数理解析研究所講究録
第2036巻 2017年 31-44

Algebraic theory of nearly holomorphic Siegel modular forms

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1. Introduction

The rationality of CM values for modular forms was studied in Shimura's work (summarized in [18]), and developed by Katz [12, 13, 14], Eischen [3, 4] and others. Furthermore, the integrality of CM values was studied by Bruinier-Ono [1] and Larson-Rolen [15] in the connection with singular moduli. The aim of this paper is to review the algebraic theory of (vector-valued) nearly holomorphic Siegel modular forms given in [11, Sections 2 and 3], and apply this theory to showing the integrality of their CM values. Another application of this theory to p-adic modular forms is given in [11, Section 4] and [16].

First, following [11] we study algebraic counterparts of nearly holomorphic Siegel modular forms as *nearly Siegel modular forms* which were considered in Darmon-Rotger [2] and Urban [19] in the elliptic modular case. Nearly Siegel modular forms are defined as global sections of certain vector bundles arising from the de Rham bundle on a Siegel modular variety. Then one can study their integrality since the Siegel modular variety has a Shimura model as the moduli space of abelian varieties. Based on results of [10], we show that the space of nearly Siegel modular forms of fixed weight is a finitely generated module, and that there exists the arithmetic Fourier expansion on this space satisfying the q-expansion principle. Furthermore, we consider the analytic realization of nearly Siegel modular forms given by the Hodge decomposition of the de Rham bundle, and show that this realization map gives an isomorphism between the spaces of integral nearly Siegel modular forms and of integral nearly holomorphic ones. By this theorem, one can study the integrality of nearly holomorphic Siegel modular forms using their Fourier expansions.

Second, we apply the above results to showing the integrality of CM values. Roughly speaking, our result is as follows:

Theorem (for the precise statement, see Theorems 3.6 and 3.7). *Any (component of) CM value for an integral nearly holomorphic Siegel modular form is integral over $\mathbb{Z}[1/d]$, where d denotes the discriminant of the corresponding CM field.*

This fact was observed in [1] and [15] for non-holomorphic modular functions with giving upper bounds of the denominators of these CM values.

2. Nearly modular forms

2.1. Representation of classical groups. Let V be a $2g$-dimensional vector space with symplectic form, and W be its anisotropic subspace of dimension g. Then $GL_g = GL(W)$ is a general linear group of rank g which is contained in a symplectic group $Sp_{2g} = Sp(V)$ of rank g as

$$GL_g \cong \left\{ \begin{pmatrix} A & O \\ O & {}^tA^{-1} \end{pmatrix} \in Sp_{2g} \,\middle|\, A \in GL_g \right\}.$$

Let B_g be the Borel subgroup of GL_g consisting of upper-triangular matrices, and B_{2g} denote the Borel subgroup of Sp_{2g} given by

$$\left\{ \begin{pmatrix} A & * \\ O & {}^tA^{-1} \end{pmatrix} \in Sp_{2g} \,\middle|\, A \in B_g \right\}.$$

Then the maximal torus $T_g \subset B_g$ of GL_g becomes that of Sp_{2g}, and $\mathbb{Z}^g$ is identified with the group $X(T_g)$ of characters of T_g as

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & t_g \end{pmatrix} \mapsto t_1^{\kappa_1} \cdots t_g^{\kappa_g}$$

for $(\kappa_1, ..., \kappa_g) \in \mathbb{Z}^g$. Then

$$X^+(T_g) = \{(\kappa_1, ..., \kappa_g) \in \mathbb{Z}^g \mid \kappa_1 \geq \cdots \geq \kappa_g \geq 0\}$$

becomes the set of dominant weights with respect to B_{2g}. Let κ be an element of $X^+(T_g)$ which is naturally regarded as regular functions on B_g and on B_{2g}. Then

$$\begin{aligned} W_\kappa &:= \mathrm{Ind}_{B_g}^{GL_g}(-\kappa) = \left\{ \phi \in \Gamma\left(\mathcal{O}_{GL_g}\right) \mid \phi(ab) = \kappa(b)\phi(a) \ (b \in B_g) \right\}, \\ V_\kappa &:= \mathrm{Ind}_{B_{2g}}^{Sp_{2g}}(-\kappa) = \left\{ \psi \in \Gamma\left(\mathcal{O}_{Sp_{2g}}\right) \mid \psi(ab) = \kappa(b)\psi(a) \ (b \in B_{2g}) \right\} \end{aligned}$$

are representation spaces of GL_g, Sp_{2g} by

$$\begin{aligned} \phi(a) &\mapsto (\alpha \cdot \phi)(a) = \phi(\alpha^{-1}a) \quad (\phi \in W_\kappa,\ \alpha \in GL_g), \\ \psi(a) &\mapsto (\alpha \cdot \psi)(a) = \psi(\alpha^{-1}a) \quad (\psi \in V_\kappa,\ \alpha \in Sp_{2g}) \end{aligned}$$

respectively. The duals W_κ^* (resp. V_κ^*) of W_κ (resp. V_κ) are called the *universal representations* of highest weight κ (cf. [9, 5.1.3 and 8.1.2]), and hence the highest weight of W_κ (resp. V_κ) are $(-\kappa_g, ..., -\kappa_1)$ (resp. κ). By construction, W_κ, V_κ give rational homomorphisms of GL_g, Sp_{2g} respectively over any base ring, and for each $h \in \mathbb{Z}$, $W_{\kappa - h(1,...,1)} \cong W_\kappa \otimes \det^{\otimes h}$. Over a field of characteristic 0, W_κ^* (resp. V_κ^*) are realized as direct summands of certain tensor products of W (resp. V) associated with κ, and hence W_κ can be regarded as a direct summand of $V_\kappa^* \cong V_\kappa$.

