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Energy-efficient Threshold Circuits Computing Generalized Symmetric Functions

Hiroki Maniwa[1], Takayuki Oki[2], Akira Suzuki[1],
Kei Uchizawa[2], and Xiao Zhou[1]

[1] Graduate School of Information Sciences, Tohoku University
[2] Graduate School of Science and Engineering, Yamagata University

1 Introduction

A threshold circuit is a generalized computational model of standard logic circuits consisting of AND, OR, NOT gates, whose basic elements compute linear threshold functions. Threshold circuits have attracted considerable attention in circuit complexity, and much research has been devoted to understand their computation for a few decades [7, 10–13]. However, the computational power of threshold circuits is still in the dark. We cannot even rule out the possibility that any decision problem in NEXP (that is, exponential analog of NP) is computable by a threshold circuit of polynomial size and depth two (e.g. see [4, 21]).

In this paper, we investigate the computational power of threshold circuits in terms of a biologically-inspired complexity measure, called *energy complexity*. As a neural network in the brain carries out information processing by conveying electrical signals (i.e., "firing") among neurons, we can view a threshold circuit as a network computing a Boolean function by conveying Boolean values (i.e., "1") among threshold gates. The *energy* e of a threshold circuit C is then defined as the maximum number of gates outputting "1" in C, where the maximum is taken over all the input assignments to C. Uchizawa *et al.* [17] introduced the energy complexity to study the computational power of neural networks with sparse activity, where a relatively small number of simultaneously firing neurons out of a large population contributes to information processing in the nervous system, since many experimental and theoretical studies support that such sparsity constitutes a general principle of neural coding employed in the brain [1, 2, 8]. More recently, the computational power of neural networks with sparse activity has growth of interest, since training a neural network so as to acquire sparsity, so-called sparse coding, is found to benefit constructing high-performance neural networks, and is now a common trick widely used in deep learning methods [3, 5, 6].

There are several known results on the computational power of threshold circuits of small energy. It is known that any Boolean function is computable by a threshold circuit of energy one if an exponential size is allowed [16]. We also know that any linear decision tree (i.e., binary decision trees whose internal nodes are labeled by linear threshold functions) of polynomial number of leaves can be simulated by a threshold circuit of polynomial size and logarithmic energy [17]. Besides, some research shows how the energy complexity is involved in other major complexity measures of threshold circuits such as size, depth and fan-in. It is known that there exists an n-variable Boolean function for which any constant-depth circuit of energy $n^{O(1)}$ requires exponential size, while the function is computable by a threshold circuit of depth two and linear size if we allow a circuit to have energy $O(n)$ [18] . There also exist tradeoffs relating size, fan-in and energy of a threshold circuit computing an explicit Boolean function [14, 15, 19, 20], and a threshold circuit thus needs more size (depth, or fan-in) to be sparse. In particular, Suzuki *et al.* consider in [14] extreme cases where threshold circuits have energy one. For a positive integer m, consider a Boolean function MOD_n^m with n variables $(n \geq m)$ defined as follows: For every $\mathbf{x} \in \{0,1\}^n$, $\mathrm{MOD}_n^m(\mathbf{x}) = 0$ if and only if the number of ones in $\mathbf{x}$ is a multiple of m. They show that any threshold circuit of energy one require size $2^{(n-m)/2}$ to compute MOD_n^m, but the function is computable by a threshold circuit of size $O(n)$ and energy two.

We here explore the computational power of threshold circuits of energy one in more detail. We first show a proposition that is useful to construct a threshold circuit of energy one, and also present a simple lemma that we obtain by formalizing and generalizing a proof method implicitly used in [15] to evaluate computational limitation of threshold circuits of energy one. We then derive lower and upper bounds for threshold circuit of energy one for several Boolean functions. Moreover, we show that these functions are computable by threshold circuits of substantially less size if energy two is allowed, which highlights the difference between circuits of energy one and two. More formally, we focus on threshold circuits computing the following three functions.

We first consider Sum-Inequality Function SIE_n defined as follows: For every pair of $\mathbf{x}, \mathbf{y} \in \{0,1\}^n$, we define $\mathrm{SIE}_n(\mathbf{x}, \mathbf{y}) = 1$ if and only if the number of ones in $\mathbf{x}$ is different from the counterpart of $\mathbf{y}$. We show that any threshold circuit of energy one, which computes SIE_n, requires size at least $\binom{2n}{n}$ and this lower bound is best possible. Moreover, we observe that SIE_n is computable by a threshold circuit of size just two if energy two is allowed.

