S}_{2k}$ (technically, we also have to take inner products between forms $f \in M_{2k}^!$ and $g \in \mathbb{S}_{2k}$, but hybrid approaches for the regularizations will work in full generality and we ignore the details here).

4.1. **Regularization of Petersson.** The idea that Petersson used to generalize (2.3) is very similar to the idea used in the regularization (3.1). Instead of cutting off the fundamental domain away from $i\infty$, one cuts out small neighborhoods around each pole $\mathfrak{z}$ of f or g and then shrinks the hyperbolic volume of the neighborhoods to zero in a limit. In particular, for $\mathfrak{z} \in \mathbb{H}$ define the ball

$$\mathcal{B}_\varepsilon(\mathfrak{z}) := \left\{z \in \mathbb{H} : r_{\mathfrak{z}}(z) < \varepsilon\right\},$$

where $r_{\mathfrak{z}}(z) := |X_{\mathfrak{z}}(z)|$ with

$$X_{\mathfrak{z}}(z) := \frac{z-\mathfrak{z}}{z-\overline{\mathfrak{z}}}.$$

The functions $r_{\mathfrak{z}}(z)$ are naturally connected to the hyperbolic distance $d(z, \mathfrak{z})$ between z and $\mathfrak{z} = \mathfrak{z}_1 + i\mathfrak{z}_2$ in $\mathbb{H}$ via the formula

$$r_{\mathfrak{z}}(z) = \tanh\left(\frac{d(z,\mathfrak{z})}{2}\right);$$

recall that the hyperbolic distance may be expressed through (see p. 131 of [1])

$$\cosh\left(d(z,\mathfrak{z})\right) = 1 + \frac{|z-\mathfrak{z}|^2}{2y\mathfrak{z}_2}. \tag{4.1}$$

Let $[z_1], \ldots, [z_r] \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ be the distinct $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of all of the poles of $f, g \in \mathbb{S}_{2k}$ and choose a fundamental domain $\mathcal{F}^*$ such that all z_ℓ lie in the interior of $\Gamma_{z_\ell}\mathcal{F}^*$, where $\Gamma_{\mathfrak{z}}$ is the stabilizer of $\mathfrak{z}$ in $\mathrm{PSL}_2(\mathbb{Z})$. Petersson's regularized inner product is then defined by

$$\langle f, g\rangle := \lim_{\varepsilon_1,\ldots,\varepsilon_r\to 0^+} \int_{\mathcal{F}^*\setminus\left(\bigcup_{\ell=1}^r \mathcal{B}_{\varepsilon_\ell}(z_\ell)\right)} f(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2}. \tag{4.2}$$

A necessary and sufficient condition for the convergence of the regularization (4.2) is given in [15, Satz 1]. Furthermore, certain Poincaré series related to the elliptic expansions (Petersson proved an elliptic coefficient formula as well; cf. [15, Satz 9]) with poles in the upper half-plane were also shown to be orthogonal to cusp forms in [15, Satz 7]. Once again, Petersson's necessary and sufficient condition implies that his regularization diverges in particular when evaluating Petersson norms for elements of $\mathbb{S}_{2k}$ which are not cusp forms.

4.2. **A new regularization.** Since Petersson's regularization still sometimes diverges, one requires a further regularization; we recall the construction from [5]. Roughly speaking, the integrand in (2.3) is multiplied by an $\mathrm{SL}_2(\mathbb{Z})$-invariant function $H_s(\tau)$ which removes the poles of the integrand whenever $\mathrm{Re}(s)$ is sufficiently large. We then take the constant term of the Laurent expansion around $s = 0$ to be our regularization. To be more precise, let $[z_1], \ldots, [z_r] \in \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$ be the distinct $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of all of the poles of f and g and define

$$\langle f, g\rangle := \mathrm{CT}_{s=0}\left(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} f(z)H_s(z)\overline{g(z)}y^{2k}\frac{dxdy}{y^2}\right), \tag{4.3}$$

where

$$H_s(z) = H_{s_1,\ldots,s_r,z_1,\ldots,z_r}(z) := \prod_{\ell=1}^r h_{s_\ell,z_\ell}(z).$$

Here

$$h_{s_\ell,z_\ell}(z) := r_{z_\ell}^{2s_\ell}(Mz),$$

with $M \in \mathrm{SL}_2(\mathbb{Z})$ chosen such that $Mz \in \mathcal{F}^*$. Moreover $\mathrm{CT}_{s=0}$ denotes the constant term in the Laurent expansion around $s_1 = s_2 = \cdots = s_r = 0$ of the meromorphic continuation (if existent).

In the same sense that the results in [3] may be viewed as an analytic definition for a regularized inner product satisfying the Bruinier–Funke pairing for arbitrary $f, g \in M_{2k}^!$, the above regularization may be viewed as an analytic definition for a regularized integral giving a similar pairing for all $f, g \in \mathbb{S}_{2k}$. However, instead of defining the pairing via the Fourier expansions, the pairing is defined via the elliptic expansions of f and a weight $2-2k$ *polar harmonic Maass form* (i.e., a harmonic Maass form with singularities in the upper half-plane) G which is a pre-image of g under the ξ-operator. To describe the pairing, the *elliptic expansion* of $f \in \mathbb{S}_{2k}$ around $\mathfrak{z} \in \mathbb{H}$ is given by

$$f(z) = (z-\overline{\mathfrak{z}})^{-2k} \sum_{n\gg-\infty} a_{f,\mathfrak{z}}(n)X_{\mathfrak{z}}^n(z). \tag{4.4}$$

For the polar harmonic Maass form G, we again denote the coefficients of its meromorphic part (i.e., of the form in (4.4)) by $a_{G,\mathfrak{z}}^+(n)$.