If a linear map $\pi : V \to W$ satisfies that $W \hookrightarrow V \xrightarrow{\pi} W$ is the identity map on W and that $\mathrm{Ker}(\pi)$ is anisotropic for the symplectic form on V, then π gives a decomposition $V = W \oplus \mathrm{Ker}(\pi)$ compatible with the symplectic form. This decomposition induces an inclusion $GL_g \hookrightarrow Sp_{2g}$, and hence by the associated pullback, one has a ring homomorphism $\Gamma\left(\mathcal{O}_{Sp_{2g}}\right) \to \Gamma\left(\mathcal{O}_{GL_g}\right)$ which gives a GL_g-equivariant map $V_\kappa \to W_\kappa$.

2.2. Modular variety. We review results of Chai-Faltings [5] on the moduli space of abelian varieties and its compactifications. For positive integers g and N, let ζ_N be a primitive Nth root of 1, and $\mathcal{A}_{g,N}$ be the moduli stack classifying principally polarized abelian schemes of relative dimension g with symplectic level N structure. Then $\mathcal{A}_{g,N}$ is a smooth algebraic stack over $\mathbb{Z}[1/N, \zeta_N]$ of relative dimension $g(g+1)/2$, and becomes a fine moduli scheme if $N \geq 3$. Furthermore, the associated complex orbifold $\mathcal{A}_{g,N}(\mathbb{C})$ is represented as the quotient space $\mathcal{H}_g/\Gamma(N)$ of the Siegel upper half space $\mathcal{H}_g$ of degree g by the integral symplectic group

$$\Gamma(N) = \left\{ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \in Sp_{2g}(\mathbb{Z}) \;\middle|\; \begin{array}{ll} A_\gamma \equiv D_\gamma \equiv 1_g & \mathrm{mod}(N) \\ B_\gamma \equiv C_\gamma \equiv 0 & \mathrm{mod}(N) \end{array} \right\}$$

of degree g and level N which acts on $\mathcal{H}_g$ as

$$\mathcal{H}_g \ni Z \mapsto \gamma(Z) = \left(A_\gamma Z + B_\gamma\right)\left(C_\gamma Z + D_\gamma\right)^{-1} \in \mathcal{H}_g \ \ (\gamma \in \Gamma(N)).$$

Let $\pi : \mathcal{X} \to \mathcal{A}_{g,N}$ be the universal abelian scheme with 0-section s, denote by $\mathbb{E}$ the *Hodge bundle* of rank g defined as $\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{A}_{g,N}}\right) = s^*\left(\Omega^1_{\mathcal{X}/\mathcal{A}_{g,N}}\right)$, and by $\omega = \det(\mathbb{E})$ the *Hodge line bundle.*

For a smooth and $GL(\mathbb{Z}^g)$-admissible polyhedral cone decomposition of the space of positive semi-definite symmetric bilinear forms on $\mathbb{R}^g$, Chai-Faltings [5, Chapter IV] construct the associated smooth compactification $\overline{\mathcal{A}}_{g,N}$ of $\mathcal{A}_{g,N}$, and the semi-abelian scheme $\mathcal{G}$ with 0-section s over $\overline{\mathcal{A}}_{g,N}$ extending $\mathcal{X} \to \mathcal{A}_{g,N}$. Then $\overline{\omega} = \det\left(s^*\left(\Omega^1_{\mathcal{G}/\overline{\mathcal{A}}_{g,N}}\right)\right)$ is an extension of $\omega = \det(\mathbb{E})$ to $\overline{\mathcal{A}}_{g,N}$, and

$$\mathcal{A}^*_{g,N} = \mathrm{Proj}\left(\bigoplus_{h \geq 0} H^0\left(\overline{\mathcal{A}}_{g,N}, \overline{\omega}^{\otimes h}\right)\right)$$

is a projective scheme over $\mathbb{Z}[1/N, \zeta_N]$ called *Satake's minimal compactification.* It is shown in [5, Chapter IV, 6.8] that any geometric fiber of $\overline{\mathcal{A}}_{g,N}$ is irreducible, and hence $\mathcal{A}_{g,N}$ has the same property.

Assume that $N \geq 3$. Then $\mathcal{A}^*_{g,N}$ contains $\mathcal{A}_{g,N}$, and its complement has a natural stratification by locally closed subschemes, each of which is isomorphic to $\mathcal{A}_{i,N}$ $(0 \leq i \leq g-1)$. Therefore, the relative codimension

$$\mathrm{codim}_{\mathbb{Z}[1/N,\zeta_N]}\left(\mathcal{A}^*_{g,N} - \mathcal{A}_{g,N},\ \mathcal{A}^*_{g,N}\right)$$

over $\mathbb{Z}[1/N, \zeta_N]$ of $\mathcal{A}^*_{g,N} - \mathcal{A}_{g,N}$ in $\mathcal{A}^*_{g,N}$ becomes

$$\frac{g(g+1)}{2} - \frac{(g-1)g}{2} = g$$

which is greater than 1 if $g > 1$. Furthermore, there is a natural morphism $\overline{\mathcal{A}}_{g,N} \to \mathcal{A}^*_{g,N}$ (which is an isomorphism if $g = 1$) extending the identity map on $\mathcal{A}_{g,N}$ such that $\overline{\omega}$ is the pullback by this morphism of the tautological line bundle ω^* on $\mathcal{A}^*_{g,N}$.

2.3. CM point. Let $\varphi : S \to \mathcal{A}_{g,N}$ be a morphism of schemes over $\mathbb{Z}[1/N, \zeta_N]$ which becomes a R-rational point on $\mathcal{A}_{g,N}$ if $S = \mathrm{Spec}(R)$ for a $\mathbb{Z}[1/N, \zeta_N]$-algebra R. Then as the associated object, there is an abelian scheme X over S with principal polarization λ and symplectic level N structure σ. A *test object* (resp. an *extended test object*) over S associated with a morphism $\varphi : S \to \mathcal{A}_{g,N}$ is the above (X, λ, σ) together with basis of regular 1-forms on X/S (resp. basis of $H^1_{\mathrm{DR}}(X/S)$). By definition, any element of $\mathcal{M}_\rho(R)$ is evaluated as an element of $\mathcal{O}_S^d$ at each test object over an R-scheme S, where this evaluation is functorial on S and equivariant for ρ under base changes of regular 1-forms.