We then consider the complement of MOD_n^m, denoted by $\overline{\mathrm{MOD}_n^m}$. Note that the complement of SIE_n is computable by a threshold circuit of size two and energy one while SIE_n requires an exponential size, which implies that a lower bound for a Boolean function does not immediately imply the one for its complement. Therefore, though Suzuki *et al.* [14] obtained an exponential lower bound for MOD_n^m, it is unclear whether a similar lower bound holds for $\overline{\mathrm{MOD}_n^m}$. In contrast to SIE_n, we prove that $\overline{\mathrm{MOD}_n^m}$ also requires size $\Omega((2^{n/(m-1)})/m)$ for threshold circuits of energy one. We also construct a threshold circuit of energy one to show that this lower bound is almost tight, and show that $\overline{\mathrm{MOD}_n^m}$ is computable by a threshold circuit of linear size and energy two.

We lastly consider Generalized Inner Product Functions GIP_n^k defined as follows: For every $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k) \in (\{0,1\}^n)^k$ where $\mathbf{x}_i \in \{0,1\}^n$, $1 \leq i \leq k$, we denote $\mathbf{x}_i[j]$ by the jth-component of $\mathbf{x}_i$, and we then define $\mathrm{GIP}_n^k(\mathbf{x}) = 1$ if and only if $\sum_{j=1}^{n} \mathbf{x}_1[j]\mathbf{x}_2[j]\cdots\mathbf{x}_k[j]$ is odd. We prove that GIP_n^k requires size at least $((k-1)^n + (k+1)^n)/2^k$ for threshold circuits of energy one, while it is computable by a threshold circuit of size $(k+1)^n$ and energy one. We then show that GIP_n^k is computable by a threshold circuit of size 2^n and energy two for any k; note that the size is independent of k.

Consequently, our results imply that increasing energy by one gives remarkable computational power to a threshold circuit of energy one under any of the three cases where the circuit has constant size, linear size or exponential size.

The rest of the paper is organized as follows. In Sections 2, we define threshold circuits, and give some proposition with regards to threshold circuits of energy one. In Section 3, we present upper and lower bounds for threshold circuits computing SIE_n, $\overline{\mathrm{MOD}_n^m}$, and GIP_n^k. In Section 4, we conclude with some remarks.

2 Preliminaries

In Section 2.1, we define some terms on threshold circuits, and provide simple propositions for threshold circuits of energy one. In Section 2.2, we give a lemma that is useful to evaluate computational limitation of threshold circuits of energy one.

2.1 Threshold Circuits

A *threshold gate* g is a logic gate computing a linear threshold function of an arbitrary integer z of inputs, which is identified by weight $\mathbf{w}(g) \in \mathbb{R}^z$ for the z inputs and an threshold $t(g) \in \mathbb{R}$, where the ith component of $\mathbf{w}(g)$, denoted by $\mathbf{w}(g)[i]$, is a weight for ith input. We define the output $g(\mathbf{x})$ of g as follows: For every $\mathbf{x} \in \{0,1\}^z$,

$$g(\mathbf{x}) = \mathrm{sign}\left(\mathbf{w}(g) \cdot \mathbf{x} - t(g)\right) \tag{1}$$

$$= \begin{cases} 1 & \text{if } \sum_{i=1}^{z} \mathbf{w}(g)[i]\mathbf{x}[i] \geq t(g); \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

where $\mathbf{w}(g) \cdot \mathbf{x}$ denotes the inner-product of $\mathbf{w}(g)$ and $\mathbf{x}$. It is known that a threshold gate is closed under complement: If a Boolean function f is computable by a threshold gate, then the complement of f is also computable by a threshold gate [9].

