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Large time behavior of solutions to the compressible Navier-Stokes equations in an infinite layer under slip boundary condition *

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1 Introduction

We study the large time behavior of solutions of the compressible Navier-Stokes equations

$$\partial_t\rho+\mathrm{div}(\rho v)=0, \tag{1}$$

$$\rho(\partial_t v+v\cdot\nabla v)-\mu\Delta v-(\mu+\mu')\nabla\mathrm{div}v+\nabla P(\rho)=0 \tag{2}$$

in an infinite layer Ω of $\mathbb{R}^2$:

$$\Omega=\{x=(x_1,x_2)\in\mathbb{R}^2;\ x_1\in\mathbb{R},0<x_2<1\}$$

under the slip boundary condition

$$\partial_{x_2}v^1|_{x_2=0,1}=0,\quad v^2|_{x_2=0,1}=0. \tag{3}$$

Here $\rho=\rho(x,t)>0$ and $v={}^\top(v^1(x,t),v^2(x,t))$ denote the unknown density and velocity, respectively, at time $t\geq 0$ and position $x\in\Omega$; $P=P(\rho)$ is the pressure that is assumed to be a smooth function of ρ satisfying $P'(\rho_*)>0$ for a given constant $\rho_*>0$; μ and μ' are viscosity coefficients that are assumed to be constants and satisfy $\mu>0,\mu+\mu'\geq 0$; div, ∇ and Δ denote the usual divergence, gradient and Laplacian with respect to x. Here and in what follows ${}^\top\cdot$ means the transposition.

We impose the initial condition

$$\rho|_{t=0}=\rho_0,\quad v|_{t=0}=v_0. \tag{4}$$

Here $\rho_0=\rho_0(x)$ and $v_0=v_0(x)$ satisfy $\rho_0(x)\to\rho_*$ and $v_0(x)\to 0$ as $|x|\to\infty$.

The aim of our research is to investigate the large time behavior of solutions to (1)-(4) around the motionless state $\rho=\rho_*$, $v=0$. We rewrite (1)-(2) into the following equations for the perturbation

$$\partial_t\phi+\gamma\mathrm{div}w=f^0(\phi,w), \tag{5}$$

*This is based on a joint work with Shota Enomoto (Graduate School of Mathematics, Kyushu University) and Professor Yoshiyuki Kagei (Faculty of Mathematics, Kyushu University).

$$\partial_t w - \nu\Delta w - \tilde{\nu}\nabla \mathrm{div} w + \gamma\nabla\phi = \tilde{f}(\phi, w). \tag{6}$$

Here $u = {}^\top(\phi, w)$ with $\phi = \frac{1}{\rho_*}(\rho - \rho_*)$ and $w = \frac{1}{\gamma}v$ denotes the perturbation from $u_s = {}^\top(\rho_*, 0)$; ν, $\tilde{\nu}$ and γ are parameters given by

$$\nu = \frac{\mu}{\rho_*}, \quad \tilde{\nu} = \frac{\mu + \mu'}{\rho_*}, \quad \gamma = \sqrt{P'(\rho_*)};$$

and $f(\phi, w) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$ denote the nonlinear terms.

The boundary condition (3) and initial condition (4) are transformed into

$$\partial_{x_2} w^1|_{x_2=0,1} = 0, \quad w^2|_{x_2=0,1} = 0 \tag{7}$$

and

$$u|_{t=0} = u_0 = {}^\top(\phi_0, w_0). \tag{8}$$

Here u_0 satisfies $u_0(x) \to 0$ as $|x| \to \infty$.

The large time behavior of solutions of the compressible Navier-Stokes equations (1)-(2) on the layer Ω was studied in [1, 2, 3, 4] under the non-slip boundary condition $v|_{x_2=0,1} = 0$. It was shown in [3] that the large time behavior of perturbations of the motionless state is described by a one-dimensional linear heat equation. In [4] the asymptotic stability of parallel flow was considered and it was proved that the large time behavior of perturbations of parallel flow is described by a one-dimensional viscous Burgers equation when the Reynolds and Mach numbers are sufficiently small. In the case of time-periodic parallel flow, the large time behavior of perturbations is also described by a one-dimensional diffusion equation ([1, 2]). In all cases of [1, 2, 3, 4], the asymptotic leading parts under the non-slip boundary condition exhibit purely diffusive phenomena. In this paper we show that the solution of (1)-(2) under the slip boundary condition (3) with (4) behaves like a superposition of one-dimensional diffusion waves as $t \to \infty$ as in the case of one-dimensional compressible Navier-Stokes equation [7, 10]. More precisely, consider the problem (5)-(8) for u. We prove that, under appropriate conditions for u_0, the solution $u(t)$ satisfies