For a field extension k of $\mathbb{Q}(\zeta_N)$, a k-rational point α on $\mathcal{A}_{g,N}$ corresponding to a CM abelian variety X is called a *CM point over* k if the following conditions hold:

- The $\mathbb{Q}$-algebra $\mathrm{End}_k(X) \otimes \mathbb{Q}$ is isomorphic to the direct sum $\bigoplus_i L_i$, where L_i are CM fields, i.e., totally imaginary quadratic extensions of totally real number fields K_i. Then $H^1_{\mathrm{DR}}(X/k)$ is an invertible $\mathrm{End}_k(X) \otimes \mathbb{Q}$-module.
- There are algebra homomorphisms $\varphi_i : L_i \otimes k \to K_i \otimes k$ such that

  $$x \otimes y \mapsto (\varphi_i(x \otimes y), \varphi_i(\iota_i(x) \otimes y)) \ \ (x \in L_i, y \in k)$$

  give rise to isomorphisms $L_i \otimes k \xrightarrow{\sim} K_i \otimes k \oplus K_i \otimes k$, where ι_i denotes the involution of L_i over K_i.

Note that any CM abelian variety can be defined over a number field, and has potentially good reduction at all finite places. Therefore, for any CM point α on $\mathcal{A}_{g,N}$ and any rational prime p, there is an (extended) test object $\widetilde{\alpha}$ associated with α over an algebra which is a finite $\mathbb{Z}_{(p)}$-module, where $\mathbb{Z}_{(p)}$ denotes the valuation ring of $\mathbb{Q}$ at p.

2.4. Modular forms. In what follows, we assume that

$$g > 1, \ N \geq 3.$$

First, following [6, 2.2.1] we give the process of twisting a locally free sheaf by a linear representation. Let X be a scheme, and $\mathcal{F}$ be a locally free sheaf on X of rank n. Take $\{U_i\}_{i \in I}$ be an open cover of X trivializing $\mathcal{F}$. Then the natural isomorphism $\mathcal{F}|_{U_i \cap U_j} \cong$

$\mathcal{F}|_{U_j\cap U_i}$ gives rise to the transition function $g_{ij} \in GL_n\left(\mathcal{O}_X|_{U_i\cap U_j}\right)$ satisfying the cocycle condition. Let $\rho : GL_n \to GL_m$ be a rational homomorphism over a $\mathbb{Z}$-algebra R. Then we construct a locally free $\mathcal{O}_X \otimes R$-module $\mathcal{F}_\rho$ on $X \otimes R$ as $\mathcal{F}_\rho|_{U_i} = \left((\mathcal{O}_X \otimes R)\,|_{U_i}\right)^m$, where the isomorphism $\mathcal{F}_\rho|_{U_i\cap U_j} \cong \mathcal{F}_\rho|_{U_j\cap U_i}$ is given by $\rho(g_{ij}) \in GL_m\left((\mathcal{O}_X \otimes R)\,|_{U_i\cap U_j}\right)$.

For a $\mathbb{Z}\,[1/N, \zeta_N]$-algebra R, a positive integer d and a rational homomorphism $\rho : GL_g \to GL_d$ over R, let $\mathbb{E}_\rho$ be the locally free sheaf on

$$\mathcal{A}_{g,N} \otimes R = \mathcal{A}_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} R$$

obtained from twisting the Hodge bundle $\mathbb{E}$ by ρ. If ρ is obtained from $\kappa \in \mathbb{Z}^g$, then we put $\mathbb{E}_\kappa = \mathbb{E}_\rho$, and denote this rank d by $d(\mathbb{E}_\kappa)$.

Definition 2.1. Let R be a $\mathbb{Z}\,[1/N, \zeta_N]$-algebra. For a rational homomorphism $\rho : GL_g \to GL_d$ over R, we put

$$\mathcal{M}_\rho(R) = H^0\left(\mathcal{A}_{g,N} \otimes R, \mathbb{E}_\rho\right),$$

and call these elements *Siegel modular forms over R of weight ρ* (and degree g, level N). If $\rho = \omega^{\otimes h} : GL_g \to \mathbb{G}_m$, then we put $\mathcal{M}_h(R) = \mathcal{M}_{\omega^{\otimes h}}(R)$, and call these elements of weight h. More generally, for an R-module M, the space of *Siegel modular forms with coefficients in M of weight ρ* is defined as

$$\mathcal{M}_\rho(M) = H^0\left(\mathcal{A}_{g,N} \otimes R, \mathbb{E}_\rho \otimes_R M\right).$$

We consider the case where $R = \mathbb{C}$. For each $Z \in \mathcal{H}_g$, let

$$\mathcal{X}_Z = \mathbb{C}^g/(\mathbb{Z}^g + \mathbb{Z}^g \cdot Z)$$

be the corresponding abelian variety over $\mathbb{C}$, and $(u_1, ..., u_g)$ be the natural coordinates on the universal cover $\mathbb{C}^g$ of $\mathcal{X}_Z$. Then $\mathbb{E}$ is trivialized over $\mathcal{H}_g$ by $du_1, ..., du_g$. For an element $\gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix}$ of $\Gamma(N)$,

$$\mathcal{X}_Z \stackrel{\sim}{\to} \mathcal{X}_{\gamma(Z)};\ {}^t(u_1, ..., u_g) \mapsto (C_\gamma Z + D_\gamma)^{-1} \cdot {}^t(u_1, ..., u_g),$$

