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EXAMPLES OF SIEGEL MODULAR FORMS IN THE KERNEL OF THE THETA OPERATOR MOD p

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ABSTRACT. We construct many examples of Siegel modular forms in the kernel of the theta operator mod p by using theta series.

1. INTRODUCTION

Ramanujan's θ operator is a familiar topic in the theory of elliptic modular forms, defined by

$$f = \sum a(n)e^{2\pi inz} \longmapsto \theta(f) = \frac{1}{2\pi i}f' = \sum na(n)e^{2\pi inz}.$$

For Siegel modular forms of degree n, the Fourier expansion runs over positive semidefinite half-integral matrices of size n and we can define several analogues of the Ramanujan θ operator: For $1 \leq r \leq n$ we may introduce

$$\begin{aligned} F = \sum_T a(T)e^{2\pi i tr(TZ)} \longmapsto \Theta^{[r]}(F) &= \frac{1}{(2\pi i)^r}\left(\frac{\partial}{\partial_{ij}}\right)^{[r]} F \\ &= \sum_T T^{[r]}a(T)e^{2\pi i tr(TZ)}, \end{aligned}$$

where, for a matrix A of size n, we denote by $A^{[r]}$ the matrix of all the determinants of its submatrices of size r and $\partial_{ij} := \frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial z_{ij}}$. In general, $\Theta^{[r]}F$ is no longer a modular form, but it is a modular form mod p (even a p-adic modular form), vector-valued if $r < n$, see [5].

Our aim in this note is to explore the existence and explicit construction of Siegel modular forms which are in the kernel of such

Θ-operators mod p, we focus on the scalar-valued case $j = n$, for details and the general case we refer to [7].
Obvious candidates for such modular forms are theta series ϑ_S^n

$$\vartheta_S^n(Z) = \sum_{X \in \mathbb{Z}^{(n,n)}} e^{\pi i tr(X^t S X Z)},$$

attached to even integral positive quadratic forms S of rank n and level being a positive power of p; we will also consider variants of this involving a harmonic polynomial:

$$\vartheta_{S,\det}^n(Z) = \sum_{X \in \mathbb{Z}^{(n,n)}} \det(X) e^{\pi i tr(X^t S X Z)}.$$

Looking at the Fourier expansion, evidently such theta series are in the kernel of $\Theta^{[n]}$ mod p. On the other hand, these theta series are not of level one and one has to do a level change to level one by increasing the weight by a suitable multiple of $p-1$. This is always possible, see [5]. This method works only for even degree n, because we otherwise enter into the realm of modular forms of half-integral weight, but a somewhat weaker variant of our method is also possible for the case of odd n.
On the other hand, for even degree, our method provides plenty of examples for level one forms F (of weight in an arbitrary congruence class modulo $p-1$) which satisfy $\Theta^{[n]} F \equiv 0 \bmod p$, the only obstruction comes from the arithmetic of quadratic forms, which puts some constraints on the existence of positive definite forms with the requested properties (e.g. for $n = 2$ there do not exist positive definite quadratic forms of level p, if $p \equiv 1 \bmod 4$).
We have to make an important comment on what we mean by "explicit construction" here: The kernel of $\Theta^{[n]}$ mod p is a notion which depends only on modular forms mod p, therefore the weight of the constructed modular form is only of interest mod $(p-1)$. We call our procedure "weak construction" if we are only interested in the weight mod $p-1$. On the other hand, one is also interested in explicit small "true weights" for which we can get modular forms in the kernel mod p. In this note we focus on the latter "strong construction". Indeed, in some cases our strong construction is the best possible in the sense that the level one form F obtained from the

theta series has the smallest possible weight among all forms congruent to it, i.e. the weight of F is equal to its "filtration" in the sense of Serre and Swinnerton-Dyer, see [6].
In the final section we also show that some of the known examples of congruences for degree two Siegel modular forms can be explained by our methods.

2. Preliminaries

For standard facts about Siegel modular forms we refer to [1, 8, 10].
The group $Sp(n,\mathbb{R})$ acts on the upper half space $\mathbb{H}_n$ in the usual way and on functions $f : \mathbb{H}_n \longrightarrow \mathbb{C}$ by the standard slash-operator.
We will mainly be concerned with modular forms for congruence subgroups of type

$$\Gamma_0^n(p) := \left\{ M = \begin{pmatrix} AB \\ CD \end{pmatrix} \mid C \equiv 0 \bmod p \right\},$$

where p is an odd prime. The nebentypus character will be trivial or the quadratic character

$$\chi_p(*) := \left(\frac{(-1)^{\frac{p-1}{2}} p}{*} \right)$$

for an odd prime p.

3. Examples

We exibit here some examples of degree 2, which should be explained by our construction.

3.1. $k = 12$, $p = 23$. Let Δ_{12} be Ramanujan's cusp form of weight 12 and $[\Delta_{12}]$ the Klingen-Eisenstein series of degree 2 attached to it. It was shown in [2] that this is not congruent zero mod 23, but

$$\Theta^{[2]}([\Delta_{12}] \equiv 0 \bmod 23$$

3.2. $k = 16$, $p = 31$. Mizumoto [12] found a similar example, by considering the unique normalized eigen cups form Δ_{16} of weight 16 and its Klingen-Eisenstein $[\Delta_{16}]$; he showed that

$$\Theta^{[2]}([\Delta_{16}] \equiv 0 \bmod 31$$

3.3. $k = 35, p = 23$. Let χ_{35} be the wellknown cusp form of degree 2 and weight 35, then it was shown in [9] by using explicit computations and a "Sturm bound" that

$$\Theta^{[2]}(\chi_{35}) \equiv 0 \bmod 23.$$

A similar example was exhibited in [11] for weight 47 and a congruence mod 31. We understand that in the meantime S.Takemori has found many more numerical examples of similar type in degree 2.
It will follow from our considerations that all these examples can be explained by our constructions, using binary quadratic forms.