$$\|\partial_x^k(u - \chi_+ \boldsymbol{a}_+ - \chi_- \boldsymbol{a}_-)(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1, \tag{9}$$

where $\boldsymbol{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given by

$$\chi_\pm(x_1, t) = z_\pm(x_1 \pm \gamma t, t). \tag{10}$$

Here $z_\pm = z_\pm(x_1, t)$ are the self-similar solutions of the viscous Burgers equations

$$\partial_t z_\pm - \frac{\nu + \tilde{\nu}}{2}\partial_{x_1}^2 z_\pm \mp c\partial_{x_1}(z_\pm^2) = 0 \tag{11}$$

satisfying

$$\int_{\mathbb{R}} z_\pm(x_1, t)\mathrm{d}x_1 = \frac{1}{2}\int_\Omega (\phi_0(x) \pm (1 + \phi_0(x))w_0^1(x))\mathrm{d}x \tag{12}$$

for some constant $c \in \mathbb{R}$.

In contrast to the case of the non-slip boundary condition, we see that a hyperbolic aspect of (1)-(2) appears in the asymptotic leading part of the solution under the slip boundary condition.

2 Main results

We set

$$H^2_* = \{w = {}^\top(w^1, w^2) \in H^2(\Omega);\ \partial_{x_2} w^1|_{x_2=0,1} = 0, w^2|_{x_2=0,1} = 0\}.$$

For $\alpha \in \mathbb{R}$, we denote by $L^1_\alpha = L^1_\alpha(\Omega)$ the weighted L^1 space with weight $(1 + |x_1|)^\alpha$, and its norm is denoted by

$$\|f\|_{L^1_\alpha} = \int_\Omega (1 + |x_1|)^\alpha |f(x)| \mathrm{d}x.$$

We now state the main results of this paper. We have the following decay estimate of the L^2 norm of the solution u.

Theorem 2.1 *There exists a positive number ε_0 such that if $u_0 = {}^\top(\phi_0, w_0) \in (H^2 \times H^2_*) \cap L^1$ with $w_0 = {}^\top(w_0^1, w_0^2)$ satisfies $\|u_0\|_{H^2 \cap L^1} \le \varepsilon_0$, then problem (5)-(8) has a unique global solution*

$$u(t) = {}^\top(\phi(t), w(t)) \in C([0, \infty); H^2 \times H^2_*)$$

and $u(t)$ satisfies

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}} \|u_0\|_{H^2 \cap L^1}$$

for $t \ge 0,\ k = 0, 1, 2$.

We next consider the asymptotic behavior of solutions.

Theorem 2.2 *In addition to the assumptions of Theorem 2.1, if $\phi_0, w_0^1 \in L^1_{1/2}$, then*

$$\|\partial_x^k(u - \chi_+ \boldsymbol{a}_+ - \chi_- \boldsymbol{a}_-)(t)\|_{L^2} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}, \quad k = 0, 1.$$

Here $\boldsymbol{a}_\pm = {}^\top(1, \pm 1, 0)$ and $\chi_\pm = \chi_\pm(x_1, t)$ are the diffusion waves given in (10)-(12).

3 Outline of the proof

3.1 Spectral properties of linearized operator

We rewrite the equation (5)-(6) as following

$$\partial_t u + Lu = F, \quad u|_{t=0} = u_0, \tag{13}$$

where $u = {}^\top(\phi, w); F = {}^\top(f^0, \tilde{f})$ with $\tilde{f} = {}^\top(f^1, f^2)$ is a given function, and L is an operator of the form

$$L = \begin{pmatrix} 0 & \gamma \mathrm{div} \\ \gamma \nabla & -\nu\Delta - \tilde{\nu}\nabla \mathrm{div} \end{pmatrix}$$

in $H^1 \times L^2$ with domain $D(L) = H^1 \times H^2_*$.

To investigate (13), we take the Fourier transform of (13) in $x_1 \in \mathbb{R}$, and then we expand $\hat{u}$ and $\hat{F}$ into the Fourier series to obtain

$$\partial_t \hat{u}_k + \hat{L}_{\xi,k}\hat{u}_k = \hat{F}_k, \tag{14}$$

where $\hat{u}_k = {}^\top(\hat{\phi}_k, \hat{w}_k^1, \hat{w}_k^2)$, $\hat{F}_k = {}^\top(\hat{f}_k^0, \hat{f}_k^1, \hat{f}_k^2)$ and

$$\hat{L}_{\xi,k} = \begin{pmatrix} 0 & i\gamma\xi & \gamma k\pi \\ i\gamma\xi & \nu(\xi^2 + k^2\pi^2) + \tilde{\nu}\xi^2 & -i\tilde{\nu}k\pi\xi \\ -\gamma k\pi & i\tilde{\nu}k\pi\xi & \nu(\xi^2 + k^2\pi^2) + \tilde{\nu}k^2\pi^2 \end{pmatrix}.$$