and hence γ acts equivariantly on the trivialization of $\mathbb{E}$ over $\mathcal{H}_g$ as the left multiplication by $(C_\gamma Z + D_\gamma)^{-1}$. Therefore, γ acts equivariantly on the induced trivialization of $\mathbb{E}_\rho$ over $\mathcal{H}_g$ as the left multiplication by $\rho\,(C_\gamma Z + D_\gamma)^{-1}$. Then $f \in \mathcal{M}_\rho(\mathbb{C})$ is a complex analytic section of $\mathbb{E}_\rho$ on $\mathcal{A}_{g,N}(\mathbb{C}) = \mathcal{H}_g/\Gamma(N)$, and hence is a $\mathbb{C}^d$-valued holomorphic function on $\mathcal{H}_g$ satisfying the ρ-automorphic condition:

$$f(Z) = \rho\,(C_\gamma Z + D_\gamma)^{-1} \cdot f\,(\gamma(Z))\ \left(Z \in \mathcal{H}_g,\ \gamma = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \in \Gamma(N)\right)$$

which is equivalent to that $f\,(\gamma(Z)) = \rho\,(C_\gamma Z + D_\gamma) \cdot f(Z)$. Furthermore, the value of f at a test object $(X, \lambda, \alpha; w_1, ..., w_g)$ over a subfield k of $\mathbb{C}$ becomes $\rho(G) \cdot f(Z) \in k^d$, where ${}^t(du_1, ..., du_g) = G \cdot {}^t(w_1, ..., w_g)$.

Let $\iota : \mathcal{A}_{g,N} \hookrightarrow \mathcal{A}^*_{g,N}$ be the natural inclusion, and let $\mathbb{E}^*_\rho$ be the direct image (or pushforward) $\iota_* (\mathbb{E}_\rho)$ which is defined as a sheaf on $\mathcal{A}^*_{g,N} \otimes R$ satisfying that $\mathbb{E}^*_\rho(U) = \mathbb{E}_\rho (\iota^{-1}(U))$ for open subsets U of $\mathcal{A}^*_{g,N} \otimes R$. This implies immediately that

$$\mathcal{M}_\rho(R) = H^0 \left(\mathcal{A}^*_{g,N} \otimes R, \mathbb{E}^*_\rho\right).$$

Furthermore, based on that $\mathrm{codim}_{\mathbb{Z}[1/N,\zeta_N]} \left(\mathcal{A}^*_{g,N} - \mathcal{A}_{g,N},\ \mathcal{A}^*_{g,N}\right) > 1$, Ghitza [7, Theorem 3] proved that $\mathbb{E}^*_\rho$ is a coherent sheaf on $\mathcal{A}^*_{g,N} \otimes R$. From this fact, it is shown in [10, Theorem 1] that $\mathcal{M}_\rho(R)$ is a finitely generated R-module, and that $\mathcal{M}_\rho(\mathbb{C})$ consists of $\mathbb{C}^d$-valued holomorphic functions on $\mathcal{H}_g$ satisfying the ρ-automorphic condition.

2.5. Fourier expansion. Let q_{ij} $(1 \le i, j \le g)$ be variables with symmetry $q_{ij} = q_{ji}$. Then in [17], Mumford constructs a semi-abelian scheme formally represented as

$$(\mathbb{G}_m)^g / \langle (q_{ij})_{1 \le i \le g} \mid 1 \le j \le g\rangle;\ (\mathbb{G}_m)^g = \mathrm{Spec}\left(\mathbb{Z}\left[x_1^{\pm 1}, ..., x_g^{\pm 1}\right]\right)$$

over

$$\mathbb{Z}\left[q_{ij}^{\pm 1}\ (i \ne j)\right] [[q_{11}, ..., q_{gg}]].$$

This becomes an abelian scheme which is called *Mumford's abelian scheme* over

$$\mathbb{Z}\left[q_{ij}^{\pm 1}\ (i \ne j)\right] [[q_{11}, ..., q_{gg}]] [1/q_{11}, ..., 1/q_{gg}]$$

with principal polarization corresponding to the multiplicative form

$$((a_1, ..., a_g), (b_1, ..., b_g)) \mapsto \prod_{1 \le i,j \le g} q_{ij}^{a_i b_j}$$

on $\mathbb{Z}^g \times \mathbb{Z}^g$. Hence for each 0-dimensional cusp c on $\mathcal{A}^*_{g,N}$, this polarized abelian scheme over

$$\mathcal{R}_{g,N} = \mathbb{Z}\left[1/N,\ \zeta_N,\ q_{ij}^{\pm 1/N}\ (i \ne j)\right] \left[\left[q_{11}^{1/N}, ..., q_{gg}^{1/N}\right]\right] [1/q_{11}, ..., 1/q_{gg}]$$

has the associated symplectic level N structure, and $\omega_i = dx_i/x_i$ $(1 \le i \le g)$ form a basis of regular 1-forms. Taking the pullback by the associated morphism $\mathrm{Spec}\left(\mathcal{R}_{g,N}\right) \to \mathcal{A}_{g,N}$, $\mathbb{E}$ is trivialized by the basis $\omega_1, ..., \omega_g$, and hence $\mathbb{E}_\rho$ is also trivialized over

$$\mathrm{Spec}\left(\mathcal{R}_{g,N} \otimes R\right) = \mathrm{Spec}\left(\mathcal{R}_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} R\right).$$

In what follows, we fix such a trivialization:

$$\mathbb{E}_\rho \times_{\mathcal{A}_{g,N} \otimes R} \mathrm{Spec}\left(\mathcal{R}_{g,N} \otimes R\right) = \left(\mathcal{R}_{g,N} \otimes R\right)^d.$$

Then for an R-module M, the evaluation on Mumford's abelian scheme gives a homomorphism

$$F_c : \mathcal{M}_\rho(M) \to \left(\mathcal{R}_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M\right)^d$$

which we call the *Fourier expansion map* associated with c. Furthermore, it is shown in [10, Theorem 2] that F_c satisfies the following q-expansion principle:

If M' is an R-submodule of M and $f \in \mathcal{M}_\rho(M)$ satisfies that

$$F_c(f) \in \left(\mathcal{R}_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M'\right)^d,$$

then $f \in \mathcal{M}_\rho(M')$.

This result was already shown by Harris [8, 4.8, Theorem] in the case where M is a field extension of a field M' containing $\mathbb{Q}(\zeta_N)$.