A *threshold circuit* C is a feedforward circuit consisting of threshold gates, and is expressed by a directed acyclic graph. Let n be the number of inputs to C, then C has n input nodes of in-degree 0, each of which corresponds to one of the n input variables $\mathbf{x}[1], \mathbf{x}[2], \cdots, \mathbf{x}[n]$, while the other nodes correspond to threshold gates. The inputs to a gate g in C consists of the inputs $\mathbf{x}[1], \mathbf{x}[2], \cdots, \mathbf{x}[n]$ and the outputs of some gates directed to g. Let g_s be one of the gates of out-degree 0, and we regard the output $g_s(\mathbf{x})$ of g_s as the *output* $C(\mathbf{x})$ *of* C, that is, $C(\mathbf{x}) = g_s(\mathbf{x})$ for every input $\mathbf{x} \in \{0,1\}^n$. We call g_s the *top gate* of C. A threshold circuit C *computes* a Boolean function $f : \{0,1\}^n \to \{0,1\}$ if $C(\mathbf{x}) = f(\mathbf{x})$ for every $\mathbf{x} \in \{0,1\}^n$. We define *size* s of C as the number of gates in C, and define the *energy* e of C as

$$e = \max_{\mathbf{x} \in \{0,1\}^n} \sum_{i=1}^{s} g_i(\mathbf{x}),$$

where $g_i(\mathbf{x})$ is the output of g_i when input of C is $\mathbf{x}$. We may assume without loss of generality that s and e are at least one.

For any integer n, we denote $[1..n] = \{1, 2, \ldots, n\}$. For a Boolean function f of n variables, we define $S_0(f) = \{\mathbf{x} \in \{0,1\}^n \mid f(\mathbf{x}) = 0\}$ and $S_1(f) = \{\mathbf{x} \in \{0,1\}^n \mid f(\mathbf{x}) = 1\}$. Similarly, for a threshold gate g of n variables, we define $S_0(g) = \{\mathbf{x} \in \{0,1\}^n \mid g(\mathbf{x}) = 0\}$ and $S_1(g) = \{\mathbf{x} \in \{0,1\}^n \mid g(\mathbf{x}) = 1\}$.

The following lemma is useful to construct a circuit of energy one:

Lemma 1. *Let f be a Boolean function of n variables. Let T be a set of threshold gates of n inputs such that*

$$\bigcup_{g \in T} S_1(g) = S_0(f). \tag{3}$$

Then, f is computable by a threshold circuit of size $|T|$ and energy one.

It is shown in [16] that any Boolean function f is computable by a threshold circuit of size $|S_0(f)| + 1$ and energy one. Using Lemma 1, we can remove the additive term $+1$, as follows.

Theorem 1. *Any Boolean function f is computable by a threshold circuit of size $|S_0(f)|$ and energy one.*

2.2 Strongly α-Fooling Set

In this section, we introduce a new notion that plays important role in deriving lower bounds for threshold circuits of energy one. For an integer $\alpha \geq 2$, we say that a set $R \subseteq S_0(f)$ such that $|R| \geq \alpha$ is *strongly α-fooling set for f* if, for any α-tuple of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_\alpha \in R$, there exists an α-tuple of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_\alpha \in S_1(f)$ such that

$$\sum_{i=1}^{\alpha} \mathbf{x}_i = \sum_{i=1}^{\alpha} \mathbf{y}_i$$

where the summation is componentwise. The following lemma claims that a strongly α-fooling set for f implies a lower bound on the size of any threshold circuit of energy one that computes f.

Lemma 2. *If there exists a strongly α-fooling set R for a Boolean function f, then any threshold circuit C of energy one that computes f has size*

$$s \geq \frac{|R| - 1}{\alpha - 1} + 1.$$

In the paper [14], they implicitly used strongly 2-fooling set to derive an exponential lower bound for MOD_n^m. We here formalize the idea, and show that the method can be generalized to an arbitrary integer $\alpha \geq 2$.

3 Lower and Upper Bounds for SIE_n, $\overline{\mathrm{MOD}_n^m}$ and GIP_n^k

In this section, we provide lower and upper bounds for threshold circuits of energy one and two. In Sections 3.1, 3.2 and 3.3 , we consider SIE_n, MOD_n^m and GIP_n^k, respectively.

3.1 Sum-Inequality SIE_n

For $\mathbf{x} \in \{0,1\}^n$, we define $\mathcal{H}(\mathbf{x})$ as the hamming weight (i.e., the number of ones) of $\mathbf{x}$. Thus, for every pair of $\mathbf{x} \in \{0,1\}^n$ and $\mathbf{y} \in \{0,1\}^n$,

$$\mathrm{SIE}_n(\mathbf{x},\mathbf{y}) = \begin{cases} 1 & \text{if } \mathcal{H}(\mathbf{x}) \neq \mathcal{H}(\mathbf{y}); \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 2, we obtain a lower bound for SIE_n.