For the the spectrum of $\sigma(-\hat{L}_{\xi,k})$, the case $|\xi| \ll 1,\ k = 0$ is the slowest decay part. In this case, the eigenvalues and eigenprojections are given by

$$\begin{aligned} \lambda_{\pm,0}(\xi) &= \pm i\gamma\xi - \frac{\nu+\tilde{\nu}}{2}\xi^2 + O(\xi^3) \quad (\xi \to 0), \\ P_{\pm,\xi} &= \tilde{P}_\pm(1 + O(\xi))\Pi, \end{aligned}$$

where

$$\tilde{P}_\pm = \begin{pmatrix} 1 & \pm 1 & 0 \\ \pm 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Pi u = \begin{pmatrix} \langle \phi \rangle \\ \langle w^1 \rangle \\ 0 \end{pmatrix}.$$

Here $u = {}^\top(\phi, w^1, w^2)$ and $\langle \phi \rangle$ is defined by $\langle \phi \rangle = \int_0^1 \phi(x_2)\mathrm{d}x_2$.

3.2 Decay estimate: Proof of Theorem 2.1

We consider the nonlinear problem

$$\begin{cases} \partial_t u + Lu = F(u), \\ u|_{t=0} = u_0. \end{cases} \tag{15}$$

Here $u = {}^\top(\phi, w)$ and $F(u) = {}^\top(f^0(\phi, w), \tilde{f}(\phi, w))$.

One can prove the local solvability for (15) as in [5].

Proposition 3.1 *Assume that* $u_0 = {}^\top(\phi_0, w_0) \in H^2 \times H_*^2$ *and* $\|\phi_0\|_\infty \le \frac{1}{2}$. *Then there exists* $T_0 > 0$ *depending on* $\|u_0\|_{H^2}$ *such that problem (15) has a unique solution* $u = {}^\top(\phi, w)$ *on* $[0, T_0]$ *satisfying* $u \in C([0, T_0]; H^2 \times H_*^2) \cap C^1([0, T_0]; L^2)$ *with* $w \in L^2(0, T_0; H^3)$ *and* $\|\phi_0(t)\|_\infty \le \frac{3}{4}$ *for* $t \in [0, T_0]$. *Furthermore, the inequality*

$$\sup_{t\in[0,T_0]} \{\|u(t)\|_{H^2} + \|\partial_t u(t)\|_{L^2}\} + \int_0^{T_0} \|w\|_{H^3}^2 \mathrm{d}t \le C_0\{1 + \|u_0\|_{H^2}^2\}^a \|u_0\|_{H^2}^2 \tag{16}$$

holds with some constants $C_0 > 0$ *and* $a > 0$.

The global existence of $u(t)$ follows in a standard manner from Proposition 3.1 and Proposition 3.4 below which provides the a priori bound $\|u(t)\|_{H^2} \le C\|u_0\|_{H^2\cap L^1}$ when $\|u_0\|_{H^2\cap L^1}$ is sufficiently small.

We next consider the a priori estimates for $u(t)$. Let r_0 be a number satisfying $0 < r_0 \le 1$. We introduce the cut-off function $\mathbf{1}_{\{|\xi|\le r_0\}}$ defined by

$$\mathbf{1}_{\{|\xi|\le r_0\}} = \begin{cases} 1 & (|\xi| < r_0), \\ 0 & (|\xi| \ge r_0). \end{cases}$$

We introduce the projections P_1 and P_∞ defined by

$$P_1 u = \mathcal{F}^{-1}\mathbf{1}_{\{|\xi|\le r_0\}}\Pi\mathcal{F}u, \qquad P_\infty = I - P_1.$$

We decompose $u = {}^\top(\phi, w)$ into

$$u = u_1 + u_\infty,$$

where

$$u_1 = P_1 u = {}^\top(\phi_1, w_1^1, w_1^2), \quad u_\infty = P_\infty u = {}^\top(\phi_\infty, w_\infty^1, w_\infty^2).$$

Proposition 3.2 *Let $u(t)$ be a solution of (15) on $[0,T]$. Assume that $u \in C([0,T]; H^2 \times H^2_*) \cap C^1([0,T]; L^2)$ with $w \in L^2(0,T;H^3)$. Then*

$$u_1 = {}^\top(\phi_1, w_1) \in C^1([0,T]; H^l(\Omega)) \quad (\forall l = 0,1,2,\cdots)$$

and

$$u_\infty = {}^\top(\phi_\infty, w_\infty) \in C([0,T]; H^2 \times H^2_*) \cap C^1([0,T]; L^2)$$

with $w_\infty \in L^2(0,T;H^3)$.