Assume that $M = \mathbb{C}$ and c is associated with $\sqrt{-1}\infty$. Then by the substitution $q_{ij} = \exp\left(2\pi\sqrt{-1}z_{ij}\right)$ for $Z = (z_{ij})_{i,j} \in \mathcal{H}_g$, Mumford's abelian scheme becomes $\mathcal{X}_Z$, and hence F_c becomes the analytic Fourier expansion map times $\rho\left(2\pi\sqrt{-1}\cdot 1_g\right)$. Since each $f(Z) \in \mathcal{M}_\rho(\mathbb{C})$ is a $\mathbb{C}^d$-valued holomorphic function of $Z \in \mathcal{H}_g$ and is invariant under $Z \mapsto Z + N \cdot I$ for any integral and symmetric $g \times g$ matrix I, and hence

$$F_c(f) = \sum_T a(T) \cdot \exp\left(2\pi\sqrt{-1}\mathrm{tr}(TZ)/N\right) = \sum_T a(T) \cdot \boldsymbol{q}^{T/N} \quad \left(a(T) \in \mathbb{C}^d\right),$$

where $T = (t_{ij})_{i,j}$ runs over half-integral symmetric $g \times g$ matrices, and

$$\boldsymbol{q}^{T/N} = \prod_{1 \leq i < j \leq g} \left(q_{ij}^{1/N}\right)^{2t_{ij}} \prod_{1 \leq i \leq g} \left(q_{ii}^{1/N}\right)^{t_{ii}}.$$

Furthermore, as is shown in the Cartan Seminar 4-04, $a(T) = 0$ if T is not positive semi-definite.

2.6. Nearly modular forms. Let $\mathcal{H}^1_{\mathrm{DR}}(\mathcal{X}/\mathcal{A}_{g,N})$ be the sheaf of de Rham cohomology groups of $\mathcal{X}/\mathcal{A}_{g,N}$, and define the *de Rham bundle* as

$$\mathbb{D} = \mathcal{R}^1_{\mathrm{DR}}\pi\left(\mathcal{X}/\mathcal{A}_{g,N}\right) = \pi_*\left(\mathcal{H}^1_{\mathrm{DR}}\left(\mathcal{X}/\mathcal{A}_{g,N}\right)\right)$$

which is a locally free sheaf on $\mathcal{A}_{g,N}$ of rank $2g$ with canonical symplectic form. Then one has a canonical exact sequence

$$0 \to \mathbb{E} \to \mathbb{D} \to \mathbb{D}/\mathbb{E} \to 0,$$

and the quotient $\mathbb{D}/\mathbb{E}$ is locally free of rank g. The Gauss-Manin connection

$$\nabla : \mathbb{D} \to \mathbb{D} \otimes \Omega_{\mathcal{A}_{g,N}}$$

defines $\mathcal{T}_{\mathcal{A}_{g,N}} \to \mathrm{End}_{\mathcal{O}_{\mathcal{A}_{g,N}}}(\mathbb{D})$ which, together with the above exact sequence, gives the Kodaira-Spencer isomorphism

$$\mathcal{T}_{\mathcal{A}_{g,N}} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{A}_{g,N}}}\left(\mathbb{E}, \mathbb{D}/\mathbb{E}\right).$$

Let κ be an element of $X^+(T_g)$, and denote by V_κ the universal representation of highest weight κ. Then one can obtain the associated locally free sheaf $\mathbb{D}_\kappa$ on $\mathcal{A}_{g,N}$ whose rank is denoted by $d(\mathbb{D}_\kappa)$. Furthermore, for $h \in \mathbb{Z}$, put

$$\mathbb{D}_{(\kappa,h)} = \mathbb{D}_\kappa \otimes \det(\mathbb{E})^{\otimes h}$$

which is also a locally free sheaf on $\mathcal{A}_{g,N}$ with rank $d(\mathbb{D}_\kappa)$.

Definition 2.2. Let R be a $\mathbb{Z}[1/N, \zeta_N]$-algebra. Then for $\kappa \in X^+(T_g)$ and $h \in \mathbb{Z}$, we put

$$\mathcal{N}_{(\kappa,h)}(R) = H^0\left(\mathcal{A}_{g,N} \otimes R, \mathbb{D}_{(\kappa,h)}\right),$$

and call these elements *nearly Siegel modular forms over R of weight* (κ, h) (and degree g, level N). More generally, for an R-module M, we call

$$\mathcal{N}_{(\kappa,h)}(M) = H^0\left(\mathcal{A}_{g,N} \otimes R, \mathbb{D}_{(\kappa,h)} \otimes_R M\right).$$

the space of *nearly Siegel modular forms with coefficients in M of weight* (κ, h).

Theorem 2.3 (cf. [11, Theorem 2.3]). *The R-module $\mathcal{N}_{(\kappa,h)}(R)$ is finitely generated.*

As in 2.5, let $\{\omega_i \mid 1 \le i \le g\}$ be the canonical basis of the Mumford's abelian scheme. Then there exist η_i $(1 \le i \le g)$ such that

$$\nabla(\omega_i) = \sum_{j=1}^{g} \frac{dq_{ij}}{q_{ij}} \eta_j,$$

and $\{\omega_i, \eta_i \mid 1 \le i \le g\}$ gives a basis of $\mathbb{D}$ over $\mathcal{R}_{g,N}$. By using this basis, one has a trivialization of $\mathbb{D}_\kappa$ over $\mathcal{R}_{g,N}$ and that of $\det(\mathbb{E})$ by $\omega_1 \wedge \cdots \wedge \omega_g$. Therefore, there exists the Fourier expansion map

$$\mathcal{F}_c : \mathcal{N}_{(\kappa,h)}(M) \to \left(\mathcal{R}_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M\right)^{d(\mathbb{D}_\kappa)}$$

which is obtained as the evaluation map on the Mumford's abelian scheme.