Denoting $\mathfrak{z}_2 := \mathrm{Im}(\mathfrak{z})$ and writing $\omega_\mathfrak{z}$ for the size of the stabilizer $\Gamma_\mathfrak{z}$ of $\mathfrak{z}$ in $\mathrm{PSL}_2(\mathbb{Z})$, the pairing is given by (see [4, Proposition 6.1])

$$\{f, G\} := \sum_{\mathfrak{z}\in \mathrm{SL}_2(\mathbb{Z})\backslash \mathbb{H}} \frac{\pi}{\mathfrak{z}_2\omega_\mathfrak{z}} \sum_{n\in\mathbb{Z}} a_{f,\mathfrak{z}}(n) a^+_{G,\mathfrak{z}}(-n-1). \tag{4.5}$$

It is again important to emphasize that the pairing gives a formula for the inner product with only finitely many coefficients in (4.5) non-zero. In comparison, Petersson evaluated his inner product (3.1) (resp. (4.2)) on [15, pages 42–43] via the Fourier (res. elliptic) coefficients of the forms f and g themselves, but his evaluation is given as an infinite sum, so one can only obtain an approximation for the inner product by computing the Fourier (resp. elliptic) coefficients. In other words, Petersson's constructions are better in the sense that they are given in terms of the coefficients of the original functions, while one is required to introduce new functions to determine (3.2) and (4.5), but the sums in these pairings are instead finite.

4.3. **Higher Greens functions.** The regularization (4.3) was used in [5] to compute the inner product between

$$f_Q(z) = f_{k,-D,[Q]}(z) := D^{\frac{k}{2}} \sum_{\mathcal{Q}\in[Q]} \mathcal{Q}(z,1)^{-k}$$

for positive-definite integral binary quadratic forms Q of discriminant $-D$. These are weight $2k$ meromorphic modular forms which have poles of order k at the unique zero τ_Q of Q in $\mathbb{H}$. The evaluation of the inner product between two such functions is done by again using Stokes Theorem to rewrite the inner product as the pairing (4.5) in terms of the elliptic coefficients of f_Q and the elliptic coefficients of the meromorphic part of a polar harmonic Maass form $\mathcal{G}_Q$ associated with f_Q via the ξ-operator. It then remains to explicitly compute the elliptic coefficients occurring in (4.5).

In particular, choosing two such binary quadratic forms Q and $\mathcal{Q}$, the inner product between f_Q and $f_\mathcal{Q}$ is related to the *higher Green's function* $G_k : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$, which is uniquely characterized by the following properties:

(1) G_k is a smooth real-valued function on $\mathbb{H}\times\mathbb{H}\setminus\{(z,\gamma z) | \gamma\in\Gamma, z\in\mathbb{H}\}$.
(2) For $\gamma_1,\gamma_2\in\Gamma$, we have $G_k(\gamma_1 z, \gamma_2\mathfrak{z}) = G_k(z,\mathfrak{z})$.
(3) Denoting $\Delta_{0,z} := -4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$, we have

$$\Delta_{0,z}\left(G_k\left(z,\mathfrak{z}\right)\right) = \Delta_{0,\mathfrak{z}}(G_k\left(z,\mathfrak{z}\right)) = k(1-k)G_k\left(z,\mathfrak{z}\right).$$

(4) As $z\to\mathfrak{z}$

$$G_k(z,\mathfrak{z}) = 2\omega_\mathfrak{z}\log\left(r_\mathfrak{z}(z)\right) + O(1).$$

(5) As z approaches a cusp, $G_k(z,\mathfrak{z})\to 0$.

These higher Green's functions have a long history, appearing as special cases of the resolvent kernel studied by Fay [8] and investigated thoroughly by Hejhal in [10], for example. Gross and Zagier [9] conjectured that their evaluations at CM-points are essentially logarithms of algebraic numbers, which has been since proven in a number of cases. To state the connection with inner products, let $\beta(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}dt$ be the beta function, and let $\mathcal{Q}_{-D}$ denote the set of positive-definite integral binary quadratic forms of discriminant $-D<0$. Evaluating the elliptic coefficients in (4.5) for f_Q and $\mathcal{G}_Q$ then yields the following theorem.

Theorem 4.1 (Theorem 1.5 of [5]). *For $Q \in \mathcal{Q}_{-D_1}$ and $\mathcal{Q} \in \mathcal{Q}_{-D_2}$ ($-D_1, -D_2 < 0$ discriminants) with $[\tau_Q] \neq [\tau_{\mathcal{Q}}]$, we have*

$$\langle f_Q, f_{\mathcal{Q}} \rangle = -\frac{\pi(-4)^{1-k}}{(2k-1)\beta(k,k)} \frac{G_k(\tau_Q, \tau_{\mathcal{Q}})}{\omega_{\tau_Q}\omega_{\tau_{\mathcal{Q}}}}.$$

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