Furthermore, u_1 and u_∞ satisfy

$$u_1 = P_1 e^{-tL}u_0 + \int_0^t P_1 e^{-(t-\tau)L}F(u(\tau))\mathrm{d}\tau, \tag{17}$$

$$\partial_t u_\infty + Lu_\infty = F_\infty, \quad u_\infty|_{t=0} = P_\infty u_0, \tag{18}$$

where $F_\infty = P_\infty F = {}^\top(f_\infty^0, \tilde{f}_\infty)$, $\tilde{f}_\infty = (f_\infty^1, f_\infty^2)$.

We define $M(t) \ge 0$ by

$$M(t) = M_1(t) + M_\infty(t) \quad (t \in [0,T]).$$

Here $M_1(t)$ and $M_\infty(t)$ are defined by

$$M_1(t) = \sup_{0\le\tau\le t}\left\{\sum_{k=0}^{2}(1+\tau)^{\frac{1}{4}+\frac{k}{2}}\|\partial_{x_1}^k u_1(\tau)\|_{L^2} + (1+\tau)^{\frac{3}{4}}\|\partial_t u_1(\tau)\|_{L^2}\right\},$$

$$M_\infty(t) = \Big(\sup_{0\le\tau\le t}(1+\tau)^{\frac{5}{2}}\{\|u_\infty(\tau)\|_{H^2}^2 + \|\partial_t u_\infty(\tau)\|_{L^2}^2\}\Big)^{\frac{1}{2}}.$$

We introduce the quantities $E_\infty(t)$ and $D_\infty(t)$ for $u_\infty(t) = {}^\top(\phi_\infty(t), w_\infty(t))$:

$$E_\infty(t) = \|u_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_{L^2}^2,$$
$$D_\infty(t) = \|\nabla\phi_\infty(t)\|_{H^1}^2 + \|\nabla w_\infty(t)\|_{H^2}^2 + \|\partial_t u_\infty(t)\|_{H^1}^2.$$

Proposition 3.3 *Let $u(t)$ be a solution of (15) on $[0,T]$. Then there exists a positive constant ε_1 such that if $\|u(t)\|_{H^2} \le \varepsilon_1$ and $M(t) \le 1$ for $t \in [0,T]$, the estimates*

$$M_1(t) \le C\{\|u_0\|_{L^1} + M(t)^2\} \tag{19}$$

and

$$\begin{aligned} &E_\infty(t) + \int_0^t e^{-a(t-\tau)} D_\infty(\tau)\mathrm{d}\tau \\ &\le C\big\{e^{-at}E_\infty(0) + (1+t)^{-\frac{5}{2}}M(t)^4 + \int_0^t e^{-a(t-\tau)}\mathcal{R}(\tau)\mathrm{d}\tau\big\} \end{aligned} \tag{20}$$

hold uniformly for $t \in [0,T]$ with $C > 0$ independent of T. Here $a = a(\nu, \tilde{\nu}, \gamma)$ is a positive constant; and $\mathcal{R}(t)$ is a function satisfying the estimate

$$\mathcal{R}(t) \le C\{(1+t)^{-\frac{5}{2}}M(t)^3 + M(t)D_\infty(t)\}. \tag{21}$$

3.3 Estimates of low and high frequency parts

We see from spectral properties of $-\hat{L}_{\xi,k}$ and the definition of Π that

$$\|\partial_{x_1}^l e^{-tL}P_1 u_0\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{l}{2}}\|u_0\|_{L^1} \tag{22}$$

for $l \ge 0$, and we thus obtain

$$\|\partial_{x_1}^k u_1(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\{\|u_0\|_{L^1} + M(t)^2\} \tag{23}$$

for $k = 0, 1, 2$.

As for the time derivative, we have

$$\|\partial_t u_1(t)\|_{L^2} \le C(1+t)^{-\frac{3}{4}}\{\|u_0\|_{L^1} + M(t)^2\}. \tag{24}$$

By (23) and (24), we obtain (19).

As for the high-frequency part $u_\infty = P_\infty u$, we apply the Matsumura-Nishida energy method to prove estimate (20) in Proposition 3.3.

From Proposition 3.3, one can show the following uniform estimate of $M(t)$ as in [4].