Theorem 2.4 (cf. [11, Theorem 2.4]). *The Fourier expansion map $\mathcal{F}_c$ satisfies the following q-expansion principle: If M' is an R-submodule of M and $f \in \mathcal{N}_{(\kappa,h)}(M)$ satisfies*

$$\mathcal{F}_c(f) \in \left(\mathcal{R}_{g,N} \otimes_{\mathbb{Z}[1/N,\zeta_N]} M'\right)^{d(\mathbb{D}_\kappa)},$$

then $f \in \mathcal{N}_{(\kappa,h)}(M')$.

Theorem 2.5 (cf. [11, Theorem 2.5]). *Let R be a $\mathbb{Z}[1/N, \zeta_N]$-algebra, and f be an element of $\mathcal{N}_{(\kappa,h)}(R)$. Then for an extended test object $\widetilde{\alpha}$ over R associated with a point α on $\mathcal{A}_{g,N}$, the evaluation $f(\widetilde{\alpha})$ of f at $\widetilde{\alpha}$ belongs to $R^{d(\mathbb{D}_\kappa)}$.*

Theorems 2.3–2.5 are easily extended for general representations of GL_g as follows.

Definition 2.6. Let R be a $\mathbb{Z}[1/N, \zeta_N]$-subalgebra of $\mathbb{C}$, and $\rho : GL_g \to GL_d$ be a rational homomorphism over R which is the direct sum of ρ_j, where ρ_j are associated with $W_{\kappa_j - h_j(1,\ldots,1)}$ for $\kappa_j \in X^+(T_g)$, $h_j \in \mathbb{Z}$. Put

$$\mathcal{N}_\rho(R) = \bigoplus_j \mathcal{N}_{(\kappa_j, h_j)}(R),$$

and call these elements *nearly Siegel modular forms over R of weight ρ* (and degree g, level N).

Remark 1. Any representation ρ of GL_g over a field of characteristic 0 is represented as above.

Theorem 2.7. *Let R and ρ be as in Definition 2.6.*

(1) *The R-module $\mathcal{N}_\rho(R)$ is finitely generated.*

(2) *The direct sum of the Fourier expansion maps on $\mathcal{N}_{(\kappa_j, h_j)}(R)$ gives rise to the Fourier expansion map on $\mathcal{N}_\rho(R)$ satisfying the q-expansion principle.*

(3) *Let f be an element of $\mathcal{N}_\rho(R)$. Then for an extended test object $\widetilde{\alpha}$ over R associated with a point α on $\mathcal{A}_{g,N}$, the evaluation $f(\widetilde{\alpha})$ of f at $\widetilde{\alpha}$ belongs to $R^{d(\rho)}$, where $d(\rho) = \sum_j d(\mathbb{D}_{\kappa_j})$.*

3. Arithmeticity in the analytic case

3.1. Differential operator. First, we recall Shimura's differential operator. Let R be a $\mathbb{Q}$-algebra, and identify the 2-fold symmetric tensor product $\mathrm{Sym}^2(R^g)$ of R^g with the R-module of all symmetric $g \times g$ matrices with entries in R. For a positive integer e, let $S_e\left(\mathrm{Sym}^2(R^g), R^d\right)$ be the R-module of all polynomial maps of $\mathrm{Sym}^2(R^g)$ into R^d homogeneous of degree e. For a rational homomorphism $\rho : GL_g \to GL_d$, let $\rho \otimes \tau^e$ and $\rho \otimes \sigma^e$ be the rational homomorphisms over R given by

$$GL_g(R) = \mathrm{Aut}_R(R^g) \to \mathrm{Aut}_R\left(S_e\left(\mathrm{Sym}^2(R^g), R^d\right)\right)$$

which are defines as

$$\left[(\rho \otimes \tau^e)(\alpha)(h)\right](u) = \rho(\alpha) h\left({}^t\alpha \cdot u \cdot \alpha\right)$$

and

$$\left[(\rho \otimes \sigma^e)(\alpha)(h)\right](u) = \rho(\alpha) h\left(\alpha^{-1} \cdot u \cdot {}^t\alpha^{-1}\right)$$

respectively for $\alpha \in GL_g(R)$, $h \in S_e\left(\mathrm{Sym}^2(R^g), R^d\right)$, $u \in \mathrm{Sym}^2(R^g)$. In particular, for $\alpha \in GL_g$, $\tau^e(\alpha)$ (resp. $\sigma^e(\alpha)$) consists of polynomials of entries of α (resp. α^{-1}). Furthermore, let

$$\theta^e : S_e\left(\mathrm{Sym}^2(R^g), S_e\left(\mathrm{Sym}^2(R^g), R^d\right)\right) \to R^d$$

be the contraction map defined in [18, 14.1] as $\theta^e(h) = \sum_i h(u_i, v_i)$, where $\{u_i\}$ and $\{v_i\}$ are dual basis of $\mathrm{Sym}^2(R^g)$ for the pairing $(u, v) \mapsto \mathrm{tr}(uv)$, namely $\mathrm{tr}(u_i v_j)$ is Kronecker's delta δ_{ij}. Then θ^e is GL_g-equivariant for the representations $\rho \otimes \sigma^e \otimes \tau^e$ and ρ.