Theorem 2. *Any threshold circuit of energy one that computes* SIE_n *has size*

$$s \geq \binom{2n}{n}.$$

Since it holds that

$$|S_0(\mathrm{SIE}_n)| = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n},$$

Theorem 1 implies that the lower bound given in Theorem 2 is best possible.

Corollary 1. SIE_n *is computable by a threshold circuit of size* $\binom{2n}{n}$ *and energy one.*

By the above theorem, any threshold circuits of energy one requires an exponential number of gates to compute SIE_n, which contrast with the the following proposition.

Proposition 1. SIE_n *is computable by a threshold circuit of size two and energy two.*

3.2 MOD function $\overline{\mathrm{MOD}_n^m}$

Recall that $\overline{\mathrm{MOD}_n^m}$ is the complement of MOD_n^m: For every $\mathbf{x} \in \{0,1\}^n$, $\overline{\mathrm{MOD}_n^m}(\mathbf{x}) = 1$ if and only if the number of ones in $\mathbf{x}$ is a multiple of m. We define

$$M_l = \{\mathbf{x} \in \{0,1\}^n \mid \mathcal{H}(\mathbf{x}) \equiv l \bmod m\}$$

for every l, $0 \leq l \leq m-1$. Using Lemma 2 with setting $\alpha = m$, we can give a lower bound for $\overline{\mathrm{MOD}_n^m}$.

Theorem 3. *Any threshold circuit of energy one that computes* $\overline{\mathrm{MOD}_n^m}$ *has size*

$$s \geq \frac{2^{\lfloor n/(m-1) \rfloor - 1} - 1}{m-1} + 1. \tag{4}$$

Theorem 1 implies that MOD_n^m is computable by a threshold circuit of size $s = |M_1 \cup M_2 \cup \cdots \cup M_{m-1}|$ and energy one. We below show that $\overline{\mathrm{MOD}_n^m}$ is computable by a threshold circuit of much smaller size and energy one.

Proposition 2. $\overline{\mathrm{MOD}_n^m}$ *is computable by a threshold circuit of size* $|M_1|$ *and energy one.*

We say that a Boolean function f is symmetric if $f(\mathbf{x})$ depends only on $\mathcal{H}(\mathbf{x})$ for every $\mathbf{x} \in \{0,1\}^n$. The following claim immediately implies that MOD_n^m is computable by a threshold circuit of size $n+1$ and energy two.

Proposition 3. *Any symmetric Boolean function of n variables is computable by a threshold circuit of size* $n+1$ *and energy two.*

3.3 Generalized Inner Product Function GIP_n^k

In this section, we consider Generalized Inner Product GIP_n^k. Similarly to the last sections, we can obtain the following lower bound by Lemma 2.

Theorem 4. *For any two positive integers $n \geq 1$ and $k \geq 2$, any threshold circuit of energy one that computes GIP_n^k has size*

$$s \geq \frac{(k-1)^n + (k+1)^n}{2^k}.$$

The following theorem shows the bound in the above theorem is asymptotically tight for constant k.

Theorem 5. *For positive integers k and n, GIP_n^k is computable by a threshold circuit of size at most $(k+1)^n$ and energy one.*

Lastly, we show that GIP_n^k is computable by a threshold circuit of size 2^n and energy two for any k. Since the size of our circuit is independent of k, and thus can be exponentially smaller if energy two is allowable.

Theorem 6. *For any positive integers k and n, GIP_n^k is computable by a threshold circuit of size 2^n and energy two.*

4 Conclusion

In this paper, we investigate threshold circuits of energy one, and show that the computational power of a threshold circuit of energy one and the counterpart of energy two are remarkably different for any of three cases where threshold circuits have constant size, linear size or exponential-size. It would be interesting to know if similar claims hold for threshold circuits of any of these cases, when we consider threshold circuits of energy e and $e+1$ for any constant $e \geq 2$.

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Graduate School of Information Sciences
Tohoku University
Sendai 980-8579
JAPAN
E-mail address: hiroki.maniwa.r1@dc.tohoku.ac.jp
a.suzuki@ecei.tohoku.ac.jp
zhou@ecei.tohoku.ac.jp

Graduate School of Science and Engineering
Yamagata University
Yamagata 992-8510
JAPAN
E-mail address: thm77478@st.yamagata-u.ac.jp
uchizawa@yz.yamagata-u.ac.jp