Proposition 3.4 *If $\|u_0\|_{H^2 \cap L^1}$ is sufficiently small, then*

$$M(t) \le C\|u_0\|_{H^2 \cap L^1}. \tag{25}$$

Theorem 2.1 now follows from Propositions 3.1 and 3.4.

3.4 Asymptotic behavior: Proof of Theorem 2.2

To prove Theorem 2.2 we rewrite (1)-(2) in the form of conservation laws.

We set

$$m = \rho v = \rho_*(1+\phi)v.$$

Then (1)-(2) is written as

$$\begin{cases} \partial_t \rho + \operatorname{div} m = 0, \\ \partial_t m - \mu\Delta(\frac{m}{\rho}) - (\mu+\mu')\nabla\operatorname{div}(\frac{m}{\rho}) + \nabla P(\rho) + \operatorname{div}(\frac{m\otimes m}{\rho}) = 0, \end{cases} \tag{26}$$

and the boundary condition (3) is transformed into

$$\partial_{x_2}\left(\frac{m^1}{\rho}\right)\Big|_{x_2=0,1} = 0, \quad m^2\big|_{x_2=0,1} = 0. \tag{27}$$

We decompose ${}^\top(\phi, m^1)$ as

$$\phi = \Phi + \Phi_\infty, \quad \Phi = \phi_1 = \tilde{P}_1\phi, \quad \Phi_\infty = \phi_\infty = \tilde{P}_\infty\phi,$$
$$m^1 = \rho_*\gamma(M + M_\infty), \quad M = \frac{1}{\rho_*\gamma}\tilde{P}_1 m^1, \quad M_\infty = \frac{1}{\rho_*\gamma}\tilde{P}_\infty m^1.$$

Note that $w^1 = \frac{M+M_\infty}{1+\phi}$. Here the operators $\tilde{P}_1$ and $\tilde{P}_\infty$ defined by

$$\tilde{P}_1\phi = \mathcal{F}^{-1}\mathbf{1}_{\{|\xi|\le r_0\}}\langle\mathcal{F}\phi\rangle, \quad \tilde{P}_\infty = I - \tilde{P}_1.$$

Applying P_1 to (26) and using (27), we have

$$\begin{cases} \partial_t\Phi + \gamma\partial_{x_1}M = 0, \\ \partial_t M - (\nu+\tilde{\nu})\partial_{x_1}^2 M + \gamma\partial_{x_1}\Phi = \partial_{x_1}\tilde{P}_1 g(U) + \partial_{x_1}\tilde{P}_1\tilde{g}. \end{cases} \tag{28}$$

Here $U = {}^\top(\Phi, M)$,

$$g(U) = -\frac{\rho_* P''(\rho_*)}{2\gamma}\Phi^2 - \gamma M^2,$$
$$\tilde{g} = \tilde{g}(x,t) = -(\nu+\tilde{\nu})\partial_{x_1}(\phi w^1) - \frac{\rho_* P''(\rho_*)}{2\gamma}(2\Phi\Phi_\infty + \Phi_\infty^2)$$
$$- \gamma(2MM_\infty + M_\infty^2) + \gamma(\phi w^1(M + M_\infty)),$$

where $\phi = \Phi + \Phi_\infty$, $w^1 = \frac{M+M_\infty}{1+\phi}$.

We write (28) in the form

$$\begin{cases} \partial_t U + L_0 U = \partial_{x_1}P_0 G(U) + \partial_{x_1}P_0\tilde{G}, \quad U = P_0 U, \\ U|_{t=0} = P_0 U_0, \end{cases} \tag{29}$$

where $U_0 = {}^\top(\phi_0, \frac{1}{\rho_*\gamma}m_0^1) = {}^\top(\phi_0, (1+\phi_0)w_0^1)$,

$$L_0 = \begin{pmatrix} 0 & \gamma\partial_{x_1} \\ \gamma\partial_{x_1} & -(\nu+\tilde{\nu})\partial_{x_1}^2 \end{pmatrix},$$

$$G(U) = \begin{pmatrix} 0 \\ g(U) \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 \\ \tilde{g} \end{pmatrix},$$

and P_0 denotes the projection defined by

$$P_0(U) = \begin{pmatrix} \tilde{P}_1 \Phi \\ \tilde{P}_1 M \end{pmatrix}$$

for $U = {}^\top(\Phi, M)$.