Let f be a $\mathbb{C}^d$-valued smooth function of $Z = (z_{ij})_{i,j} = X + \sqrt{-1}Y \in \mathcal{H}_g$. Then following [18, Chapter III, 12], define $S_1\left(\mathrm{Sym}^2(\mathbb{C}^g), \mathbb{C}^g\right)$-valued smooth functions $(Df)(u)$, $(Cf)(u)$ $\left(u = (u_{ij})_{i,j} \in \mathrm{Sym}^2(\mathbb{C}^g)\right)$ of $Z \in \mathcal{H}_g$ as

$$\begin{aligned}
(Df)(u) &= \sum_{1 \le i \le j \le g} u_{ij} \frac{\partial f}{\partial(2\pi\sqrt{-1}z_{ij})}, \\
(Cf)(u) &= (Df)\left(\left(Z - \overline{Z}\right) u \left(Z - \overline{Z}\right)\right),
\end{aligned}$$

and define $S_e\left(\mathrm{Sym}^2(\mathbb{C}^g), \mathbb{C}^g\right)$-valued analytic functions $C^e(f)$, $D_\rho^e(f)$ of $Z \in \mathcal{H}_g$ as

$$\begin{aligned}
C^e(f) &= C\left(C^{e-1}(f)\right), \\
D_\rho^e(f) &= (\rho \otimes \tau^e)\left(Z - \overline{Z}\right)^{-1} C^e\left(\rho\left(Z - \overline{Z}\right) f\right).
\end{aligned}$$

It is shown in [18, Chapter III, 12.10] that if f satisfies the ρ-automorphic condition for $\Gamma(N)$, then $D_\rho^e(f)(u)$ satisfies the $\rho \otimes \tau^e$-automorphic condition.

Remark 2. The above D_ρ^e becomes $\left(2\pi\sqrt{-1}\right)^{-e}$ times Shimura's original operator given in [18].

Let $u_1, \dots, u_g$ be the standard coordinates on $\mathbb{C}^g$, and α_i, β_i $(1 \le i \le g)$ be relative 1-forms on $\mathcal{X}_Z$ $(Z = (z_{ij}) \in \mathcal{H}_g)$ given by

$$\alpha_i\left(\sum_{j=1}^g a_j \boldsymbol{e}_j + \sum_{j=1}^g b_j \boldsymbol{z}_j\right) = a_i, \ \beta_i\left(\sum_{j=1}^g a_j \boldsymbol{e}_j + \sum_{j=1}^g b_j \boldsymbol{z}_j\right) = b_i$$

for each $a_j, b_j \in \mathbb{R}$, where $\boldsymbol{e}_j = (\delta_{ij})_{1 \le i \le g}$ and $\boldsymbol{z}_j = (z_{j1}, \dots, z_{jg})$. Since α_i, β_i have constant periods for all $\mathcal{X}_Z$, $\nabla(\alpha_i) = \nabla(\beta_i) = 0$. Furthermore, one has

$$du_i = \alpha_i + \sum_{j=1}^g z_{ij}\beta_j, \ d\overline{u_i} = \alpha_i + \sum_{j=1}^g \overline{z_{ij}}\beta_j$$

which implies that

$${}^t(du_1, \dots, du_g) \equiv \left(Z - \overline{Z}\right) \cdot {}^t(\beta_1, \dots, \beta_g) \mod \left(H^{0,1}\left(\mathcal{X}/\mathcal{H}_g\right)\right).$$

Then

$$\omega_i = d\log(x_i) = 2\pi\sqrt{-1}du_i \ (1 \le i \le g),$$

and hence

$$\nabla(\omega_i) = 2\pi\sqrt{-1}\nabla(du_i) = 2\pi\sqrt{-1}\sum_{j=1}^{g} dz_{ij}\cdot\beta_j = 2\pi\sqrt{-1}\sum_{j=1}^{g}\frac{dq_{ij}}{q_{ij}}\beta_j$$

which implies

$$\eta_i = \beta_i \ (1 \le i \le g).$$

The following proposition was obtained by Harris [8, Section 4] substantially, and shown by Eischen [3, Proposition 8.5] in the unitary modular case.

Proposition 3.1 (cf. [11, Proposition 3.1]). *Let $\pi : \mathcal{X} \to \mathcal{H}_g$ be the family of complex abelian varieties given by*

$$\pi^{-1}(Z) = \mathcal{X}_Z = \mathbb{C}^g/(\mathbb{Z}^g + \mathbb{Z}^g\cdot Z) \ (Z \in \mathcal{H}_g).$$

Then D_ρ^e is obtained from the composition

$$\mathbb{E}_\rho \to \mathbb{E}_\rho \otimes \left(\Omega^1_{\mathcal{H}_g}\right)^{\otimes e} \to \mathbb{E}_\rho \otimes \left(\mathrm{Sym}^2\left(\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right)\right)\right)^{\otimes e}.$$

Here the first map is given by the Gauss-Manin connection

$$\nabla : H^1_{\mathrm{DR}}(\mathcal{X}/\mathcal{H}_g) \to H^1_{\mathrm{DR}}(\mathcal{X}/\mathcal{H}_g) \otimes \Omega^1_{\mathcal{H}_g}$$

together with the projection $H^1_{\mathrm{DR}}(\mathcal{X}/\mathcal{H}_g) \to \pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right)$ derived from the Hodge decomposition

$$H^1_{\mathrm{DR}}(\mathcal{X}/\mathcal{H}_g) \cong H^{1,0}(\mathcal{X}/\mathcal{H}_g) \oplus H^{0,1}(\mathcal{X}/\mathcal{H}_g) = \pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right) \oplus \overline{\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right)},$$

and the second map is given by the Kodaira-Spencer isomorphism

$$\Omega^1_{\mathcal{H}_g} \cong \mathrm{Sym}^2\left(\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right)\right).$$

Let $\kappa \in X^+(T_g)$ be as above. Then the Gauss-Manin connection gives

$$\mathbb{D}_\kappa \to \mathbb{D}_\kappa \otimes \left(\Omega^1_{\mathcal{A}_{g,N}}\right)^e.$$

This, together with the Kodaira-Spencer isomorphism

$$\Omega^1_{\mathcal{A}_{g,N}} \cong \mathrm{Sym}^2\left(\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{A}_{g,N}}\right)\right),$$

gives rise to

$$\mathbb{D}_\kappa \to \mathbb{D}_\kappa \otimes \left(\mathrm{Sym}^2\left(\pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{A}_{g,N}}\right)\right)\right)^e$$

which we denote by $\mathcal{D}^e_\kappa$.