3.4. **Siegel Eisenstein series:** By quite different methods the last author showed that for even degree n and primes p with $p > n + 3$ and $p \equiv (-1)^{\frac{n}{2}} \bmod 4$ the Siegel Eisenstein series of degree n and weight $\frac{n}{2} + t \cdot \frac{p-1}{2}$ is in the kernel of Θ if t is an odd positive integer, see [13].

3.5. **A very simple construction.** We have to mention that we can get examples by starting from an arbitrary Siegel modular form f of level one, we may then consider $g(z) := f(pz)$; this is a modular form for the congruence subgroup $\Gamma_0(p)$ and therefore g is congruent to a level one form G; this G is in the kernel mod p of the theta operator.
This construction shows that (at least for even degree) for all weights mod p-1 we can get modular forms in the kernel of Theta operators mod p.

4. Main Results

The theorems below produce a lot of examples of level one Siegel modular forms of explicit low weight in the kernel of $\Theta^{[n]}$ mod p. Note however that the arithmetic of quadratic forms puts constraints on the existence of the positive even integral quadratic forms used.

Theorem 4.1. Let S be an even integral positive definite quadratic form of even rank n with determinant p^2 and level p with $p \geq n$ if

$p \equiv 1 \bmod 4$ and $p \geq n+3$ in general.
Then there exists a modular form F of level one and weight

$$k = \frac{n}{2} + (p-1)$$

such that $F \equiv \vartheta_S^n(Z)$.

Theorem 4.2. Let S be an even integral positive definite quadratic form of even rank n with determinant p and level p where p is a prime with $p \geq 2n+3$.

Then there exists a modular form F of level one and weight $k = \frac{n}{2} + \frac{p-1}{2}$ such that $F \equiv \vartheta_S^n(Z)$.

Both these theorems follow directly from Theorem 4 and Theorem 5 (corrected version) in [4].
The next result is concerned with similar results for theta series with harmonic polynomial:

Theorem 4.3. Let S be an even integral positive definite quadratic form with determinant p. Then there exists a cusp form F of level one of weight

$$k = \frac{n}{2} + 1 + 3 \cdot \frac{p-1}{2}$$

with $F \equiv \vartheta_{S,\det}^n \bmod p$.

Remark 4.4. Even if such S exists, one has to impose two additional conditions on the integral automorphisms of S to assure that $\vartheta_{S,\det}^n$ is not congruent zero mod p: There should not exist improper integral automorphisms of S and S should not have an integral automorphism of order p.

Remark 4.5. The results above can be refined to give a lower bound for the number of mod p linearly independent elements in the kernel of $\Theta^{[n]}$ mod p.

5. About the proofs

In most cases, the proof relies on finding a "suitable" modular form $\mathcal{E}$ of level p such that $\mathcal{E} \equiv 1 \bmod p$ and such that $\mathcal{E}$ has "good properties" mod p in all cusps. Then one can hope that for a given modular form f for $\Gamma_0(p)$ the function

$$F := trace(f \cdot \mathcal{E}^t)$$

has the desired property mod p for a suitable power of $\mathcal{E}$. Here the trace denotes a summation over representatives of $\Gamma_0(p)\backslash Sp(n,\mathbb{Z})$ and in our special case f denotes the theta series in question. There are several variants of this basic procedure, see e.g. [4, 5]. In particular, for Theorem 4.3., a very delicate choice of $\mathcal{E}$ is necessary. It is essential here to understand the p-adic properties of

$$(f \cdot \mathcal{E}^t) \mid \omega_j \qquad (0 \leq j \leq n)$$

where the matrices

$$\omega_j := \begin{pmatrix} 0_j & 0 & -1_j & 0 \\ 0 & 1_{n-j} & 0 & 0_{n-j} \\ 1_j & 0 & 0_j & 0 \\ 0 & 0_{n-j} & 0 & 1_{n-j} \end{pmatrix}$$

parametrize the (zero-dimensional) cusps of $Sp(n,\mathbb{Z})$.
We emphasize that the choice of $\mathcal{E}$ is easier, if we do not worry about the weight of F.

6. The examples mod 23

The class number of $\mathbb{Q}(\sqrt{-23})$ is 3, the three inequivalent binary quadratic forms of discriminant -23 are given by

$$S_0 = \begin{pmatrix} 2 & 1 \\ 1 & 12 \end{pmatrix}, \quad S_{\pm 1} = \begin{pmatrix} 4 & \pm 1 \\ \pm 1 & 6 \end{pmatrix},$$

where S_1 and S_{-1} are improperly equivalent. By Theorem 4.2. the modular forms

$$\vartheta^2_{S_0} \pm \vartheta^2_{S_1}$$

are congruent mod 23 to level one forms of weight $12 = 1 + \frac{23-1}{2}$. This should be compared with example 3.1 and with the case of Siegel Eisenstein series from 3.4.
Furthermore, by Theorem 4.3. the weight 2 theta series

$$\vartheta^2_{S_1,\det}$$

is congruent mod 23 to a level one cusp form of weight

$$35 = 2 + 3 \cdot \frac{23-1}{2}.$$

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