We see from the spectral properties of $-\hat{L}_{\xi,k}$ that

$$e^{-tL_0} = \mathcal{F}^{-1}(e^{\lambda_+ t} P_+ + e^{\lambda_- t} P_-)\mathcal{F},$$

where

$$\lambda_\pm = \lambda_{\pm,0} = -\frac{1}{2}(\nu + \tilde{\nu})\xi^2 \pm \frac{1}{2}\sqrt{(\nu + \tilde{\nu})^2 \xi^4 - 4\gamma^2 \xi^2},$$
$$P_\pm = \pm \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} -\lambda_\mp & i\gamma\xi \\ i\gamma\xi & \lambda_\pm \end{pmatrix}.$$

We observe that, for $|\xi| \ll 1$,

$$\lambda_\pm = -\frac{\nu + \tilde{\nu}}{2}\xi^2 \pm i\gamma\xi + O(\xi^3),$$
$$P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}(1 + O(\xi)).$$

We define $S(t)$ and $S_\pm(t)$ by

$$S(t) = S_+(t) + S_-(t),$$
$$S_\pm(t) = \mathcal{F}^{-1}\hat{S}_\pm(t)\mathcal{F},$$
$$\hat{S}_\pm(t) = \frac{1}{2}e^{-\frac{\nu+\tilde{\nu}}{2}\xi^2 t \pm i\gamma\xi t}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

Clearly, $e^{-tL_0}P_0$ has the same estimate as that for $e^{-tL}P_1$ such as (22). Furthermore, $e^{-tL_0}P_0$ is approximated by $S(t)$ in the following way. We define Π_0 by

$$\Pi_0 U_0 = {}^\top(\langle \phi_0 \rangle, \langle M_0 \rangle) \quad \text{for} \quad U_0 = {}^\top(\phi_0, M_0).$$

Note that $\Pi_0 P_0 = P_0 \Pi_0 = P_0$.

We denote by $U^{(0)}(t) = {}^\top(\phi^{(0)}(x_1, t), M^{(0),1}(x_1, t))$ the solution of the following integral equation:

$$U^{(0)}(t) = S(t)\Pi_0 U_0 + \int_0^t S(t - \tau)\partial_{x_1} G(U^{(0)}(\tau))\mathrm{d}\tau. \tag{30}$$

We see from (29) that $U(t)$ is written as

$$U(t) = e^{-tL_0}P_0 U_0 + \int_0^t e^{-(t-\tau)L_0} P_0 \partial_{x_1}(G(U) + \tilde{G})(\tau)\mathrm{d}\tau. \tag{31}$$

we have the following estimates for $U^{(0)}(t)$.

Proposition 3.5 *If* $\|U_0\|_{H^2\cap L^1} \ll 1$, *then (30) has a unique solution* $U^{(0)}(t)$ *that satisfies*

$$\|\partial_{x_1}^k U^{(0)}(t)\|_{L^2} \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|U_0\|_{H^2\cap L^1},\ k=0,1,2, \tag{32}$$

$$\|\partial_{x_1}^k U^{(0)}(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|U_0\|_{H^2\cap L^1},\ k=0,1. \tag{33}$$

We have the following estimate for $U(t)-U^{(0)}(t)$.

Theorem 3.6 *If* $\|U_0\|_{H^2\cap L^1} \ll 1$, *then*

$$\|\partial_{x_1}^k (U(t)-U^{(0)}(t))\|_{L^2} \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}+\delta}\|U_0\|_{H^2\cap L^1},\ k=0,1,$$

for any $\delta>0$.

Proof of Theorem 2.2. It suffices to show that $\|\partial_{x_1}^k (U^{(0)}-\chi_+\boldsymbol{b}_+ - \chi_-\boldsymbol{b}_-)(t)\|_{L^2}$ for $k=0,1$, where $\boldsymbol{b}_\pm = {}^\top(1,\pm 1)\in\mathbb{R}^2$. Here $\chi_\pm=\chi_\pm(x_1,t)$ is the diffusion waves given in (10)-(12) with $c=\frac{1}{2}(a+b), a=-\frac{\rho_* P''(\rho_*)}{2\gamma}, b=-\gamma$. We follow the arguments in [7, 6]. We write U_0 as

$$U_0 = U_{0+}+U_{0-},$$

where

$$U_{0\pm} = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1\end{pmatrix}\Pi_0 U_0 = \frac{1}{2}\langle \phi_0 \pm \frac{1}{\rho_*\gamma} m_0^1\rangle \boldsymbol{b}_\pm.$$

It then follows that

$$U^{(0)}(t) = S_+(t)U_{0+} + S_-(t)U_{0-} + I_{1,+}(t) + I_{1,-}(t),$$

where

$$I_{1,\pm}(t) = \int_0^t S_\pm(t-\tau)\partial_{x_1}\begin{pmatrix} 0 \\ a(\phi^{(0)})^2 + b(M^{(0),1})^2\end{pmatrix} \mathrm{d}t.$$