Proposition 3.2 (cf. [11, Proposition 3.2]). *Let $\rho : GL_g \to GL_d$ be the rational homomorphism associated with W_κ. Then via the projection $H^1_{\mathrm{DR}}(\mathcal{X}/\mathcal{H}_g) \to \pi_*\left(\Omega^1_{\mathcal{X}/\mathcal{H}_g}\right)$ derived from the Hodge decomposition, $\mathcal{D}^e_\kappa$ gives D^e_ρ.*

3.2. Nearly holomorphic modular forms. We recall the definition of nearly holomorphic Siegel modular forms by Shimura.

Definition 3.3. Let R be a $\mathbb{Z}[1/N, \zeta_N]$-subalgebra of $\mathbb{C}$. A $\mathbb{C}^d$-valued smooth function f of $Z = X + \sqrt{-1}Y \in \mathcal{H}_g$ is defined to be *nearly holomorphic over R* if f has the following expression

$$f(Z) = \sum_T q\left(T, \pi^{-1}Y^{-1}\right) \cdot \exp\left(2\pi\sqrt{-1}\mathrm{tr}(TZ)/N\right),$$

where $q\left(T, \pi^{-1}Y^{-1}\right)$ are vectors of degree d whose entries are polynomials over R of the entries of $(4\pi Y)^{-1}$. For a rational homomorphism $\rho : GL_g \to GL_d$ over R, denote by $\mathcal{N}^{\mathrm{hol}}_\rho(R)$ the R-module of all $\mathbb{C}^d$-valued smooth functions which are nearly holomorphic over R with ρ-automorphic condition for $\Gamma(N)$. Call these elements *nearly holomorphic Siegel modular forms over R of weight ρ* (and degree g, level N).

Theorem 3.4 (cf. [11, Theorem 3.4]). *Let R be a $\mathbb{Z}[1/N, \zeta_N]$-subalgebra of $\mathbb{C}$, and $\rho : GL_g \to GL_d$ be a rational homomorphism over R associated with $W_{\kappa-h(1,\dots,1)}$ for $\kappa \in X^+(T_g)$, $h \in \mathbb{Z}$. Then there exists a natural R-linear isomorphism*

$$\Phi : \mathcal{N}_{(\kappa,h)}(R) \xrightarrow{\sim} \mathcal{N}^{\mathrm{hol}}_\rho(R).$$

Consequently, $\mathcal{N}^{\mathrm{hol}}_\rho(R)$ is a finitely generated R-module, and

$$\mathcal{N}^{\mathrm{hol}}_\rho(R) \otimes_R \mathbb{C} = \mathcal{N}^{\mathrm{hol}}_\rho(\mathbb{C}).$$

Theorem 3.5 (cf. [11, Theorem 3.5]). *Let R be a $\mathbb{Z}[1/N, \zeta_N]$-subalgebra of $\mathbb{C}$, and $\rho : GL_g \to GL_d$ be a rational homomorphism over R associated with $W_{\kappa-h(1,\dots,1)}$ as in Theorem 3.4. Let $\widetilde{\alpha}$ be a test object over R corresponding to a CM abelian variety X, and assume that one can extend the basis of $\Omega^1_{X/R}$ to a basis of $H^1_{\mathrm{DR}}(X/R)$ which gives a projection $H^1_{\mathrm{DR}}(X/R) \to \Omega^1_{X/R}$ compatible with the action of $\mathrm{End}(X)$. Then for any $f \in \mathcal{N}^{\mathrm{hol}}_\rho(R)$, the evaluation $f(\widetilde{\alpha})$ of f at $\widetilde{\alpha}$ belongs to R^d.*

Theorem 3.6. *Let R and ρ be as above, and $\widetilde{\alpha}$ be a test object over R corresponding to a CM abelian variety X satisfying*

- $H^1_{\mathrm{DR}}(X/R)$ *is a free* $\mathrm{End}(X) \otimes R$*-module of rank* 1,
- R *contains all the ring of integers of the Galois closures of* L_i*, where* L_i *are CM fields such that* $\mathrm{End}(X) \otimes \mathbb{Q} \cong \bigoplus_i L_i$,
- *All the discriminants of the above* L_i *over* $\mathbb{Q}$ *are invertible in* R.

Then for any $f \in \mathcal{N}^{\mathrm{hol}}_\rho(R)$*, the evaluation* $f(\widetilde{\alpha})$ *of* f *at* $\widetilde{\alpha}$ *belongs to* R^d.

Proof. By assumption, embeddings of $L_i \hookrightarrow \mathbb{C}$ give rise to an R-isomorphism $\mathrm{End}(X) \otimes R \cong R^{2g}$, and $H^1_{\mathrm{DR}}(X/R)$ is an invertible R^{2g}-module. Then by [14, Lemma 2.0.8], there exits a projection $H^1_{\mathrm{DR}}(X/R) \to \Omega^1_{X/R}$ compatible with the action of $\mathrm{End}(X)$. $\square$

Results in this section can be extended by Theorem 3.4 for general representations of GL_g as follows.

Theorem 3.7. *Let* R *be a* $\mathbb{Z}[1/N, \zeta_N]$*-subalgebra of* $\mathbb{C}$*, and* $\rho : GL_g \to GL_d$ *be a rational homomorphism over* R *which is the direct sum of* ρ_j*, where* ρ_j *are associated with* $W_{\kappa_j - h_j(1,\ldots,1)}$ *for* $\kappa_j \in X^+(T_g)$, $h_j \in \mathbb{Z}$.

(1) *The* R*-module* $\mathcal{N}_\rho(R)$ *is finitely generated, and there exists an* R*-linear isomorphism* $\Phi : \mathcal{N}_\rho(R) \xrightarrow{\sim} \mathcal{N}^{\mathrm{hol}}_\rho(R)$*. Consequently,* $\mathcal{N}^{\mathrm{hol}}_\rho(R)$ *is a finitely generated* R*-module, and* $\mathcal{N}^{\mathrm{hol}}_\rho(R) \otimes_R \mathbb{C} = \mathcal{N}^{\mathrm{hol}}_\rho(\mathbb{C})$.

(2) *The assertions of Theorems 3.5 and 3.6 hold.*

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