We write $I_{1,\pm}(t)$ as

$$I_{1,\pm} = \pm\frac{1}{2}\int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}\big(a(\phi^{(0)})^2 + b(M^{(0),1})^2\big)\mathrm{d}\tau\boldsymbol{b}_\pm,$$

where

$$e^{-tL_\pm}u_0 = \mathcal{F}^{-1}\big[e^{(-\frac{\nu+\tilde{\nu}}{2}\xi^2 \pm i\gamma\xi)t}\hat{u}_0\big].$$

We note that $e^{-tL_\pm}$ satisfies the same estimates as those for $S_\pm(t)$.

We define $V(t)={}^\top(\eta(t),\zeta(t))$ by

$$\begin{aligned} U^{(0)}(t) &= \chi_+(t)\boldsymbol{b}_+ + \chi_-(t)\boldsymbol{b}_- + V(t) \\ &= \begin{pmatrix} \chi_+ + \chi_- + \eta \\ \chi_+ - \chi_- + \zeta \end{pmatrix}, \end{aligned}$$

and introduce

$$Y(t) = \sup_{0\le\tau\le t}\{(1+\tau)^{\frac{1}{2}}\|V(\tau)\|_{L^2} + (1+\tau)\|\partial_{x_1}V(\tau)\|_{L^2}\}.$$

We write

$$(\phi^{(0)})^2 = \chi_+^2 + \chi_-^2 + 2\chi_+\chi_- + \sigma_1\eta,$$
$$(M^{(0),1})^2 = \chi_+^2 + \chi_-^2 - 2\chi_+\chi_- + \sigma_2\zeta,$$

where $\sigma_1 = \chi_+ + \chi_- + \phi^{(0)}$ and $\sigma_2 = \chi_+ - \chi_- + M^{(0),1}$. It then follows that $I_{1,\pm}(t)$ is written in the following forms

$$I_{1,\pm}(t) = \pm\frac{1}{2}\int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}\Big((a+b)(\chi_+^2+\chi_-^2) + 2(a-b)\chi_+\chi_- + a\sigma_1\eta + b\sigma_2\zeta\Big)\mathrm{d}\tau\boldsymbol{b}_\pm.$$

Since $\chi_\pm$ satisfies

$$\chi_\pm(t) = e^{-tL_\pm}\chi_{0\pm} \pm \frac{a+b}{2}\int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}(\chi_\pm^2)(\tau)\mathrm{d}\tau,$$

where $\chi_{0\pm} = \chi_\pm(0)$, we see that

$$\begin{aligned}
V(t) &= U^{(0)}(t) - \chi_+(t)\boldsymbol{b}_+ - \chi_-(t)\boldsymbol{b}_- \\
&= S_+(t)(U_{0+} - \chi_{0+}\boldsymbol{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\boldsymbol{b}_-) + I_{1,+} + I_{1,-} \\
&\quad - \frac{a+b}{2}\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\chi_+^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ + \frac{a+b}{2}\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\chi_-^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&= S_+(t)(U_{0+} - \chi_{0+}\boldsymbol{b}_+) + S_-(t)(U_{0-} - \chi_{0-}\boldsymbol{b}_-) \\
&\quad + \frac{1}{2}(a+b)\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\chi_-^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - \frac{1}{2}(a+b)\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\chi_+^2)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&\quad + (a-b)\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\chi_+\chi_-)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - (a-b)\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\chi_+\chi_-)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&\quad + \frac{1}{2}a\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_1\eta)(\tau)\mathrm{d}\tau\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - \frac{1}{2}a\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_1\eta)(\tau)\mathrm{d}\tau\boldsymbol{b}_- \\
&\quad + \frac{1}{2}b\int_0^t e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_2\zeta)(\tau)\mathrm{d}\tau\boldsymbol{b}_+ \\
&\quad - \frac{1}{2}b\int_0^t e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_2\zeta)(\tau)\mathrm{d}\tau\boldsymbol{b}_-.
\end{aligned}$$

It then follows that

$$\|\partial_{x_1}^k V(t)\|_{L^2} \le \sum_{j=\pm}\|\partial_{x_1}^k S_j(t)(U_{0j} - \chi_{0j}\boldsymbol{b}_j)\|_{L^2}$$

$$
\begin{aligned}
&+ C_1\big(\|\partial_{x_1}^k w_+(t)\|_{L^2} + \|\partial_{x_1}^k w_-(t)\|_{L^2}\big)\\
&+ C_2\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+}\partial_{x_1}(\chi_+\chi_-)(\tau)\|_{L^2}\mathrm{d}\tau\\
&+ C_3\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-}\partial_{x_1}(\chi_+\chi_-)(\tau)\|_{L^2}\mathrm{d}\tau\\
&+ C_4\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_1\eta)(\tau)\|_{L^2}\mathrm{d}\tau\\
&+ C_5\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_1\eta)(\tau)\|_{L^2}\mathrm{d}\tau\\
&+ C_6\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_+}\partial_{x_1}(\sigma_2\zeta)(\tau)\|_{L^2}\mathrm{d}\tau\\
&+ C_7\int_0^t \|\partial_{x_1}^k e^{-(t-\tau)L_-}\partial_{x_1}(\sigma_{L^2}\zeta)(\tau)\|_{L^2}\mathrm{d}\tau\\
&=: \sum_{j=\pm}\|\partial_{x_1}^k S_j(t)(U_{0j}-\chi_{0j}\boldsymbol{b}_j)\|_{L^2} + \sum_{j=1}^{7} I_j.
\end{aligned}
$$

where

$$
w_\pm(t) = \int_0^t e^{-(t-\tau)L_\pm}\partial_{x_1}(\chi_\mp^2)(\tau)\mathrm{d}\tau,
$$

$$
C_1 = \frac{1}{2}|a+b|,\ C_2 = C_3 = |a-b|,\ C_4 = C_5 = \frac{1}{2}|a|,\ C_6 = C_7 = \frac{1}{2}|b|.
$$

Since

$$
\begin{aligned}
&\int_{\mathbb{R}}(U_{0\pm}-\chi_{0\pm}\boldsymbol{b}_\pm)\mathrm{d}x_1\\
&= \left[\frac{1}{2}\int_\Omega\Big(\phi^{(0)} \pm \frac{1}{\rho_*\gamma}m_0^1\Big)\mathrm{d}x - \int_{\mathbb{R}}\chi_{0\pm}\mathrm{d}x_1\right]\boldsymbol{b}_\pm = 0,
\end{aligned}
$$

we have

$$
\|\partial_{x_1}^k S_\pm(t)(U_{0\pm}-\chi_{0\pm}\boldsymbol{b}_\pm)\|_{L^2} \le Ct^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{L^1_{1/2}}.
$$

As for I_1, we apply the estimates for $w_\pm$ by T.-P. Liu [8] (see also [6, Lemma 4.2]) to obtain

$$
I_1 \le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2.
$$

We next estimate I_2. For $1 \le p \le \infty$ and $l \ge 0$, we have

$$
\|\partial_x^l(\chi_+\chi_-)(t)\|_{L^1} \le Ce^{-ct}\|u_0\|_{H^2\cap L^1}^2. \tag{34}
$$

It then follows from (34) that

$$
I_2 \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2.
$$

Similarly, we have $I_3 \le C(1+t)^{-\frac{3}{4}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}^2$.

As for I_4, we have

$$\begin{aligned} I_4 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}\|\sigma_1\eta(\tau)\|_{L^1}\mathrm{d}\tau \\ &+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{3}{4}}\|\partial_{x_1}^k(\sigma_1\eta)(\tau)\|_{L^2}\mathrm{d}\tau \\ &+C\int_0^{t}e^{-c_0(t-\tau)}(t-\tau)^{-\frac{1}{2}}\|\partial_{x_1}^k(\sigma_1\eta)(\tau)\|_{L^2}\mathrm{d}\tau \\ &=: I_{41}+I_{42}+I_{43}. \end{aligned}$$

By applying Proposition 3.5 and the following estimate

$$\|\partial_{x_1}^k\chi_\pm(t)\|_{L^2}\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_{L^1}, \tag{35}$$

we see that $\|\sigma_1(\tau)\|_{L^2}\le C(1+\tau)^{-\frac{1}{4}}\|u_0\|_{H^2\cap L^1}$. Since $\|\sigma_1\eta\|_{L^1}\le\|\sigma_1\|_{L^2}\|\eta\|_{L^2}$, we have

$$\begin{aligned} I_{41}&\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t), \\ I_{42}&\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t), \\ I_{43}&\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t). \end{aligned}$$

We thus obtain $I_4\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1}Y(t)$. We can obtain the estimates for I_5, I_6, I_7 in a similar manner. It then follows that if $\|u_0\|_{H^2\cap L^1}\ll 1$, we have

$$\|\partial_x^kV(t)\|_{L^2}\le C(1+t)^{-\frac{1}{2}-\frac{k}{2}}\|u_0\|_{H^2\cap L^1} \tag{36}$$

for $k=0,1$.

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